International Journal of Solids and Structures 47 (2010) 3214–3220

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Fundamental formulation for transformation toughening

Lifeng Ma *

S&V Lab, Department of Engineering Mechanics, Xi'an Jiaotong University, 710049, China

ARTICLE INFO

Article history:
Received 20 April 2010
Received in revised form 6 July 2010
Available online 10 August 2010

Keywords:
Transformation toughening
Fundamental solution
Green's function

ABSTRACT

In this paper, the transformation toughening problem is addressed in the framework of plane strain. The fundamental solution for a transformed strain nucleus located in an infinite plane is derived first. With this solution, the transformed inclusion problems are formulated by a Green's function method, and the interaction of a crack tip with a single transformation source is found. On the basis of this solution, the fundamental formulations for toughening arising from martensitic and ferroelastic transformation are formulated also using the Green's function method. Finally, some examples are provided to demonstrate the validity and relevance of the fundamental formulations proposed in the paper.

1. Introduction

There is much experimental evidence that the toughness of some ceramics can be substantially enhanced through the controlled use of martensitic transformation (see e.g., Garvie et al., 1975; Claussen, 1976; Gupta et al., 1977; Hannink, 1978; Evans and Heuer, 1980; Lange, 1982; Munz and Fett, 1998; Hannink et al., 2000; Rauchs et al., 2001, 2002; Kelly and Rose, 2002; Magnani and Brillante, 2005) or ferroelastic transformation (Clussen et al., 1984). In the case of martensitic transformation, experiments reveal that the toughening is due to a crack tip stress induced phase transformation. When the stresses in the region near the crack tip reach a critical value, zirconia inclusions particle transform from tetragonal to monoclinic, accompanied by a volume increment of 4% and a shear strain of 16%. These strains induce further stress in the crack tip region, and the stress intensity factors at the crack tip may be reduced. Therefore, the fracture toughness of the ceramic is effectively enhanced, since it takes higher applied loads to raise the stress intensity factor back up to the critical level required to cause continued crack propagation. In contrast to the martensitic case, ferroelastic transformations typically have only a shear component. Ferroelastic toughening is attributed to domain switching in the crack front and crack wake, inducing stress intensity factor reductions (see, e.g. Yang and Zhu, 1998; Wang et al., 2004; Jones et al., 2005, 2007; Jones and Hoffman, 2006; Pojprapai et al., 2008).

In addition to experimental assessments, transformation toughening has been also the subject of numerous modeling studies. Three main approaches to model transformation toughening have been used. One is an Eshelby-type approach (e. g. McMeeking and Evans, 1982; Yang and Zhu, 1998; Yi and Gao, 2000; Yi et al., 2001; Li and Yang, 2002; Fischer and Boehm, 2005). Another is the finite element method (FEM), (e.g., Zeng et al., 1999, 2004; Vena et al., 2006). The third approach is a Green's function method (e.g. Budiansky et al., 1983; Lambropoulos, 1986; Rose, 1987; Tsukamoto and Kotousov, 2006). The Eshelby-type approach can be used for analysis of interactions between a crack tip and a discrete transformed zone, but it is not convenient when multiple transformed zones are involved. FEM can be used effectively for arbitrary complex geometry of transformed zones and also with complex material constitutive laws, but multiple analyses must be undertaken to obtain adequate coverage of the appropriate parameter range. The Green's function method is convenient and straightforward when used for many kinds of geometries of transformed regions. In this paper, we concentrate on the Green's function method to provide improved tools for the analysis of transformation toughening.

Considerable progress has been achieved in the application of Green's function methods to transformation toughening problems. For example, Hutchinson (1974) solved the plane problem of the interaction of a semi-infinite crack in an infinite body and two transformed circular "spots" symmetrically located relative to the crack plane. Based on this solution, Budiansky et al. (1983) obtained results for the problem of a continuum transformation zone surrounding the crack tip with a focus on the effect of its dilatation. Thereafter, this approach has been used frequently to study transformation toughening (e.g., Lambropoulos, 1986; Tsukamoto and Kotousov, 2006). However, these solutions are only good for the problems with transformed zone that are *symmetrical* with respect to the crack plane. Separately, Rose (1987) represented both dilatant and deviatoric transformed strain components with a set of fundamental singular solutions such as a force-doublet, similar to the work of Love (1927). His methodology is rigorous, but not

* Tel.: +86 29 82663861; fax: +86 29 82668751.
E-mail address: malf@mail.xjtu.edu.cn

0020-7683/$ - see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijsolstr.2010.08.002

straightforward and is inconvenient for application. Using Rose's solution, a detailed study on transformation toughing has been conducted recently (Karihaloo and Andreasen, 1996). In the present paper we carry out a thorough and systematic study of the Green's function method for formulating problems of transformation toughening to enable more powerful tools for that application.

The aim of the present study is to develop new fundamental solutions, from which modeling of transformation toughening by Green's function methods can be easily performed. Dilatant and deviatoric strain components of transformation are to be derived and analyzed in full.

The formulation is constructed in the following steps. Firstly, a fundamental solution for a transformed strain nucleus located within an infinite plane is derived in Section 2. Based on this solution, the transformed inclusions problems are formulated first in Section 3, and the interaction of a transformed strain nucleus with a semi-infinite crack is studied in Section 4. Subsequently in Section 5, fundamental formulations for transformation toughening for martensitic and ferroelastic phenomena are developed with the Green's function method. To demonstrate the validity and relevance of these formulations, some simple but typical transformation toughening examples are studied in Section 6, and finally conclusions are drawn in Section 7.

2. Fundamental solution for a transformed strain nucleus located in an infinite plane solid

In this section, we seek the *Muskhelishvili potentials* for a transformed strain nucleus located in an infinite plane solid. The strain nucleus will be modeled in terms of the mathematical edge dislocation solutions.

2.1. Muskhelishvili formulation and Muskhelishvili potentials for an edge dislocation

In the Muskhelishvili complex formulation of plane elasticity, all components of stress and displacements are expressed in terms of two potential functions, $\Phi(z)$ and $\Omega(z)$ as follows (Suo, 1989; Muskhelishvili, 1953),

$$\begin{aligned}
&\sigma_{11}+\sigma_{22}=2[\Phi(z)+\overline{\Phi(z)}]\\
&\sigma_{22}-i\sigma_{12}=[\Phi(z)+\overline{\Omega(z)}+(z-\bar{z})\overline{\Phi'(z)}]\\
&2\mu(u_{1,1}+iu_{2,1})=\kappa\Phi(z)-\overline{\Omega(z)}-(z-\bar{z})\overline{\Phi'(z)}
\end{aligned}\tag{2.1}$$

where, $i=\sqrt{-1}$, $z = x_1 + ix_2$, $\Phi'(z) = d\Phi(z)/dz$, μ is shear modulus, $\kappa = 3 - 4\nu$ for plane strain, the comma followed by a subscript i indicates differentiation with respect to x_i, and the bar over a function denotes its complex conjugate. It is known that the Muskhelishvili potentials for an edge dislocation with Burgers vector B with magnitude b, located at point s within an infinite plane solid, can be expressed as (Suo, 1989):

$$\begin{aligned}
&\Phi(z)=F\frac{B}{z-s}\\
&\Omega(z)=FB\frac{(\bar{s}-s)}{(z-s)^2}-F\frac{\overline{B}}{z-s}\\
&B=be^{i\psi}\\
&F=\frac{\mu}{\pi i(1+\kappa)}
\end{aligned}\tag{2.2}$$

From the results above, we will derive the potentials of a transformed strain nucleus as follows.

2.2. Muskhelishvili potentials for a transformed strain nucleus

Consider a differential element with an area $dA(= dx_0 dy_0)$, which undergoes an unconstrained irreversible transformation with two *principal* strains ε_{x0} and ε_{y0} expressed in local *principal* coordinates x_0, y_0 as shown in Fig. 1. The origin of the local coordinate system x_0, y_0 lies at s in the global coordinate system x, y, and ψ is the orientation angle of the x_0 axis (associated with the *principal* strain ε_{x0}) with respect to the global x-coordinate axis. Next, given the physical meaning of an edge dislocation, an infinitesimal element with transformation strain can be represented by an assembly of four dislocations as shown in Fig. 1. The potentials for the four dislocations in the global coordinate system can be written as:

$$\begin{aligned}
&\Phi_1(z)=F\frac{B_1}{z-s_1},\quad \Omega_1(z)=F\frac{B_1(\overline{s_1}-s_1)}{(z-s_1)^2}-F\frac{\overline{B_1}}{z-s_1}\\
&\Phi_2(z)=F\frac{B_2}{z-s_2},\quad \Omega_2(z)=F\frac{B_2(\overline{s_2}-s_2)}{(z-s_2)^2}-F\frac{\overline{B_2}}{z-s_2}\\
&\Phi_3(z)=F\frac{B_3}{z-s_3},\quad \Omega_1(z)=F\frac{B_3(\overline{s_3}-s_3)}{(z-s_3)^2}-F\frac{\overline{B_3}}{z-s_3}\\
&\Phi_4(z)=F\frac{B_4}{z-s_4},\quad \Omega_4(z)=F\frac{B_4(\overline{s_4}-s_4)}{(z-s_4)^2}-F\frac{\overline{B_4}}{z-s_4}
\end{aligned}\tag{2.3}$$

where

$$\begin{aligned}
&B_1=e^{i\psi}(\varepsilon_{x_0}dx_0),\quad B_3=e^{i\pi}e^{i\psi}(\varepsilon_{x_0}dx_0),\\
&B_2=e^{-i\frac{\pi}{2}}e^{i\psi}(\varepsilon_{y_0}dy_0),\quad B_4=e^{i\frac{\pi}{2}}e^{i\psi}(\varepsilon_{y_0}dy_0),\\
&s_1=s-e^{i\psi}e^{i\frac{\pi}{2}}\left(\frac{dy_0}{2}\right),\quad s_3=s+e^{i\psi}e^{i\frac{\pi}{2}}\left(\frac{dy_0}{2}\right)\\
&s_2=s-e^{i\psi}\left(\frac{dx_0}{2}\right),\quad s_4=s+e^{i\psi}\left(\frac{dx_0}{2}\right)
\end{aligned}\tag{2.4}$$

The subscripts on dislocation parameters and potentials in the above expressions refer to the denoted dislocation as numbered in Fig. 1. The corresponding Burgers vector B_i in (2.4) is bestowed with a special meaning which represents the residual deformation of the differential element due to transformation.

The potentials for an infinite plane due to a transformation strain with principal values ε_{x0}, ε_{y0}, oriented in the direction shown

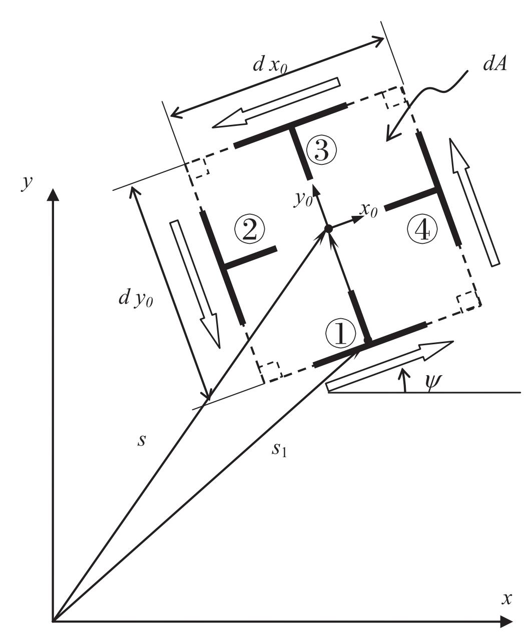

Fig. 1. A concentrated transformed strain located in an infinite plane solid.

in Fig. 1 within the differential element area $dA = dx_0dy_0$, can be obtained by superposing the four potentials together as

$$\begin{aligned}\Phi(z) &= [\Phi_1(z) + \Phi_3(z)] + [\Phi_2(z) + \Phi_4(z)]\\ \Omega(z) &= [\Omega_1(z) + \Omega_3(z)] + [\Omega_2(z) + \Omega_4(z)]\end{aligned} \tag{2.5}$$

After a lengthy but straightforward manipulation, we get

$$\begin{aligned}\Phi(z) &= iFe^{2i\psi}\frac{(\varepsilon_{y0}-\varepsilon_{x0})}{(z-s)^2}dx_0dy_0\\ &= iFe^{2i\psi}\frac{(\varepsilon_{y0}-\varepsilon_{x0})}{(z-s)^2}dA\\ \Omega(z) &= iF\left\{\frac{2(\varepsilon_{x0}+\varepsilon_{y0})-e^{2i\psi}(\varepsilon_{y0}-\varepsilon_{x0})}{(z-s)^2}+\frac{2e^{2i\psi}(\varepsilon_{y0}-\varepsilon_{x0})}{(z-s)^3}(\bar{s}-s)\right\}dx_0dy_0\\ &= iF\left\{\frac{2(\varepsilon_{x0}+\varepsilon_{y0})-e^{2i\psi}(\varepsilon_{y0}-\varepsilon_{x0})}{(z-s)^2}+\frac{2e^{2i\psi}(\varepsilon_{y0}-\varepsilon_{x0})}{(z-s)^3}(\bar{s}-s)\right\}dA\end{aligned} \tag{2.6}$$

Clearly the potentials in Eq. (2.6) are of infinitesimal order. Finite order potentials for the effect per unit area thus characterize the transformed strain *nucleus*, located at *s* in an infinite plane, and are

$$\begin{aligned}\Phi_0(z) &= iFe^{2i\psi}\frac{(\varepsilon_{y0}-\varepsilon_{x0})}{(z-s)^2}\\ \Omega_0(z) &= iF\left\{\frac{2(\varepsilon_{x0}+\varepsilon_{y0})-e^{2i\psi}(\varepsilon_{y0}-\varepsilon_{x0})}{(z-s)^2}+\frac{2e^{2i\psi}(\varepsilon_{y0}-\varepsilon_{x0})}{(z-s)^3}(\bar{s}-s)\right\}\end{aligned} \tag{2.7}$$

For convenience, we may rewrite Eq. (2.7) in a compact form as

$$\begin{aligned}\Phi_0(z) &= \frac{C_1}{(z-s)^2}\\ \Omega_0(z) &= \frac{C_2}{(z-s)^2}+C_3\frac{(\bar{s}-s)}{(z-s)^3}\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}C_1 &= iFe^{2i\psi}(\varepsilon_{y0}-\varepsilon_{x0})\\ C_2 &= iF[2(\varepsilon_{x0}+\varepsilon_{y0})-e^{2i\psi}(\varepsilon_{y0}-\varepsilon_{x0})]\\ C_3 &= 2iFe^{2i\psi}(\varepsilon_{y0}-\varepsilon_{x0})\end{aligned} \tag{2.9}$$

The set of potentials given by (2.8), (2.9) gives the fundamental solution for a transformation-induced strain *nucleus* in an infinite plane. This is the basic result on which the further analysis presented in this paper is built.

When $\varepsilon_{y0} = \varepsilon_{x0}$, namely, for a purely dilatational transformation, Eq. (2.8) will be independent of the orientation angle ψ, consistent with the expected isotropy of the effects of such a transformation. On the other hand, when $\varepsilon_{y0} = -\varepsilon_{x0}$, the solution for purely shear transformation is reached. It can be easily proved that the solutions of the two extreme cases degenerated from (2.8) are consistent with the ones obtained by Rose (1987).

3. Formulation of transformed inclusion problems by the Green's function method

In this section, we will use the fundamental solution in the above Section to formulate the transformed inclusion problems. Before doing so, we must emphasize that the following is a continuum description of polycrystalline systems (e.g. ceramics) and is based on averaging over sufficient numbers of grains within the matrix, including transformed particles. Also, we assume that the elastic properties of the transformed zone (inclusion) are identical to those prior to the transformation, following McMeeking and Evans (1982), and Budiansky et al. (1983).

3.1. Influence function for transformed inclusions

Submitting Eq. (2.8) into Eq. (2.1) we may obtain the stress components denoted as $\sigma^0_{11}(z,s)$, $\sigma^0_{12}(z,s)$ and $\sigma^0_{22}(z,s)$ at point *z*, due to the presence of a *point* transformed region at position *s*. We denote these *influence functions* by

$$\begin{aligned}f_1(x,y;x_s,y_s) &= \sigma^0_{11}(z,s)\\ f_2(x,y;x_s,y_s) &= \sigma^0_{22}(z,s)\\ f_3(x,y;x_s,y_s) &= \sigma^0_{12}(z,s)\end{aligned} \tag{3.1}$$

where $z = x + iy$ and $s = x_s + iy_s$ in the global coordinate system. The so-called influence functions in Eqs. (3.1) are used to calculate the stress field due to presence of transformed inclusions in an infinite plane solid.

3.2. Formulation of the stress field due to transformed inclusions by the Green's function method

Now consider an infinite plane solid in which some portions suffer transformation due to external load (Fig. 2) and the transformed region corresponds to area $A = \sum_{k=1}^{N} A_k$. Generally speaking, the transformed particles could not fully occupy the *nominal* region *A*, as in practice the material transformation does not run to full 100%. To reflect this, we define a *transformation density function* $D(x_s,y_s)$, to describe the extent of transformation in the *nominal* area, using the value in the range $0 \leqslant D(x_s,y_s) \leqslant 1$. Then the expression for the stress at any point (x,y) in the plane can be formulated by the Green's function method as

$$\begin{aligned}\sigma_{11}(x,y) &= \int\int_A f_1(x,y;x_s,y_s)D(x_s,y_s)dx_sdy_s\\ \sigma_{22}(x,y) &= \int\int_A f_2(x,y;x_s,y_s)D(x_s,y_s)dx_sdy_s\\ \sigma_{12}(x,y) &= \int\int_A f_3(x,y;x_s,y_s)D(x_s,y_s)dx_sdy_s, \quad x_s,y_s \in A\end{aligned} \tag{3.2}$$

where the influence functions $f_1(x,y;x_s,y_s)$, $f_2(x,y;x_s,y_s)$, $f_3(x,y;x_s,y_s)$ are given by (3.1). If the transformation density function $D(x_s,y_s)$ is a constant, the stress field will depend only on the influence function and the transformation zone shape and extent. Eq. (3.2) is the fundamental solution for the transformed inclusion problems.

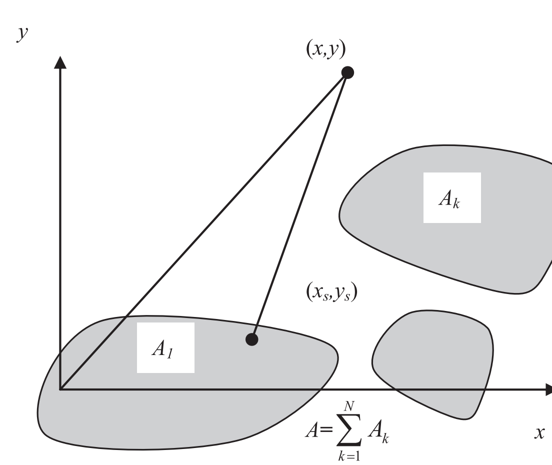

Fig. 2. Transformation strain zones in an infinite plane.

4. A transformed strain nucleus interacting with a semi-infinite crack

In this section the basic problem of a transformed strain nucleus interacting with a semi-infinite crack will be studied. It is assumed that the size of the transformed area is small compared with the crack length and other dimensions. The solution will be used to obtain the influence functions to formulate the problems about transformation toughening in Section 5.

4.1. Complex potentials for the model

Consider a transformed strain source located at s having the orientation angle ψ, interacting with a semi-infinite crack shown in Fig. 3. In the previous section, we obtained the complex potentials $\Phi_0(z,s)$, $\Omega_0(z,s)$ for a source in an infinite plane solid without a crack. Using the superposition principle and following the procedure described previously (Ma et al., 2006), one can find the exact complex potentials for the interaction of a transformation strain source with a semi-infinite crack. Omitting lengthy details, we present the general form of the result as follows:

$$\begin{aligned}\Omega(z) &= \frac{1}{\sqrt{z}}\left\{\frac{1}{2\pi}\int_{-\infty}^{0}\frac{-\left[\overline{\Phi_0}(t)+\Omega_0(t)\right]\sqrt{|t|}}{t-z}dt\right\}+\Omega_0(z)\\ \Phi(z) &= \frac{1}{\sqrt{z}}\left\{\frac{1}{2\pi}\int_{-\infty}^{0}\frac{-\left[\Phi_0(t)+\overline{\Omega_0}(t)\right]\sqrt{|t|}}{t-z}dt\right\}+\Phi_0(z)\end{aligned} \tag{4.1}$$

Direct substitution of Eq. (2.8) into Eq. (4.1) gives

$$\begin{aligned}\Omega(z) =& \frac{1}{\sqrt{z}}\left\{-\frac{1}{2\pi}\int_{-\infty}^{0}\left[\frac{\overline{C_1}}{(t-\bar{s})^2}+\frac{C_2}{(t-s)^2}+C_3\frac{(\bar{s}-s)}{(t-s)^3}\right]\sqrt{|t|}\frac{1}{t-z}dt\right\}\\ &+\frac{C_2}{(z-s)^2}+C_3\frac{(\bar{s}-s)}{(z-s)^3}\\ \Phi(z) =& \frac{1}{\sqrt{z}}\left\{-\frac{1}{2\pi}\int_{-\infty}^{0}\left[\frac{C_1}{(t-s)^2}+\frac{\overline{C_2}}{(t-\bar{s})^2}-\overline{C_3}\frac{(\bar{s}-s)}{(t-\bar{s})^3}\right]\sqrt{|t|}\frac{1}{t-z}dt\right\}\\ &+\frac{C_1}{(z-s)^2}\end{aligned} \tag{4.2}$$

After some manipulation, the complex functions for the interaction of a transformation strain source with a semi-infinite crack are finally obtained as:

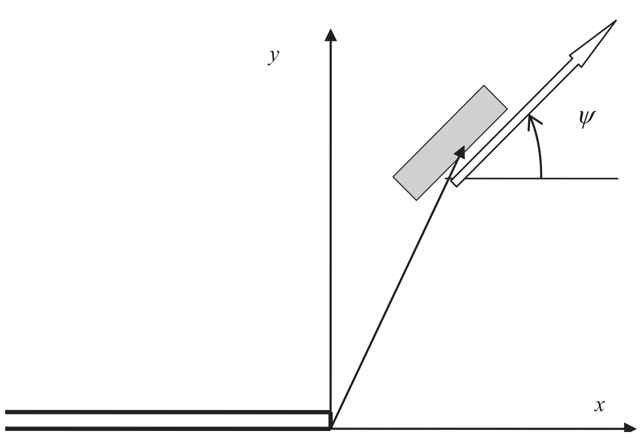

Fig. 3. A transformation strain source interacting with a semi-infinite crack.

$$\begin{aligned}\Omega(z) &= \frac{1}{2\sqrt{z}}\left\{\begin{array}{l}\frac{\overline{C_1}}{2\sqrt{\bar{s}}\left(\sqrt{\bar{s}}+\sqrt{z}\right)^2}+\frac{C_2}{2\sqrt{s}\left(\sqrt{s}+\sqrt{z}\right)^2}\\ -C_3(\bar{s}-s)\frac{\left(3\sqrt{s}+\sqrt{z}\right)}{8(s)^{\frac{3}{2}}\left(\sqrt{s}+\sqrt{z}\right)^3}\end{array}\right\}+\frac{C_2}{(z-s)^2}+C_3\frac{(\bar{s}-s)}{(z-s)^3}\\ \Phi(z) &= \frac{1}{2\sqrt{z}}\left\{\begin{array}{l}\frac{C_1}{2\sqrt{s}\left(\sqrt{s}+\sqrt{z}\right)^2}+\frac{\overline{C_2}}{2\sqrt{\bar{s}}\left(\sqrt{\bar{s}}+\sqrt{z}\right)^2}\\ +\overline{C_3}(\bar{s}-s)\frac{\left(3\sqrt{\bar{s}}+\sqrt{z}\right)}{8(\bar{s})^{\frac{3}{2}}\left(\sqrt{\bar{s}}+\sqrt{z}\right)^3}\end{array}\right\}+\frac{C_1}{(z-s)^2}\end{aligned} \tag{4.3}$$

4.2. Influence function for stress intensity factors

By virtue of the second equation of (2.1), the definition of the stress intensity factor (SIF) becomes,

$$\begin{aligned}K = K_I + iK_{II} &= \lim_{x\to 0^+}\sqrt{2\pi x}[\sigma_{22}+i\sigma_{12}]\\ &= \lim_{x\to 0^+}\sqrt{2\pi x}[\overline{\Phi}(x)+\Omega(x)]\end{aligned} \tag{4.4}$$

Inserting Eq. (4.3) into (4.4), we obtain the SIF due to the presence of the strain source as

$$K = K_I + iK_{II} = \frac{\sqrt{2\pi}}{2}\left[\overline{C_1}\frac{1}{\bar{s}\sqrt{\bar{s}}}+C_2\frac{1}{s\sqrt{s}}-C_3(\bar{s}-s)\frac{3\sqrt{s}}{4s^3}\right] \tag{4.5}$$

or

$$K = K_I + iK_{II} = \frac{\mu}{\sqrt{2\pi}(1+\kappa)}\left[\begin{array}{l}\frac{e^{-2i\psi}(\varepsilon_{y0}-\varepsilon_{x0})}{\bar{s}\sqrt{\bar{s}}}+\frac{\left[2(\varepsilon_{x0}+\varepsilon_{y0})-e^{2i\psi}(\varepsilon_{y0}-\varepsilon_{x0})\right]}{s\sqrt{s}}\\ -2e^{2i\psi}(\varepsilon_{y0}-\varepsilon_{x0})(\bar{s}-s)\frac{3\sqrt{s}}{4s^3}\end{array}\right] \tag{4.6}$$

Since the *stress intensity factors* (4.5) or (4.6) are induced by a single point source of transformation (namely, a strain nucleus), the SIF due to the presence of extended transformed areas can be calculated by integrating the transformation strain contribution over the source region, i.e. the transformation zone area. The expression in Eq. (4.6) will play the role of the integral kernel. We denote this point influence function for the SIF by

$$f_4(s) = \frac{\mu}{\sqrt{2\pi}(1+\kappa)}\left[\begin{array}{l}\frac{e^{-2i\psi}(\varepsilon_{y0}-\varepsilon_{x0})}{\bar{s}\sqrt{\bar{s}}}+\frac{\left[2(\varepsilon_{x0}+\varepsilon_{y0})-e^{2i\psi}(\varepsilon_{y0}-\varepsilon_{x0})\right]}{s\sqrt{s}}\\ -2e^{2i\psi}(\varepsilon_{y0}-\varepsilon_{x0})(\bar{s}-s)\frac{3\sqrt{s}}{4s^3}\end{array}\right] \tag{4.7}$$

5. Formulation of the transformation toughening problems with the Green's function method

Transformation-induced strain influence functions of different kinds have been derived in the previous sections. In this section, we use the influence functions to formulate the transformation toughening problems with Green's function method.

5.1. Martensitic transformation toughening

The problem geometry is shown in Fig. 4. As before, the transformation density function over the transformed zone $A=\sum_{k=1}^{N}A_k$ is denoted by $D(x_s,y_s)$. The net SIF at the crack tip due to the presence of the transformation zone is given by

$$\Delta K = \Delta K_I + \Delta K_{II} = \int\int_A f_4(x_s,y_s)D(x_s,y_s)dA \tag{5.1}$$

Here $f_4(x_s,y_s) = f_4(s)$ given in Eq. (4.7).

Generally, when remote loading is applied so as to create stress intensity K_∞, the material in the vicinity of the crack tip may undergo transformation under the high local stresses. Assuming the transformation zone and the transformation strain are known,

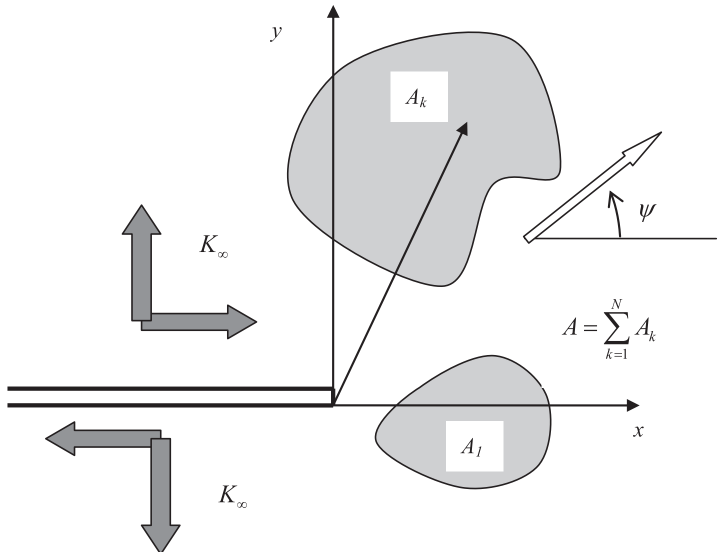

Fig. 4. Transformed zone in front of a crack tip under remote load.

Eq. (5.1), allows the evaluation of the crack tip "transformation shielding" effect in terms of SIF reduction. This allows the stability of the cracked body to be assessed.

5.2. Ferroelastic transformation toughening

The toughening mechanism of ferroelastic transformation results from the 90° ferroelastic domain switching (interchanging of the ferroelastic long and short axes) (Jones et al., 2007). This procedure can be briefly illustrated with Fig. 5.

The mechanism of ferroelastic transformation is different from martensitic transformation, but, mathematically, this case is a special transformation toughening case of the above general solution. Let us consider one elementary domain which is involved in the switching phenomenon in the plane strain regime as shown in Fig. 6. The domain has an original orientation ψ. We suppose that there is no residual stress present prior to domain switching. After domain switching from ψ to $\psi+\frac{\pi}{2}$, we can find that

$$\varepsilon_{x0} = -\varepsilon_0, \quad \varepsilon_{y0} = \varepsilon_0 \tag{5.2}$$

where $\varepsilon_0 = (c-a)/a \ll 1$.

Substituting (5.2) into (4.7), the influence function for the net SIF due to the elementary domain switching is obtained:

$$f_5(s) = \frac{2\mu\varepsilon_0}{\sqrt{2\pi}(1+\kappa)}\left[\frac{e^{-2i\psi}}{\bar{s}\sqrt{\bar{s}}} + \frac{1}{2}\left(1-3\frac{\bar{s}}{s}\right)\frac{e^{2i\psi}}{s\sqrt{s}}\right] \tag{5.3}$$

or

$$f_5(r,\theta,\psi) = \frac{\mu\varepsilon_0}{\sqrt{2\pi}(1+\kappa)r^{\frac{3}{2}}}\left\{\begin{array}{l}\left[3\cos\left(\frac{3}{2}\theta-2\psi\right)-3\cos\left(\frac{7}{2}\theta-2\psi\right)\right]\\ +i\left[\sin\left(\frac{3}{2}\theta-2\psi\right)+3\sin\left(\frac{7}{2}\theta-2\psi\right)\right]\end{array}\right\} \tag{5.4}$$

Similarly, if the transformed zone is known as shown in Fig. 4, the net SIF due to domain switching can be computed by the Green's function method:

$$\Delta K = \Delta K_I + i\Delta K_{II} = \int\int_A f_5(r,\theta,\psi)D(r,\theta)dA \tag{5.5}$$

where $D(r,\theta)$ is the transformation density function.

6. Examples

The purpose of the present paper is to establish a mathematical framework for the solution of *direct* transformation toughening problems, i.e. the computation of the effects of transformation-induced strain on the crack tip SIF. The questions about the region and the extent of transformation are not addressed here. It is clear, however, that in practice, prior to the evaluation of SIF incorporating the effects of transformation toughening, the transformed region and transformation strain must be determined first. In the literature, several criteria have been proposed for material transformation, such as the mean stress criterion (see e.g. Budiansky

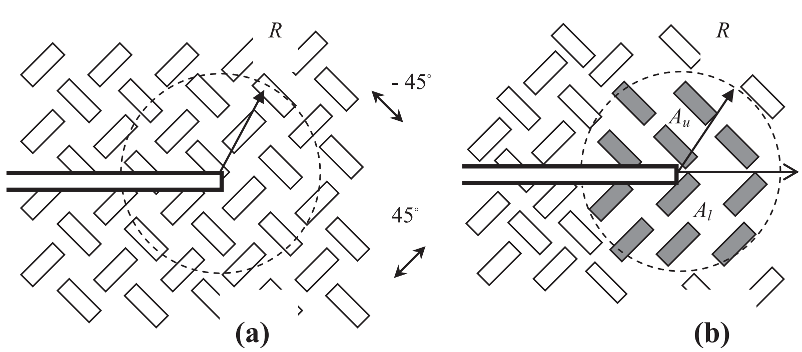

Fig. 5. Schematic demonstration of the toughening mechanism due to 90°ferroelastic domain switching near a crack tip: (a) Pre-switching; (b) Post-switching (in dashed circle).

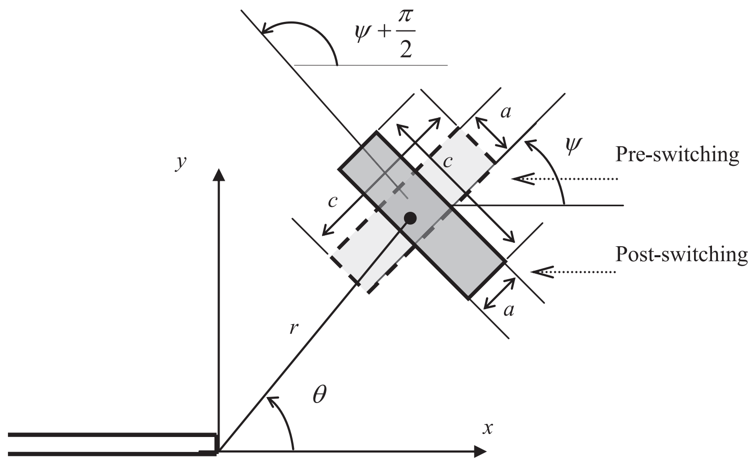

Fig. 6. A 90° switching elementary domain with an original orientation ψ near the crack tip.

et al., 1983), the shear stress criterion (Evans and Cannon, 1986), the transformation strain energy criterion (see, e.g. Lambropoulos, 1986), the energy switching criterion (Hwang et al., 1995) and other criteria for transformation toughening, which are reviewed in the reference (Hannink et al., 2000). According to different material transformation mechanisms, the corresponding criterion can be used to evaluate the transformed zone and assess the degree of transformation toughening. While the problem of transformation criterion remains a controversial issue, its further discussion lies outside the scope of this paper.

In this section, our aim is to verify the validity of the obtained fundamental solutions and to demonstrate the efficiency of the formulations obtained above. Three simple but representative transformation toughening examples will be investigated for given transformation zones. In order to attack the inverse problem of identifying the most appropriate transformation criterion, different transformation zones can be considered and compared using the explicit solutions given below.

6.1. Example 1: interaction between an infinite crack and a transformed spot

This example has been studied by McMeeking and Evans (1982) through an Eshelby-type approach. So the existing solutions of the problem can be regarded as a standard solution to verify the validity of the obtained fundamental solutions in this paper.

A transformed circular region of radius R and center at (r,θ), $R \ll r$, interacts with an infinite crack, as shown in Fig. 7. Suppose the transformation strain within the region is dilatational,

$$\varepsilon_{x0} = \varepsilon_{y0} = \varepsilon_0 \tag{6.1}$$

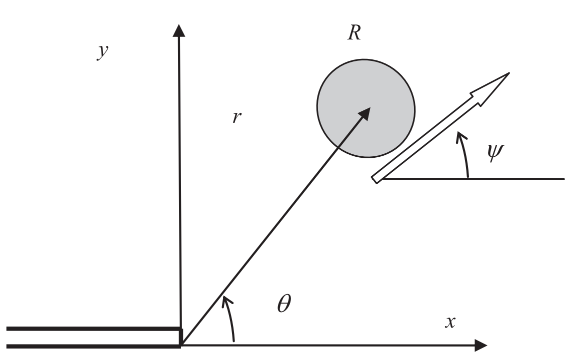

Fig. 7. Interaction between an infinite crack and a transformed spot (McMeeking and Evans, 1982).

and $D(x_s,y_s) = 1$. Inserting Eq. (6.1) into Eq. (4.7) we get

$$f_4(s) = \frac{4\varepsilon_0\mu}{\sqrt{2\pi}(1+\kappa)}\frac{1}{s\sqrt{s}} \tag{6.2}$$

Substituting Eq. (6.2) into Eq. (5.1), we get ΔK of the crack tip due to presence of the transformed spot as

$$\begin{aligned}\Delta K = \Delta K_I + i\Delta K_{II} &= \int\int_A f_4(x_s,y_s)D(x_s,y_s)dA \\ &= \frac{4\varepsilon_0\mu\pi R^2}{\sqrt{2\pi}(1+\kappa)}\left[\frac{1}{s\sqrt{s}}\right] \\ &= \frac{4\varepsilon_0\mu\pi R^2}{\sqrt{2\pi}(1+\kappa)}r^{-\frac{3}{2}}\left[\cos\frac{3}{2}\theta - i\sin\frac{3}{2}\theta\right]\end{aligned} \tag{6.3}$$

This result is completely consistent with the results obtained by McMeeking and Evans (1982).

6.2. Example 2: a semi-infinite crack enclosed by a transformed wake

Consider a semi-infinite crack enclosed by a transformed wake as shown in Fig. 8. Suppose the radius of the transformed circular area in front of crack tip is R, the infinite transformed area is A, transformation is a dilatational transformation ($\varepsilon_{x0} = \varepsilon_{y0} = \varepsilon_0$), and $D(r,\theta) = 1$. The influence function is identical to Eq. (6.2) which can be rewritten as

$$f_4(r,\theta) = \frac{4\varepsilon_0\mu}{\sqrt{2\pi}(1+\kappa)}r^{-\frac{3}{2}}\left[\cos\frac{3}{2}\theta - i\sin\frac{3}{2}\theta\right] \tag{6.4}$$

Substituting Eq. (6.4) into Eq. (5.1), after some straightforward manipulation, we get

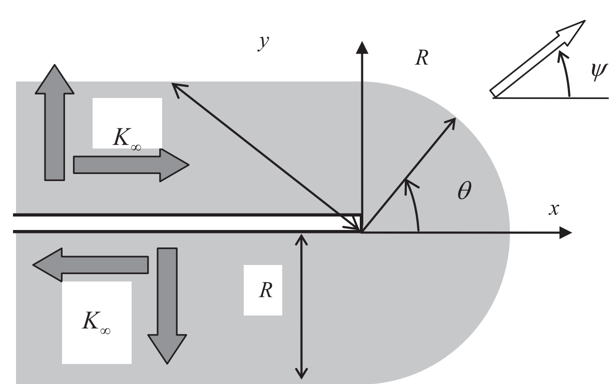

Fig. 8. A crack is enclosed by a transformed wake.

$$\Delta K = \Delta K_I + i\Delta K_{II} = \int\int_A f_4(r,\theta)D(r,\theta)dA = -\left(\frac{8\sqrt{2}}{3}+16\right)\frac{\mu\varepsilon_0}{(1+\kappa)}\sqrt{\frac{R}{\pi}} \quad (6.5)$$

It can be seen from this example that the proposed formulation is also efficient for an infinite transformation area. While the shape of the transformation zone is particularly simple in this case, in principle the formulation can be used for more complex zone shapes, e.g. of discretization is employed.

6.3. Example 3: ferroelastic transformation of a circular area enclosing a crack tip

We revisit the transformation problem shown in Fig. 5. We consider the intertwining domain distribution with orientation angles of 45° and −45°, respectively, within a circular area (see Fig. 5(a)). Suppose that *only* the domains with the orientation angle −45° in the upper circular area A_u suffer a 90°-switching after loading, while the domains with orientation angle 45° in the lower circular area A_l suffer 90° switching, as shown in Fig. 5(b). The radius of the circular area is R and $D(r,\theta) = 1/2$. Substituting the influence function (5.4) into Eq. (5.5), after a straightforward manipulation, we get

$$\Delta K = \Delta K_I + i\Delta K_{II} = \int\int_{A_u} f_5\left(r,\theta,-\frac{\pi}{4}\right)D(r,\theta)dA + \int\int_{A_l} f_5\left(r,\theta,\frac{\pi}{4}\right)D(r,\theta)dA = -\frac{16}{7}\frac{\mu\varepsilon_0\sqrt{R}}{\sqrt{2\pi}(1+\kappa)} \quad (6.6)$$

It can be seen from (6.6) that $\Delta K_{II} = 0$ and a mode I opening crack is obtained.

The proposed formulation is thus demonstrated to be also efficient for ferroelastic transformation toughening problems with partial transformation.

7. Conclusions

The fundamental solutions have been obtained for a transformed strain nucleus located in an infinite plane and in a plane containing a semi-infinite crack. On the basis of these results, the solutions have been developed and presented in the simple forms for toughening induced by the martensitic transformation and the ferroelastic transformation. Finally, some simple examples have been presented to demonstrate that the proposed fundamental formulation will pave the way for more rigorous studies of transformation toughening problems.

Acknowledgements

The author thanks Professor Robert M. McMeeking for his stimulation, encouragements, and comments throughout the course of the work. The author is also indebted to helpful discussions with Professor Alexander M. Korsunsky. This work has been partially supported by the National Basic Research Program of China (Grant No. 2007CB70770-5)

References

Budiansky, B., Hutchinson, J.W., Lambropoulos, J.C., 1983. Continuum theory of dilatant transformation toughness in ceramics. Int. J. Solids Struct. 19, 337–355.
Claussen, N., 1976. Fracture toughness of Al_2O_3 with an unstabilized ZrO_2 dispersed phase. J. Am. Ceram. Soc. 59, 49–51.
Clussen, N., Ruehle, M., Heuer, H., 1984. Science and technology of zirconia II. Advances in Ceramics, 12. American Ceramic Society, Columbus, OH.
Evans, A.G., Cannon, R.M., 1986. Toughening of brittle solids by martensitic transformations. Acta Metall. Mater. 34, 761–800.
Evans, A.G., Heuer, A.H., 1980. Review-transformation toughening in ceramics: martensitic transformations in crack-tip stress fields. J. Am. Ceram. Soc. 63, 241–248.
Fischer, F.D., Boehm, H.J., 2005. On the role of the transformation eigenstrain in the growth or shrinkage of spheroidal isotropic precipitations. Acta Mater. 53, 367–374.
Garvie, R.C., Hannink, R.H.J., Pascoe, R.T., 1975. Ceramic steel? Nature 258, 703–704.
Gupta, T.K., Bechtold, J.H., Kuznicki, R.C., Cadoff, L.H., Rossing, B.R., 1977. Stabilization of tetragonal phase in polycrystalline zirconia. J. Mater. Sci. 12, 2421–2426.
Hannink, R.H.J., 1978. Growth morphology of the tetragonal phase in partially stabilized zirconia. J. Mater. Sci. 13, 2487–2496.
Hannink, R.H.J., Kelly, P.M., Muddle, B.C., 2000. Transformation toughening in zirconia-containing ceramics. J. Am. Ceram. Soc. 83, 461–487.
Hutchinson, J.W., 1974. On Steady Quasi-Static Crack Growth. Harvard University Report DEAP S-8 (AFOSR-TR-74-1042).
Hwang, S.C., Lynch, C.S., McMeeking, R.M., 1995. Ferroelectric/ferroelastic interactions and a polarization switching model. Acta Metall. Mater. 43, 2073–2084.
Jones, J.L., Hoffman, M., 2006. R-curve and stress-strain behavior of ferroelastic ceramics. J. Am. Ceram. Soc. 89, 3721–3727.
Jones, J.L., Salz, C.R.J., Hoffman, M., 2005. Ferroelastic fatigue of a soft PZT ceramic. J. Am. Ceram. Soc. 88, 2788–2792.
Jones, J.L., Motahari, S.M., Varlioglu, M., Lienert, U., Bernier, J.V., Hoffman, M., Uestuendag, E., 2007. Crack tip process zone domain switching in a soft lead zirconate titanate ceramic. Acta Mater. 55, 5538–5548.
Karihaloo, B.L., Andreasen, J.H., 1996. Mechanics of transformation toughening and related topics. North-Holland Series in Applied Mathematics and Mechanics, 40. Elsevier, Amsterdam.
Kelly, P.M., Rose, L.R.F., 2002. The martensitic transformation in ceramics-its role in transformation toughening. Progr. Mater. Sci. 47, 463–557.
Lambropoulos, J.C., 1986. Effect of nucleation on transformation toughening. J. Am. Ceram. Soc. 69, 218–222.
Lange, F.F., 1982. Transformation toughening-part 2, contribution to fracture toughness. J. Mater. Sci. 17, 235–239.
Li, Z.H., Yang, L.H., 2002. The application of the Eshelby equivalent inclusion method for unifying modulus and transformation toughening. Int. J. Solids Struct. 39, 5225–5240.
Love, A.E.H., 1927. Mathematical Theory of Elasticity. Cambridge University Press.
Ma, L., Lu, T.J., Korsunsky, A.M., 2006. Vector J-integral analysis of crack interaction with pre-existing singularities. J. Appl. Mech. 73, 876–883.
Magnani, G., Brillante, A., 2005. Effect of the composition and sintering process on mechanical properties and residual stresses in zirconia-alumina composites. J. Eur. Ceram. Soc. 25, 3383–3392.
McMeeking, R.M., Evans, A.G., 1982. Mechanics of transformation-toughening in brittle materials. J. Am. Ceram. Soc. 65, 242–246.
Munz, D., Fett, T., 1998. Ceramics-mechanical properties failure behaviour material selection. Springer Series in Material Sciences. Springer, Germany.
Muskhelishvili, N.I., 1953. Some problems of mathematical theory of elasticity. (English translation from the third Russian edition. Noordhoff Ltd. Groningen,).
Pojprapai, S., Jones, J.L., Studer, A.J., Russell, J., Valanoor, N., Hoffman, M., 2008. Ferroelastic domain switching fatigue in lead zirconate titanate ceramics. Acta Mater. 56, 1577–1587.
Rauchs, G., Fett, T., Munz, D., Oberacker, R., 2001. Tetragonal-to-monoclinic phase transformation in CeO_2-stabilized zirconia under uniaxial loading. J. Eur. Ceram. Soc. 21, 2229–2241.
Rauchs, G., Fett, T., Munz, D., Oberacker, R., 2002. Tetragonal-to-monoclinic phase transformation in CeO_2-stabilized zirconia under multiaxial loading. J. Eur. Ceram. Soc. 22, 841–849.
Rose, L.R.F., 1987. The mechanics of transformation toughening. Proc. R. Soc. Lond. A. 412, 169–197.
Suo, Z., 1989. Singularities interacting with interfaces and cracks. Int. J. Solids Struct. 25, 1133–1142.
Tsukamoto, H., Kotousov, A., 2006. Transformation toughening in zirconia-enriched composites: micromechanical modeling, Int. J. Fract. 139, 161–168.
Vena, P., Gastaldi, D., Contro, R., Petrini, L., 2006. Finite element analysis of the fatigue crack growth rate in transformation toughening ceramics. Int. J. Plast. 22, 895–920.
Wang, J., Shi, S.Q., Chen, L.Q., Li, Y., Zhang, T.Y., 2004. Phase field simulations of ferroelectric/ferroelastic polarization switching. Acta Mater. 52, 749–764.
Yang, W., Zhu, T., 1998. Switch-toughening of ferroelectrics subjected to electric fields. J. Mech. Phys. Solids 46, 291–311.
Yi, S., Gao, S., 2000. Fracture toughening mechanism of shape memory alloys due to martensite transformation. Int. J. Solids Struct. 37, 5315–5327.
Yi, S., Gao, S., Shen, L.X., 2001. Fracture toughening mechanism of shape memory alloys under mixed-mode loading due to martensite transformation. Int. J. Solids Struct. 38, 4463–4476.
Zeng, D., Katsube, N., Soboyejo, W.O., 1999. Simulation of transformation toughening of heterogeneous materials with randomly distributed transforming particles by hybrid FEM. Comput. Mech. 23, 457–465.
Zeng, D., Katsube, N., Soboyejo, W.O., 2004. Discrete modeling of transformation toughening in heterogeneous materials. Mech. Mater. 36, 1057–1071.