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The principle of equivalent eigenstrain for inhomogeneous inclusion problems

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ABSTRACT

In this paper, based on the principle of virtual work, we formulate the equivalent eigenstrain approach for inhomogeneous inclusions. It allows calculating the elastic deformation of an arbitrarily connected and shaped inhomogeneous inclusion, by replacing it with an equivalent homogeneous inclusion problem, whose eigenstrain distribution is determined by an integral equation. The equivalent homogeneous inclusion problem has an explicit solution in terms of a definite integral. The approach allows solving the problems about inclusions of arbitrary shape, multiple inclusion problems, and lends itself to residual stress analysis in non-uniform, heterogeneous media. The fundamental formulation introduced here will find application in the mechanics of composites, inclusions, phase transformation analysis, plasticity, fracture mechanics, etc.

1. Introduction

Inhomogeneous inclusion refers to a given region (called inclusion) in an infinite linear elastic space, and the region has elastic properties that are different from the surroundings (called matrix). Of interest are the elastic fields inside and outside region generated by the prescribed eigenstrain in the region or by remotely applied stresses. This problem is the so-called inhomogeneous inclusion problem. If the elastic properties of the inclusion are identical to the matrix ones, it degenerates to homogenous inclusion problem. Due to its numerous valuable engineering applications in the problems such as thermal expansion, phase transformation, reinforcing phases, material inhomogeneities, precipitates, defects, plastic strain or misfit strain voids etc., the inhomogeneous problem has become one of the most attractive topics of solid mechanics for more than 50 years, and is the subject of constant studies (see, e.g. Mindlin (1936), Goodier (1937), Mindlin and Cheng (1950), Sen (1951), Eshelby (1957, 1959, 1961), Jaswon and Bhargava (1961), Mori and Tanaka (1973), Willis (1981), Mura (1987), Ru (1999, 2000, 2003), Andrianov et al. (2008), Zou et al. (2011), Chen et al. (2011, 2014), etc. among them). Actually, hundreds of references have been devoted to this study, which can be partly found in the review papers by Mura and his co-workers (1988, 1996), Zhou et al. (2013) and the books by Christensen (1979), Mura (1987) and Nemat-Nasser and Hori (1999).

The two classical papers by Eshelby (1957, 1959) generalized the earlier episodic results pertaining to elliptical inhomogeneities (e.g. Inglis, (1913)) by proving the uniform nature of the internal strain field in all such inhomogeneities when the inclusion eigenstrain is uniform. This powerful result laid the foundation for systematic exploration of problems about inclusions in elastic solids (see, e.g., Mori and Tanaka (1973), Hutchinson (1976) and it is now commonly referred to as the Eshelby property. Inspirited by Eshelby's work and practically urgent need, great efforts have been devoted to extend the problem to a more generalized one, in which: (i) the inclusion can be of non-ellipsoidal shape and (ii) the eigenstrain in the inclusion can be non-uniform. For the homogenous inclusion problems, it can be said that this target has been achieved with the Green's function method or other techniques (see e.g. Mura (1987), Ru (2000), Li and Anderson (2001), Kuvshinov (2008), Ma (2010), Avazmohammadi et al. (2010), Zou et al. (2011), Ma et al. (2013)). However, for the inhomogeneous inclusion problems, the theoretical challenges are still there.

It should be mentioned that, to solve the *inhomogeneous ellipsoidal* inclusion problems with prescribed uniform eigenstrain, the *equivalent inclusion method* (EIM) was proposed by Eshelby (1957) and reformulated by Mura (1987). This method establishes that the *inhomogeneous ellipsoidal* inclusion problem can be transformed into a *homogeneous* ellipsoidal *inclusion* problem. Since the latter problem pertains to a homogeneous material, it can be readily solved. The efficiency and elegance of the EIM has meant that attempts have been made to study the more general

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inhomogeneous and non-ellipsoidal inclusion problems, although the validity of extending the approach to such problems remains the subject of theoretical research. Recently, Zheng and his co-authors (Zheng et al., 2006; Zou et al., 2010) found that the Eshelby's tensor obtained via EIM for convex inclusions led to a valid solution with small errors, while for non-convex inclusions large errors were observed. Clearly, the appropriate theoretical treatment for the inhomogeneous inclusion problems with non-ellipsoidal shape and/or non-uniform eigenstrain is still needed to be done.

This study aims to propose an all-rounded approach to treat the inhomogeneity problems with non-ellipsoidal shape and/or non-uniform eigenstrain. It should be explained that we deliberately title this study with *the equivalent eigenstrain principle* in order to distinguish it from Eshelby's cutting-straining-re-welding technique and the associated EIM. The present derivation based on the *principle of virtual work* may have wide applications to arbitrarily shaped inclusions, multiple inclusions, arbitrary non-uniform residual strain distributions, etc.

The structure of this study is as follows: in Section 2, the basic equations are presented for next use. In Section 3, the principle of equivalent eigenstrain is initially discussed and the procedure of equivalence transformation of homogeneity-to-inhomogeneity is performed. In Section 4 the principle of equivalent eigenstrain is further studied and the equivalence transformation of inhomogeneity-to-homogeneity with consideration of residual strain in the inhomogeneity is studied. Subsequently, in Section 5, with consideration of both residual strain in inhomogeneity and far-field load, the equivalence transformation of inhomogeneity-to-homogeneity problem is studied. In Section 6, the implementation of the principle is explained and a simple example is given for demonstrating the application. In Section 7, the conclusions are addressed.

2. Elementary knowledge of inclusion mechanics

In this section, the field equations for the elasticity theory with particular reference to solving eigenstrain problems will be reviewed. It includes Hooke's law, equilibrium conditions, and compatibility condition, as well as the solution for homogenous inclusion problems.

2.1. Hooke's law and compatibility condition

For infinitesimal deformations, the total strain ε_{ij} is regarded as the sum of the elastic strain e_{ij} and eigenstrain ε^*_{ij}

$$\varepsilon_{ij} = e_{ij} + \varepsilon^*_{ij} \tag{2.1}$$

The total strain ε_{ij} must be compatible and in terms of displacement u_i,

$$\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2 \tag{2.2}$$

The elastic strain is related to stress σ_{ij} by Hooke's law,

$$\sigma_{ij} = C_{ijkl}e_{kl} = C_{ijkl}(\varepsilon_{kl} - \varepsilon^*_{kl}) = C_{ijkl}(u_{k,l} - \varepsilon^*_{kl}) \tag{2.3}$$

where C_{ijkl} are the elastic moduli and the summation convention for the repeated indices is employed. Since $C_{ijkl} = C_{ijlk}$, we have $C_{ijkl}u_{k,l} = C_{ijkl}u_{l,k}$. In the region where $\varepsilon^*_{kl} = 0$, Eq. (2.3) becomes:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} = C_{ijkl}u_{k,l} \tag{2.4}$$

The inverse expression of (2.3) is

$$(\varepsilon_{ij} - \varepsilon^*_{ij}) = C^{-1}_{ijkl}\sigma_{kl} = S_{ijkl}\sigma_{kl} \tag{2.5}$$

where $S_{ijkl} = C^{-1}_{ijkl}$ is the elastic compliance.

2.2. Equilibrium conditions

The equations of equilibrium are

$$\sigma_{ij,j} + f_i = 0, \quad i = 1, 2, 3 \tag{2.6}$$

where f_i is body force. The boundary conditions for external surface forces T_i are

$$T_i = \sigma_{ij}n_j, \tag{2.7}$$

where n_j is the exterior unit normal vector on the boundary of the body.

2.3. Solution for homogenous inclusion by Green's function method

A homogeneous inclusion is embedded in an infinite solid (see Fig. 1). If the profile of the inclusion is given, without going to details, then its deformation field can be obtained through Green's function method as follows:

2.3.1. 3-D inclusion (Mura, 1987)

The displacement field in the inclusion will be

$$u_i(\mathbf{x}) = -\int_{-\infty}^{\infty} C_{jlmn}\varepsilon^*_{mn}(\mathbf{x}')G_{ij,l}(\mathbf{x} - \mathbf{x}')d\mathbf{x}', \quad \mathbf{x} \in \Omega \tag{2.8}$$

where the function $G_{ij}(\mathbf{x} - \mathbf{x}')$ is Green's function and sometimes it is called the influence function. It has $G_{ij,l}(\mathbf{x} - \mathbf{x}') = \frac{\partial}{\partial x_l}G_{ij}(\mathbf{x} - \mathbf{x}') = -\frac{\partial}{\partial x'_l}G_{ij}(\mathbf{x} - \mathbf{x}')$. The corresponding expressions for the strain and stress are as follows (Mura, 1987)

$$\varepsilon_{ij}(\mathbf{x}) = -\frac{1}{2}\int_{-\infty}^{\infty} C_{klmn}\varepsilon^*_{mn}(\mathbf{x}')\{G_{ik,lj}(\mathbf{x} - \mathbf{x}') + G_{jk,li}(\mathbf{x} - \mathbf{x}')\}d\mathbf{x}' \tag{2.9}$$

and

$$\sigma_{ij}(\mathbf{x}) = C_{ijkl}\int_{-\infty}^{\infty} C_{pqmn}\{G_{kp,qn}(\mathbf{x} - \mathbf{x}')\varepsilon^*_{ml}(\mathbf{x}') - G_{kp,ql}(\mathbf{x} - \mathbf{x}')\varepsilon^*_{mn}(\mathbf{x}')\}d\mathbf{x}' \tag{2.10}$$

For the isotropic materials, the expression of the Green's function is

$$G_{ij}(\mathbf{x} - \mathbf{x}') = \frac{1}{4\pi\mu}\frac{\delta_{ij}}{|\mathbf{x} - \mathbf{x}'|} - \frac{1}{16\pi\mu(1-\nu)}\frac{\partial^2}{\partial x_i \partial x_j}|\mathbf{x} - \mathbf{x}'| \tag{2.11}$$

where δ_{ij} is the Kronecker delta, $|\mathbf{x} - \mathbf{x}'|^2 = (x_i - x'_i)(x_i - x'_i)$, μ is the material's shear modulus, ν is material's Poisson ratio. Solution (2.11) was found by Lord Kelvin.

2.3.2. 2-D inclusion

Similarly, we can get the deformation field for the two-dimensional problem with Green's function method (see, Mura (1987), Ru (1999), Ma (2010), Ma et al. (2013)). Since its expression is a bit complex and lengthy, for simplicity, we will not list it here but it can be found in the above listed references.

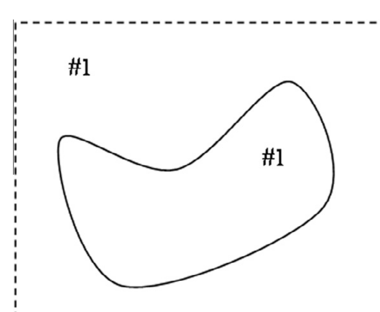

Fig. 1. A homogeneous inclusion in an infinite elastic solid. The material properties of matrix and inclusion are identical.

In all, the homogeneous inclusion problems, in theory, are solvable with Green's function method.

3. The principle of equivalent eigenstrain: equivalence transformation from homogeneity to inhomogeneity

Theorem 1. *The material, of an arbitrary region Ω within a homogenous body under far-field load, can be replaced by an inhomogeneous material accompanying an eigenstrain distribution in it. The stress and total strain distributions in the body do not change.*

The principle of virtual work will be used to prove this theorem, which can be expressed as

$$\int_v f_i \delta u_i dv + \int_s T_i \delta u_i ds = \int_v \sigma_{ij} \delta \varepsilon_{ij} dv \tag{3.1}$$

where f_i is body force. δu_i and $\delta \varepsilon_{ij}$ are the virtual displacement and strain.

In the following, we will prove the Theorem 1. Assume that there are two systems as shown in Fig. 2 (a) and (b). Fig. 2 (a) is the original *homogenous* inclusion problem and Fig. 2(b) is the counterpart virtual *inhomogeneous* inclusion problem. The shape of the inclusion can be arbitrary. Two systems are identical except the inclusion regions. We imaginarily cut out the inclusion regions from the corresponding systems as shown in Fig. 2. For the inclusion regions (a) and (b), the principle of virtual work for them can be written, respectively, as

$$\int_v f_i \delta u_i dv + \int_s T_i \delta u_i ds = \int_v \sigma_{ij} \delta \varepsilon_{ij} dv \quad \text{for the region } \Omega \text{ in (a)} \tag{3.2}$$

$$\int_v f_i^* \delta u_i dv + \int_s T_i^* \delta u_i ds = \int_v \sigma_{ij}^* \delta \varepsilon_{ij} dv \quad \text{for the region } \Omega \text{ in} \tag{3.3}$$

where f_i, T_i, σ_{ij} are the body force, boundary traction, and stress of the inclusion (a), and f_i^*, T_i^*, σ_{ij}^* are the body force, boundary traction, and stress of the inclusion (b); δu_i, $\delta \varepsilon_{ij}$ are virtual displacements and strains respectively. Eq. (3.2) minus (3.3) gives

$$\int_v (f_i - f_i^*) \delta u_i dv + \int_s (T_i - T_i^*) \delta u_i ds = \int_v (\sigma_{ij} - \sigma_{ij}^*) \delta \varepsilon_{ij} dv \tag{3.4}$$

Enforcing $f_i = f_i^*, T_i = T_i^*$, and because $\delta \varepsilon_{ij}$ can be of any value, Eq. (3.4) implies that the stresses in the two inclusions are identical as

$$\sigma_{ij} = \sigma_{ij}^* \tag{3.5}$$

From the above manipulation, it can be also found that the identical inclusion boundary traction will lead identical stress distribution in entire two systems (a) and (b) indeed. We know that

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl} \quad \text{for region } \Omega \text{ in system (a)} \tag{3.6}$$

$$\sigma_{ij}^* = C_{ijkl}^* (\varepsilon_{kl} - \varepsilon_{kl}^0) \quad \text{for region } \Omega \text{ in system (b)} \tag{3.7}$$

where C_{ijkl} is the elastic constant of homogeneous inclusion and matrix, ε_{ij} in (3.6) and (3.7) is the *total strain*; ε_{kl}^0 is the imaginary eigenstrain in Ω of (b) to be specified, C_{ijkl}^* is the pre-given elastic constant of inhomogeneous inclusion.

It should be explained here again that: (i) the identical tractions at the inclusion boundary for both systems ($f_i = f_i^*, T_i = T_i^*$) guarantees the stress states in the two systems identical; (ii) Identity of *total strain* ε_{ij} in (3.6) and (3.7), indeed, is a nature requirement for displacement identity in regions Ω of (a) and (b). So that, under the identical boundary traction and body force in (b) and (a), the elastic deformation difference due to different elastic constants of the inclusions can be compensated by distributing appropriate eigenstrain ε_{kl}^0 in Ω of (b). Inserting (3.6) and (3.7) into (3.5) leads

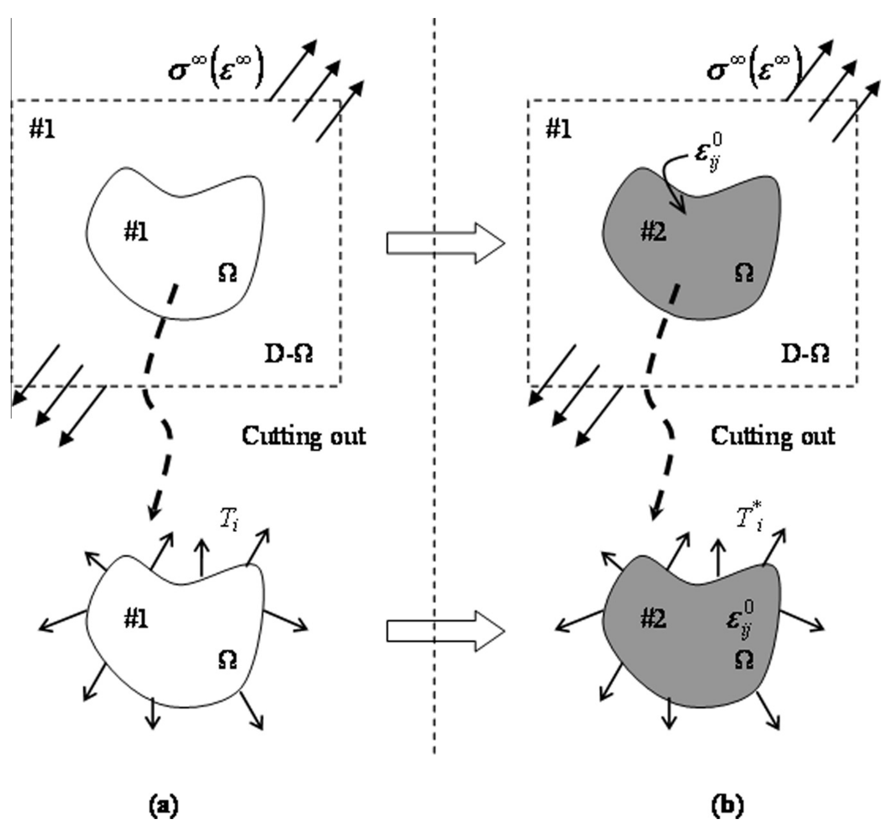

Fig. 2. The principle of equivalent eigenstrain: transformation from homogeneity to inhomogeneity. (a) The original homogeneous inclusion system, (b) the counterpart virtual inhomogeneous inclusion system.

$$C_{ijkl}\varepsilon_{kl} = C^*_{ijkl}(\varepsilon_{kl} - \varepsilon^0_{kl}) \tag{3.8}$$

Then the eigenstrain ε^0_{kl} in Ω of (b) is obtained as

$$\varepsilon^0_{ij} = S^*_{ijkl}(C^*_{klpq} - C_{klpq})\varepsilon_{pq} \tag{3.9}$$

In other words, if the eigenstrain is prescribed by Eq. (3.9), the deformation in system (b) will be completely identical to system (a). This is the equivalence transformation from homogeneity to inhomogeneity. Strain ε^0_{ij} is named as equivalent eigenstrain in this paper. Up to now, the Theorem 1 has been proved.

If ε_{pq} in Eq. (3.9) is a uniform strain due to remote load, we denote it as ε^∞_{pq}. Thus under uniform load in Fig. 2 (a), the equivalent eigenstrain distribution required in (b) will be uniform, as

$$\varepsilon^0_{ij} = S^*_{ijkl}(C^*_{klpq} - C_{klpq})\varepsilon^\infty_{pq} \tag{3.10}$$

We should keep in mind that Eq. (3.10) holds for any shape inclusion. This result will be used later.

4. The principle of equivalent eigenstrain: equivalence transformation from inhomogeneity to homogeneity

Theorem 2. *An arbitrary inhomogeneous inclusion Ω, with residual strain embedded within a homogenous body, can be equaled by a homogeneous material identical to matrix material accompanying an equivalent eigenstrain distribution in it. The stress and total strain distributions in the body do not change.*

As before, the Theorem 2 can be proved in the following. Suppose that two systems as shown in Fig. 3 (a) and (b) are identical except the inclusion regions. Fig. 3 (a) is the original *inhomogeneous* inclusion system, and Fig. 3 (b) is the counterpart virtual *homogeneous* inclusion system. We imaginarily cut out the inclusion regions from the corresponding systems. For the inclusion regions (a) and (b), the principle of virtual work for them can be written respectively as

$$\int_v f^*_i \delta u_i dv + \int_s T^*_i \delta u_i ds = \int_v \sigma^*_{ij}\delta\varepsilon_{ij} dv \text{ for the region } \Omega \text{ in (a)} \tag{4.1}$$

$$\int_v f_i \delta u_i dv + \int_s T_i \delta u_i ds = \int_v \sigma_{ij}\delta\varepsilon_{ij} dv \quad \text{for the region } \Omega \text{ in (b)} \tag{4.2}$$

where δu_i, $\delta\varepsilon_{ij}$ are virtual displacements and strains respectively.

Again, subtraction of (4.1) from (4.2) gives

$$\int_v (f_i - f^*_i)\delta u_i dv + \int_s (T_i - T^*_i)\delta u_i ds = \int_v (\sigma_{ij} - \sigma^*_{ij})\delta\varepsilon_{ij} dv \tag{4.3}$$

Enforcing $f_i = f^*_i, T_i = T^*_i$, and since $\delta\varepsilon_{ij}$ can be of any value, Eq. (4.3) spontaneously implies

$$\sigma^*_{ij} = \sigma_{ij} \tag{4.4}$$

We know that

$$\sigma^*_{ij} = C^*_{ijkl}(\varepsilon_{kl} - \varepsilon^*_{kl}) \text{ for region } \Omega \text{ in system (a)} \tag{4.5}$$

$$\sigma_{ij} = C_{ijkl}(\varepsilon_{kl} - \varepsilon^0_{kl}) \text{ for region } \Omega \text{ in system(b)} \tag{4.6}$$

where ε^*_{kl} is the real residual strain in Ω of (a), ε^0_{kl} is the equivalent eigenstrain in Ω of (b), and ε_{kl} is the total strain in (4.5) and (4.6), C^*_{ijkl} is the elastic constant of the inhomogeneous inclusion, C_{ijkl} is the elastic constant of the homogeneous inclusion and matrix. Similarly, condition $f_i = f^*_i$, $T_i = T^*_i$ guarantees the stress states identical in two systems (a) and (b). The total strains ε_{ij} in (4.5) and (4.6) are identical, which is a nature requirement for displacement identity in Ω of both systems. Thus, under identical boundary traction and body force, the *total* deformation of the inhomogeneous inclusion in system (a) is equaled to the one in system (b) by distributing an appropriate equivalent eigenstrain ε^0_{kl}. In other words, the inhomogeneous inclusion in system (a) can be transformed into the corresponding homogeneous inclusion in system (b) with appropriate equivalent eigenstrain ε^0_{kl}. Inserting (4.5) and (4.6) into (4.4) leads

$$C^*_{ijkl}(\varepsilon_{kl} - \varepsilon^*_{kl}) = C_{ijkl}(\varepsilon_{kl} - \varepsilon^0_{kl}) \tag{4.7}$$

Then the equivalent eigenstrain ε^0_{kl} in Ω of (b) is

$$\varepsilon^0_{ij} = S_{ijkl}(C_{klpq} - C^*_{klpq})\varepsilon_{pq} + S_{ijkl}C^*_{klpq}\varepsilon^*_{pq} \text{ in } \Omega \tag{4.8}$$

When the equivalent eigenstrain ε^0_{ij} is prescribed according to (4.8), one can get the deformation state in system (a) by alternatively analyzing the stress state in the homogeneous inclusion system (b). Since the system (b) is a homogenous inclusion problem, as mentioned in Section 1, it is theoretically solvable. Up to now, the Theorem 2 has been proved.

Additionally, it should be noted that the total strain ε_{pq} in Eq. (4.8) can be expressed in terms of an integral function of eigenstrain ε^0_{ij} according to Eq. (2.9) as,

$$\varepsilon_{pq}(\mathbf{x}) = -\frac{1}{2}\int_{-\infty}^{\infty} C_{klij}\varepsilon^0_{ij}(\mathbf{x}')\{G_{pk,lq}(\mathbf{x}-\mathbf{x}') + G_{qk,lp}(\mathbf{x}-\mathbf{x}')\}d\mathbf{x}' \tag{4.9}$$

For simplicity, we denote (4.9) as

$$\varepsilon_{pq} = F_{pq}(\varepsilon^0_{ij}) \tag{4.10}$$

It can be formulated by Green's function method. So, we may rewrite Eq. (4.8) as

$$\varepsilon^0_{ij} = S_{ijkl}(C_{klpq} - C^*_{klpq}) \cdot F_{pq}(\varepsilon^0_{ij}) + S_{ijkl}C^*_{klpq}\varepsilon^*_{pq} \tag{4.11}$$

Eq. (4.11) is an equation for solving ε^0_{ij}. Once we obtain ε^0_{ij}, and the total strain ε_{ij} is known through (4.9), the stress field in the inhomogeneous inclusion can be evaluated as

$$\sigma_{ij} = C_{ijkl}(\varepsilon_{kl} - \varepsilon^0_{kl}) \text{ in } \Omega \tag{4.12}$$

and

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \text{ in D-}\Omega \tag{4.13}$$

This procedure is the equivalence transformation from inhomogeneity to homogeneity.

5. The principle of equivalent eigenstrain: inclusion with residual strain and far-field load

5.1. Inhomogeneous inclusion under far-field load

In this sub-section we study the inhomogeneous inclusion problem under far-field load. As before, we consider two systems as shown in Fig. 4(a) and (b), and they are identical except the inclusion regions. Fig. 4(a) is the original inhomogeneous inclusion problem, and Fig. 4(b) is the counterpart virtual homogeneous inclusion problem. First, let's suppose the stress distribution and total deformation in two systems are complete identical, and we try to find the equivalent eigenstrain ε^0_{ij} in system (b) in order to get the solution of Fig. 4(a).

The homogeneous inclusion problem in Fig. 4(b) can be decomposed into two subproblems: a homogenous inclusion problem with uniform remote load as shown in Fig. 5(a) and a homogenous inclusion problem with residual strain in it as shown in Fig. 5(b). Correspondingly, the problem Fig. 4(a) can be also decomposed into two subproblems: an inhomogeneous inclusion problem with uniform remote load and a residual strain ε^b_{ij} as shown Fig. 5(A) and an inhomogeneous inclusion problem only with a residual strain $-\varepsilon^b_{ij}$ as shown Fig. 5(B).

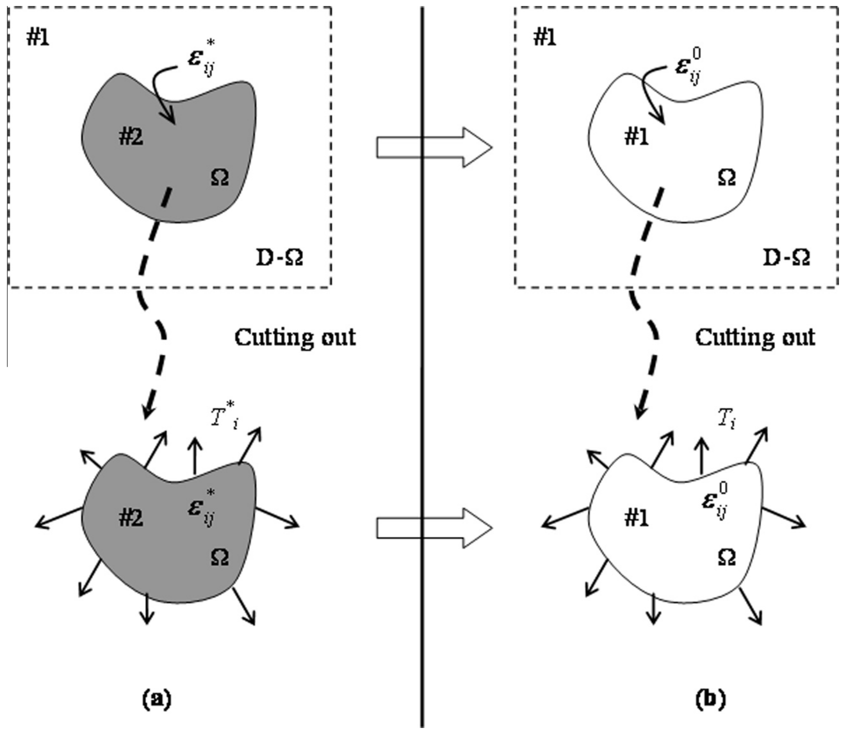

Fig. 3. The principle of equivalent eigenstrain: Transformation from inhomogeneity to homogeneity. (a) Original inhomogeneous inclusion system, (b) counterpart virtual homogeneous inclusion system.

The solution of subproblem in Fig. 5(a) is uniform and can be denoted as σ^{∞}_{ij} and $\varepsilon^{\infty}_{ij}$. To solve the subproblem in Fig. 5(b) is a bit complex. The equivalent eigenstrain ε^0_{ij} in Fig. 5(b) can be solved in the following steps, where the solutions of subproblems in Fig. 5(A) and Fig. 5(B) will be employed:

(i) Employing Theorem 1 in Section 3, we perform equivalence transformation from *homogeneity to inhomogeneity* shown from Fig. 5(a) to Fig. 5(A) by Eq. (3.10), and a transition equivalent eigenstrain ε^b_{ij} in Fig. 5 (A) is

$$\varepsilon^b_{ij} = S^*_{ijkl}(C^*_{klpq} - C_{klpq})\varepsilon^{\infty}_{pq} \tag{5.1}$$

(ii) Based on the superposition principle, we deliberately set inhomogeneous inclusion shown in Fig. 5(B) with residual strain $-\varepsilon^b_{ij}$ to counteract the residual strain ε^b_{ij} in Fig. 5(A) to satisfy the original problem of Fig. 4(a). Then, employing Theorem 2 in Section 4, we perform equivalent transformation from *inhomogeneity to homogeneity* shown from Fig. 5(B) to Fig. 5(b) by virtue of Eq. (4.8). Namely, replacing ε^*_{ij} in Eq. (4.8) with $-\varepsilon^b_{ij}$, we get ε^0_{ij} in Fig. 5(b) as

$$\varepsilon^0_{ij} = S_{ijkl}(C_{klpq} - C^*_{klpq})\varepsilon_{pq} - S_{ijkl}C^*_{klpq}\varepsilon^b_{pq} \tag{5.2}$$

For the homogenous inclusion problem in Fig. 5(b), the total strain ε_{pq} in (5.2) is an integral function of only equivalent eigenstrain ε^0_{ij} as Eq.(4.9), denoted as

$$\varepsilon_{pq} = F_{pq}(\varepsilon^0_{ij}) \tag{5.3}$$

Now, inserting (5.1) into (5.2) we get

$$\varepsilon^0_{ij} = S_{ijkl}\left(C_{klpq} - C^*_{klpq}\right)\varepsilon_{pq} + S_{ijkl}\left(C_{klpq} - C^*_{klpq}\right)\varepsilon^{\infty}_{pq} \tag{5.4}$$

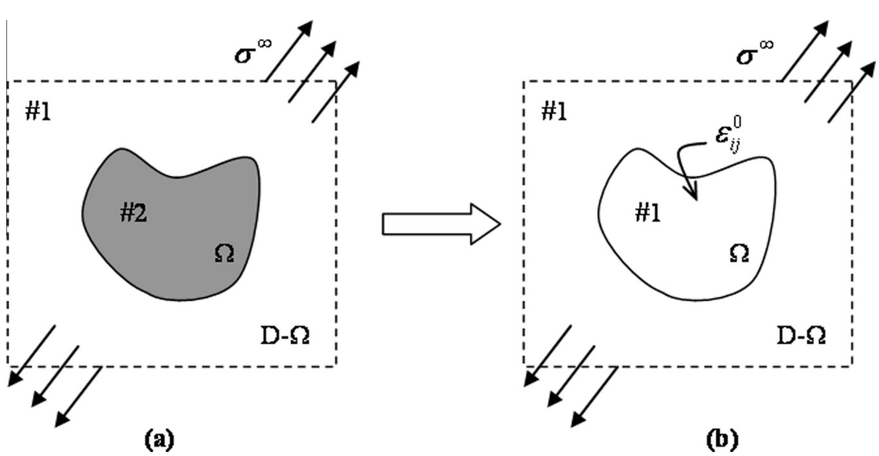

Fig. 4. An inhomogeneous inclusion under far-field load. (a) Original inhomogeneous inclusion system (with no residual strain), (b) the counterpart virtual homogeneous inclusion system with equivalent eigenstrain.

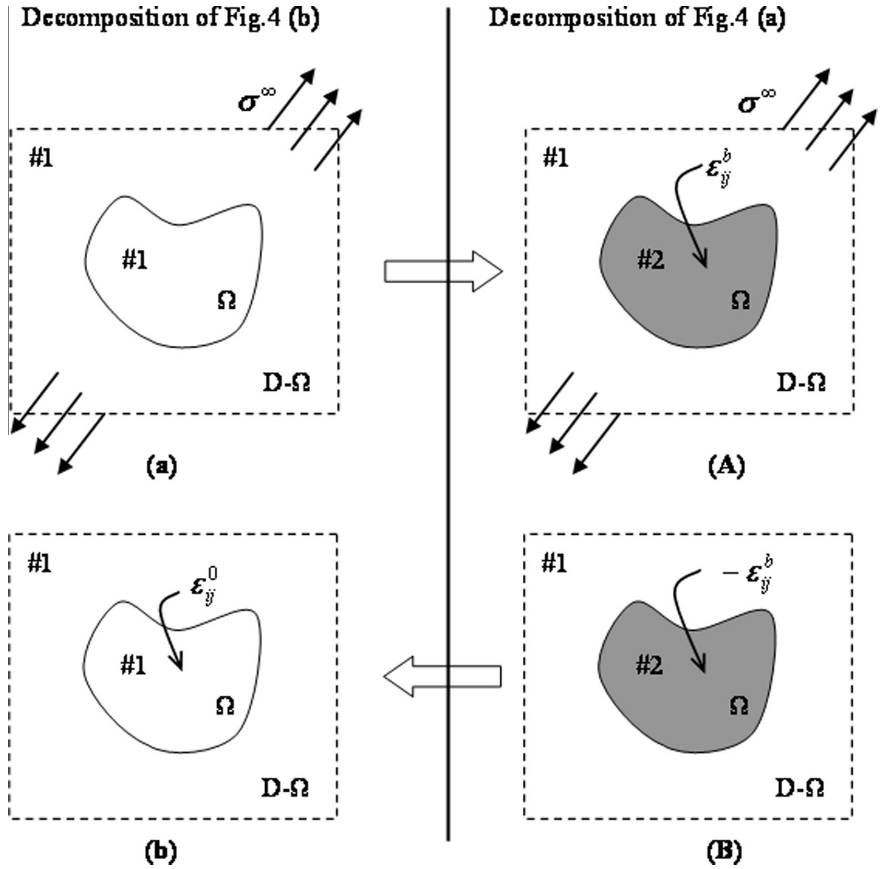

Fig. 5. Superposition principles to the problems in Fig. 4. Problem in Fig. 4(b) can be decomposed into sub-problems: Fig 5 (a) + Fig. 5 (b); Problem in Fig. 4(a) can be decomposed into sub-problems: Fig 5 (A) + Fig. 5 (B).

This is the equivalence eigenstrain for the counterpart virtual homogeneous inclusion problem in Fig. 4(b). In other words, by the above manipulation, we have transformed the original inhomogeneous inclusion problem shown in Fig. 4(a) into the homogeneous inclusion problem shown Fig. 4(b).

Eq. (5.4) is an equation for unknown variable ε^0_{ij}. Once it is solved from (5.4), we can get the stress field in the inhomogeneous inclusion in Fig. 4(a) by superposing the solutions of the two subproblems shown in Fig. 5(a) and (b) as

$$\sigma_{ij} = C_{ijkl}(\varepsilon_{kl} - \varepsilon^0_{kl}) + C_{ijkl}\varepsilon^\infty_{kl} \quad \text{in } \Omega \tag{5.5}$$

and

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} + C_{ijkl}\varepsilon^\infty_{kl} \quad \text{in D} - \Omega \tag{5.6}$$

It should be pointed out that in this sub-section we studied the situation where there is no real residual strain in Fig. 4(a). Indeed, the expression (5.4) can be directly obtained from Eq. (4.8), in which we just extract the far-field load contribution ε^∞_{ij} from the total strain ε_{pq} and let new ε_{pq} represent the contributions from the real residual strain (here we suppose $\varepsilon^*_{pq} = 0$ in (4.8)). Furthermore, compared with the second term at the right hand in Eq. (4.8), it can be found that the far-field uniform load in Eq. (5.4) plays a similar uniform eigenstrain role for any shape inhomogeneous inclusion problems.

5.2. An inhomogeneous inclusion with residual strain and also under far-field load

In this sub-section we study the inhomogeneous inclusion problem, as shown in Fig. 6(a), under far-field load and with the residual strain in the inclusion.

In the same way, the equivalent eigenstrain ε^0_{ij} in Fig. 6(b) due to the presence of both residual strain ε^*_{ij} and far-field load in Fig. 6(a) can be directly obtained by superposition of previous solutions in the framework of linear mechanics as

$$\varepsilon^0_{ij} = S_{ijkl}(C_{klpq} - C^*_{klpq})\varepsilon_{pq} + S_{ijkl}(C_{klpq} - C^*_{klpq})\varepsilon^\infty_{pq} + S_{ijkl}C^*_{klpq}\varepsilon^*_{pq} \tag{5.7}$$

where ε_{pq} is merely an integral function of the equivalent eigenstrain ε^0_{ij} as Eq. (4.9).

Equation (5.7) indeed transforms the inhomogeneous inclusion problems in Fig. 6(a) into the homogenous inclusion problems in Fig. 6(b). Once ε^0_{ij} is solved from (5.7) with (4.9), the stress state in the inhomogeneous inclusion will be

$$\sigma_{ij} = C_{ijkl}(\varepsilon_{kl} - \varepsilon^0_{kl}) + C_{ijkl}\varepsilon^\infty_{kl} \text{ in } \Omega \tag{5.8}$$

and the stress state outside of the inhomogeneous inclusion will be

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} + C_{ijkl}\varepsilon^\infty_{kl} \text{ in D} - \Omega. \tag{5.9}$$

6. Implementation

In this section, the numerical implementation is discussed and a simple example is provided in order to demonstrate its procedure.

6.1. The general form for implementation

Here, let's consider an inhomogeneous inclusion embedded in an infinite solid as shown in Fig. 7. The known residual strain $\varepsilon^*_{mn}(\mathbf{x})$ in the inclusion is non-uniform, and the whole system is under far-field load $\sigma^\infty_{ij}(\varepsilon^\infty_{ij})$. Then equivalent eigenstrain can be calculated as follows.

First we rewrite Eq. (2.9) as

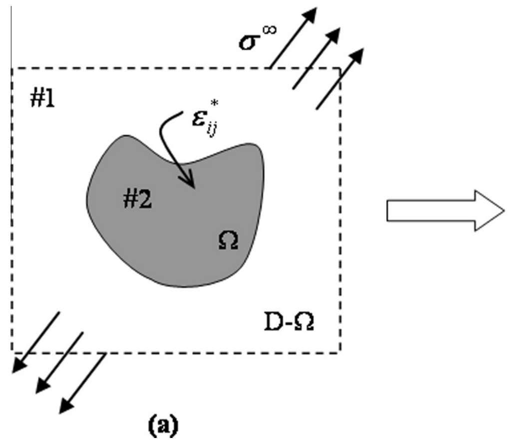

Fig. 6. An inhomogeneous inclusion with residual strain in it and under far-field load. (a) The original inhomogeneous inclusion system (with residual strain), (b) The counterpart virtual homogeneous inclusion system with equivalent eigenstrain.

$$\varepsilon_{ij}(\mathbf{x}) = \int_\Omega C_{klmn}\varepsilon^0_{mn}(\mathbf{x}')K_{ijkl}(\mathbf{x}-\mathbf{x}')d\mathbf{x}' \tag{6.1}$$

where $\varepsilon^0_{mn}(\mathbf{x})$ is the equivalent eigenstrain to be solved and

$$K_{ijkl}(\mathbf{x}-\mathbf{x}') = -\frac{1}{2}\{G_{ik,lj}(\mathbf{x}-\mathbf{x}') + G_{jk,li}(\mathbf{x}-\mathbf{x}')\} \tag{6.2}$$

Inserting (6.1) into (5.7), we get

$$\varepsilon^0_{ij}(\mathbf{x}) = S_{ijkl}(C_{klpq} - C^*_{klpq})\int_\Omega C_{rsmn}\varepsilon^0_{mn}(\mathbf{x}')K_{pqrs}(\mathbf{x}-\mathbf{x}')d\mathbf{x}' + S_{ijkl}(C_{klpq} - C^*_{klpq})\varepsilon^\infty_{pq} + S_{ijkl}C^*_{klpq}\varepsilon^*_{pq}(\mathbf{x}), \quad \mathbf{x}\in\Omega \tag{6.3}$$

Eq. (6.3) is an equation for solving the unknown equivalent strain $\varepsilon^0_{ij}(\mathbf{x})$. After it is solved, and by inserting it into Eqs. (5.8) and (5.9), the stress state of the inhomogeneous inclusion system is obtained.

It should be pointed out that since Eq. (6.3) is a singular linear integral equation indeed and it is theoretically solvable. For example, some closed solution for the equivalent eigenstrain $\varepsilon^0_{ij}(\mathbf{x})$ can be obtained by Fourier transform approach (Mura, 1987; Ma et al., 2012). However, during practical application, it is still difficult to get their explicit solutions since the inhomogeneous inclusions may have irregular shape, and numerical calculation possibly has to be performed. The numerical quadratures for solving (6.3) for the 1-D, 2-D and 3-D cases, respectively, are the next step research topics. In this paper, our study is limited to theoretical basis for treating inhomogeneous inclusion problems. The aspects on numerically solving (6.3) will be presented later elsewhere. Additionally, it should be noted that even though $\varepsilon^*_{pq}(\mathbf{x})$ is a uniform distribution in (6.3), $\varepsilon^0_{ij}(\mathbf{x})$ is unlikely to be uniform due to the arbitrary inclusion's profile. The Eshelby's tensor, it is even harder to be constant for inhomogeneous inclusion problems. The solution of Eshelby's inhomogeneous ellipsoidal inclusion is a very special one which has exceptional properties.

6.2. A simple example: stresses induced by an inhomogeneous sphere embedded in an infinite 3-D solid

We intend to provide a simple example to further demonstrate how to implement the formulae obtained in this paper, and also to validate the theoretical results. For simplicity and clearness, an inhomogeneous sphere embedded in an infinite solid with nominal interference fit strain $\varepsilon^* = \Delta/a$ as shown in Fig. 8 is studied, assuming that both material#1 and material#2 are dissimilar and isotropic.

The nominal residual strain in Fig. 8 is

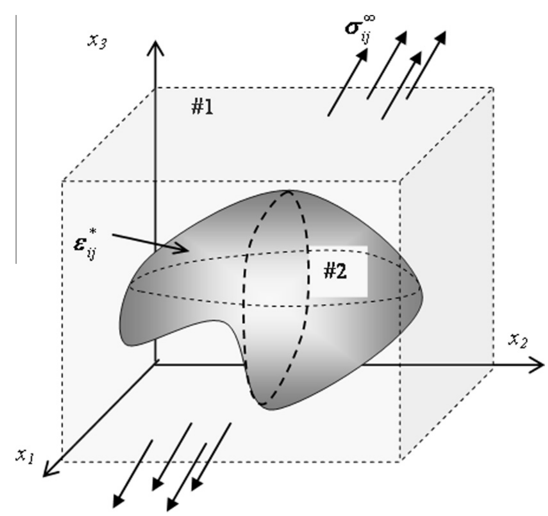

Fig. 7. An inhomogeneous inclusion embedded in an infinite solid with non-uniform residual strain in it and also under far-field load.

$$\varepsilon^*_{ij} = \delta_{ij}\varepsilon^* = \delta_{ij}\Delta/a \text{ in } \Omega \tag{6.4}$$

From Eq. (4.8), we know that

$$C_{ijml}\varepsilon^0_{ml} = (C_{ijml} - C^*_{ijml})\varepsilon_{ml} + C^*_{ijml}\varepsilon^*_{ml} \text{ in } \Omega \tag{6.5}$$

For isotropic materials, the anisotropic elastic constant can be expressed in terms of the Lamé constants $\lambda(\lambda^*)$ and μ (μ^*) as

$$\begin{cases} C_{ijml} = \lambda\delta_{ij}\delta_{ml} + \mu(\delta_{im}\delta_{jl} + \delta_{il}\delta_{jm}) \text{ for material\#1} \\ C^*_{ijml} = \lambda^*\delta_{ij}\delta_{ml} + \mu^*(\delta_{im}\delta_{jl} + \delta_{il}\delta_{jm}) \text{ for material\#2} \end{cases} \tag{6.6}$$

Inserting (6.6) into (6.5) leads

$$\lambda\delta_{ij}\varepsilon^0_{mm} + 2\mu\varepsilon^0_{ij} = [(\lambda-\lambda^*)\delta_{ij}\varepsilon_{mm} + 2(\mu-\mu^*)\varepsilon_{ij}] + (\lambda^*\delta_{ij}\varepsilon^*_{mm} + 2\mu^*\varepsilon^*_{ij}) \text{ in } \Omega \tag{6.7}$$

Since this is a sphere polar symmetric problem and the nominal residual strain is expressed in terms of (6.4), we may write the equivalent eigenstrain ε^0_{ij} and total strain ε_{ij}, respectively, as

$$\varepsilon^0_{ij} = \delta_{ij}\varepsilon_0, \quad \varepsilon_{ij} = \delta_{ij}\varepsilon \tag{6.8}$$

Thus Eq. (6.7) can be rewritten as

$$(3\lambda+2\mu)\varepsilon_0 = [3(\lambda-\lambda^*) + 2(\mu-\mu^*)]\varepsilon + (3\lambda^*+2\mu^*)\varepsilon^* \text{ in } \Omega \tag{6.9}$$

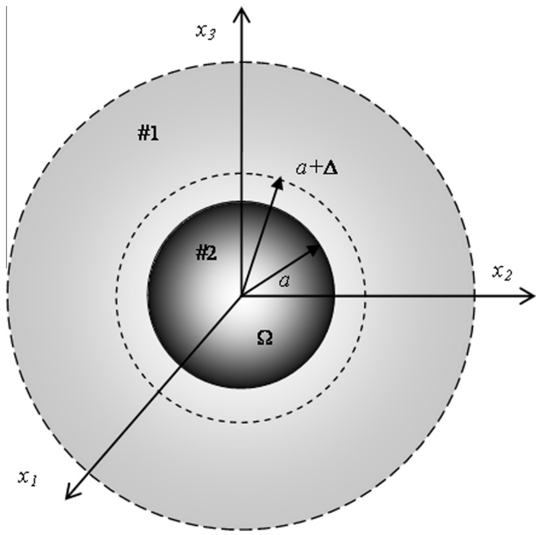

Fig. 8. An inhomogeneous sphere embedded in an infinite solid with nominal interference fit strain $\varepsilon^* = \Delta/a$.

in which ε is a function of ε_0 as the form of Eq. (4.9), and it can be found with Eq. (6.1) as (Mura, 1987)

$$\varepsilon = \frac{(1+\nu)}{3(1-\nu)}\varepsilon_0 = c\varepsilon_0 \text{in } \Omega \tag{6.10}$$

Then, ε_0 can be explicitly resolved from (6.9) as

$$\varepsilon_0 = \frac{(3\lambda^* + 2\mu^*)}{\{(3\lambda + 2\mu)(1-c) + (3\lambda^* + 2\mu^*)c\}}\varepsilon^* \tag{6.11}$$

So the stress in the inclusion can be evaluated with (4.12), (6.8) and (6.11) as

$$\begin{aligned}\sigma_{ij} &= C_{ijml}(\varepsilon_{ml} - \varepsilon^0_{ml}) = [\lambda\delta_{ij}\delta_{ml} + \mu(\delta_{im}\delta_{jl} + \delta_{il}\delta_{jm})](\varepsilon_{ml} - \varepsilon^0_{ml}) \\ &= (3\lambda + 2\mu)\delta_{ij}(\varepsilon - \varepsilon_0) = (3\lambda + 2\mu)\delta_{ij}(c-1)\varepsilon_0 \\ &= -\delta_{ij}\frac{(3\lambda^* + 2\mu^*)4\mu}{[4\mu + (3\lambda^* + 2\mu^*)]}\varepsilon^*\end{aligned} \tag{6.12}$$

This solution can be found in elastic mechanics text books, which is derived by other method.

This example not only may provide a simple demonstration of the application of the results obtained in this paper, but also may partially validate the manipulations in the previous sections.

7. Remarking conclusion

The principle of equivalent eigenstrain for inhomogeneous inclusion problems has been proposed based on the principle of virtual work. It enables the inhomogeneous inclusion problems to be transformed into homogenous inclusion problems whose solutions are maturely studied. A simple example for interference fit has been provided. It demonstrates that the fundamental formulation proposed in this paper may pave the way to a systematic study of inhomogeneous inclusion problems. The formulation allows solving the problems about inclusions of arbitrary shape, multiple inclusion problems, and lends itself to residual stress analysis in non-uniform, heterogeneous media. It is expected to apply in the mechanics of composites, inclusions, phase transformation analysis, plasticity, fracture mechanics, etc.

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References

Andrianov, B.V., Argatov, I.I., Weichert, D., 2008. On the absence of the Eshelby property for slender non-ellipsoidal inhomogeneities. Proc. R. Soc. A 464, 1079–1088.
Avazmohammadi, R., Hashemi, R., Shodja, H.M., Kargarnovin, M.H., 2010. Ellipsoidal domain with piecewise nonuniform eigenstrain field in one of joined isotropic half-spaces. J. Elast. 98, 117–140.
Chen, C.Q., Cui, J.Z., Duan, H.L., Feng, X.Q., He, L.H., Hu, G.K., Huang, M.J., Huo, Y.Z., Ji, B.H., Liu, B., Peng, X.H., Shi, H.J., Sun, Q.P., Wang, J.X., Wang, Y.S., Zhao, H.P., Zhao, Y.P., Zheng, Q.S., Zou, W.N., 2011. Perspectives in mechanics of heterogeneous solids. Acta Mech. Solida Sin. 24, 1–26.
Chen, Q.D., Xu, K.Y., Pan, E., 2014. Inclusion of an arbitrary polygon with graded eigenstrain in an anisotropic piezoelectric half plane. Int. J. Solids Struct. 51, 53–62.
Christensen, R.M., 1979. Mechanics of Composite Materials. Wiley, New York.
Eshelby, J.D., 1957. The determination of the elastic field of an ellipsoidal inclusion, and related problems. Proc. R. Soc. Lond. A 241, 376–396.
Eshelby, J.D., 1959. The elastic field outside an ellipsoidal inclusion. Proc. R. Soc. Lond. A. 252, 561–569.
Eshelby, J.D., 1961. Elastic inclusions and inhomogeneities. In: Sneddon, I.N., Hill, R. (Eds.), Progress in Solid Mechanics, vol. 2. North-Holland, Amsterdam, pp. 89–140.
Goodier, J.N., 1937. On the integration of the thermoelastic equations. Phil. Mag. 7, 1017–1032.
Hutchinson, J.W., 1976. Bounds and self-consistent estimates for creep of polycrystalline materials. Proc. R. Soc. Lond. A 348, 101–127.
Inglis, C.E., 1913. Stresses in a plate due to the presence of cracks and sharp corners. Inst. Naval Archit. Lond. 55, 219–230.
Jaswon, M.A., Bhargava, R.D., 1961. Two-dimensional elastic inclusion problems. Proc. Camb. Phil. Soc. 57, 669–680.
Kuvshinov, B.N., 2008. Elastic and piezoelectric fields due to polyhedral inclusions. Int. J. Solids Struct. 45, 1352–1384.
Li, Q., Anderson, P.M., 2001. A compact solution for the stress field from a cuboidal region with a uniform transformation strain. J. Elast. 64, 237–245.
Ma, L.F., 2010. Fundamental formulation for transformation toughening. Int. J. Solids Struct. 47, 3214–3220.
Ma, L.F., Zhao, J., Korsunsky, A.M., 2012. Fundamental solutions for singularities within a layered solid. Eur. J. Mech. A 35, 37–46.
Ma, L.F., Korsunsky, A.M., McMeeking, R.M., 2013. The fundamental formulation for residual stress problems in anisotropic solids. J. Appl. Mech. 80, 051001.
Mori, T., Tanaka, K., 1973. Average stress in matrix and average elastic energy of materials with misfitting inclusions. Acta Metall. 21, 571–574.
Mindlin, R.D., Cheng, D.H., 1950. Thermoelastic stress in the semi-infinite solid. J. Appl. Phys. 21, 931–933.
Mindlin, R.D., 1936. Force at a point in the interior of a semi-infinite solid. Physics 7, 195–202.
Mura, T., 1987. Micromechanics of Defects in Solids, second ed. Martinus Nijhoff, Dordrecht, Netherlands.
Mura, T., 1988. Inclusion problems. Appl. Mech. Rev. 41, 15–20.
Mura, T., Shodja, H.M., Hirose, Y., 1996. Inclusion problems. Appl. Mech. Rev. 49, S118–S127.
Nemat-Nasser, S., Hori, M., 1999. Micromechanics: Overall Properties of Heterogeneous Materials, second ed. Elsevier, Amsterdam, Netherlands.
Ru, C.Q., 1999. Analytical solution for Eshelby's problem of an inclusion of arbitrary shape in a plane or half-plane. J. Appl. Mech. 66, 315–322.
Ru, C.Q., 2000. Eshelby's problem for two-dimensional piezoelectric inclusions of arbitrary shape. Proc. Roy. Soc. A 456, 1051–1068.
Ru, C.Q., 2003. Eshelby inclusion of arbitrary shape in an anisotropic plane or halfplane. Acta Mech. 160, 219–234.
Sen, B., 1951. Note on stresses produced by nuclei of thermo-elastic strain in a semi-infinite elastic solid. Quart. Appl. Math. 8, 365–369.
Willis, J.R., 1981. Variational and related methods for the overall properties of composites. Adv. Appl. Mech. 21, 1–78.
Zheng, Q.S., Zhao, Z.H., Du, D.X., 2006. Irreducible structure, symmetry and average of Eshelby's tensor fields in isotropic elasticity. J. Mech. Phys. Solids 54, 368–383.
Zhou, K., Hoh, H.J., Wang, X., Keer, L.M., Pang, J.H.L., Song, B., Wang, Q.J., 2013. A review of recent works on inclusions. Mech. Mater. 60, 144–158.
Zou, W.N., He, Q.C., Huang, M.J., Zheng, Q.S., 2010. Eshelby's problem of non-elliptical inclusions. J. Mech. Phys. Solids 58, 346–372.
Zou, W.N., He, Q.C., Zheng, Q.S., 2011. General solution for Eshelby's problem of 2D arbitrarily shaped piezoelectric inclusions. Int. J. Solids Struct. 48, 2681–2694.