Chapter 1

BASIC CONCEPTS

1.1 Introduction to QFT and the Lectures

READING MATERIAL:wiki: "quantum field theory"

Why QFT

1. classical physics + QM are not enough to describe all phenomenology, especially microcosmic phenomenology.
   - in microcosmic, particles are very small and moves at very high speed (QM+RST)
   - describe a process with creative or annihilate of particles
2. field need quantization when space-time interval tending to zero
   - review: radiations in QM
3. wave-particle duality
   - in classical physics (wave or particle)
   - in QM (duality, more quanta)
   - unity of description in QFT

4. a updated language to describe updated nature
   - what are elementary particles and interactions
   - description of new phenomenology: dark matter, dark energy, ...
   - develop physics to an unified theory

Introduction to QFT

- history:
  - 1925: Born and Jordan, to calculate in quantum transition
  - 1926: Born, Heisenberg, and Jordan, a quantum theory for free EM field; Dirac, solve some problem
  - the early motivation to develop QFT in history: solve the problem of many particle interactions (more from experiment)
  - the second motivation: combine SRT and QM (more from theory)
- axiomatic system:
  - 3 in QM and 2 in SRT
  - compared with classical physics $\cdots$
- wave-particle duality:
  - before the broth of QFT, wave theory vs. particle theory
  - wave theory: Maxwell's EM theory and GRT – core: field
  - particle theory: QM – core: quanta
  - But, probability of wave function is essentially field!
- lacks of QM and SR
  - QM: negative energy, non-conservation of number of particle, negative probability problem in relativistic QM
  -

- description language and basic concepts:
  - updated field concept
  - vacuum
  - Lagrangian
  - symmetry
- some successful examples:
  - μ anomaly magnetic moment

$$\begin{aligned} a|_{th} = \frac{\alpha}{4\pi} &= 0.00115965218178(77) \\ a|_{exp} &= 0.00115965218073(28) \end{aligned}$$

  - and Lamb shift
  - comprehension electric charge from renormalization
- development in future
  - SM and its triumph
  - physics beyond SM: neutrino mass, dark matter, dark energy, gravitation theory...
  - possible candidates: ED, SUSY, Superstring
- the relationship between particle physics and QFT

On the Course

- contents (see contents for details)
- references

(Das) Ashok Das, Lectures on Quantum Field Theory, Univ. of Rochester, USA, 2008.

(Zee) A. Zee

(Zuber) Zuber,

(Ramond) Ramond

- test and result: 30% from 3 quizzes + 70% from open final test
- questions and answers outside class:
  - email: hepzhy@mail.xjtu.edu.cn
  - office: Room 1108 in Main Building

1.2 Lecture 2: Review of QM and SR

1.2.1 5 hypothesis

- Hilbert space
- Hermitian operator
- statistic hypothesis
- The speed of light postulate
- principle of special relativity

Hypothesis : Hilbert space

description of a state

- review of the description of a state in classical physics:
  - a particle at the position of x at the time of t: $x(t)$
  - its momentum is $p(t)$ at the time t
  - a curve in the plane of $x-p$, which means a precise trajectory $\{x(t), p(t)\}$
  - characteristics:
    * describe a particle by physical observables, like the position x, the momentum p
    * a certain trajectory $\{x(t), p(t)\}$ - phase figure
- the description is invalid in micro-motion. review the diffraction of a electron
  - diffraction device
  - electron motion is a line in classical mechanics; a point at the screen
  - reduce the size of hole; diffraction pattern
  - control electron current; single electron; pattern at the screen
  - randomization of a single electron; probability of many electrons;

  - the position is not a certain quantity

- review interference experiment

- A novel way to describe a state.

  - describe the state of a particle
  - $x(t) \to |x\rangle$ at the time of t
    * $|\rangle$ labels a object (particle) state
    * x in the $|\rangle$ is a value that refers to ...
    * $\{|\, x\rangle\}$ form a set of physical state
  - $p(t) \to |p\rangle$ at the time t
  - examples:
    * spin state: label spin-up state (in z direction) $|\,+\rangle$ and spin-down state $|\,-\rangle$. The means are ...
    * describe pages of a book as $|\; page = x\rangle$ with the mean of ...
    * describe a classmate in our lesson
      · by name: $|\; name\rangle$
      · by ID: $|\; ID\rangle$
      · relationship: $|\; name\rangle \longleftrightarrow |\; ID\rangle$
      · by column and row in the class $|\; column, row\rangle$
    * find more examples HOMEWORK
  - new characteristics:
    * physical quantity vs. physical value
    * determinacy vs. probability
    * how to understand probability in $|\; f(x,y)\rangle$?
      · in terms of the distribution function ...
    * how to explain probability?
      · natural? or compromised?
      · Copenhagen School and Copenhagen explain

· more...

- Some reasons to choose the novel sign.
  - to describe the diffraction of electron
  - what's the definition of a particle in classical physics?
    - with well defined mass, spin, energy, momentum, and so on)
    - the characteristics: *described by* many physics quantities
    - or, we can say, a particle is *a state with* a set of many certain physics quantities

Definition and Properties

hypothesis I: physical state corresponds to a vector in Hilbert space.

- Hilbert space: a complex vector space with well-defined inner product
- three kinds of operation:
  - addition
  - number product
  - inner product
- some properties:

$$\psi + \phi = \phi + \psi, \quad (\psi + \phi) + \chi = \psi + (\phi + \chi), \quad a(\psi + \phi) = a\psi + a\phi, \tag{1.1}$$

$$(a + b)\psi = a\psi + b\psi, \quad (\psi, \phi) = (\phi, \psi)^*, \quad (\psi, \phi + \chi) = (\psi, \phi) + (\psi + \chi), \tag{1.2}$$

$$(\psi, a\phi) = a(\psi, \phi), \quad (a\psi, \phi) = a^*(\psi, \phi), \quad (\psi, \psi) = |\psi|^2 \tag{1.3}$$

- some concepts: linearly independence; base vector; complete set; infinite dimension vs. finite dimension
- base vector and matrix representation
- what physical information can be obtained from a vector in Hilbert space?
- NOTE: left-vector and right-vector

Dirac notation

- origin: 1. left-vector $\neq$ right-vector; 2. they locate in the same bracket; 3. right-vector $\psi \to$ left-vector $\psi^\dagger$
- left-vector $\to$ bra $\langle\psi|$ and right-vector $\to$ ket $|\phi\rangle$
- inner product in Dirac notation $\langle\psi| \cdot |\phi\rangle = \langle\psi \mid \phi\rangle$
- denote Hermitian eigenstates $\{\psi_i\}$ by corresponding eigenvalues $\{i\}$
- some composition Dirac notation: $\mid \alpha\rangle \mid \beta\rangle$, $\mid \alpha\rangle\langle\beta \mid$, $\langle\alpha \mid \beta\rangle$
- what is Dirac notation advantage??
- an example: expand a random state into Hermitian eigenstates

$$\mid \Psi\rangle = \sum_i c_i \mid i\rangle = \sum_i \langle i \mid \Psi\rangle \mid i\rangle = \sum_i \mid i\rangle\langle i \mid \Psi\rangle$$

  1. $\langle i \mid \Psi\rangle$ is coefficient; 2. $\mid i\rangle\langle i \mid$ is projection operator; 3. complete relation $\sum_i \mid i\rangle\langle i \mid= 1$.

- the physical means of $\langle i \mid \Psi\rangle$, $\langle\phi \mid \Psi\rangle$, $\mid i, t\rangle$ and $\langle i, t \mid \Psi\rangle$

Hypothesis 2: Hermitian Operator

- $\mid \alpha\rangle$ is a state, what do we know from the state?
- review classical observables
- in QM, we need an operator

What Is An Operator

- a general operator: $A\psi = \phi$ or A let $\psi \to \phi$
- eigenvalues (some special values): $A\psi_i = a_i\psi_i$ corresponding to special states (eigenstates)
- linear operator: $A(a\psi + b\phi) = aA\psi + bA\phi$

- unity operator: $A\psi = \psi$
- inverse operator: $A\psi = \phi$, $A^{-1}\phi = \psi$
- unitary operator: $AA^\dagger = A^\dagger A = 1$

What Is A Hermitian Operator

- Hermitian operator: $A = A^\dagger$
- three important properties of Hermitian operator:
  - real eigenvaleus $a(\psi, \psi) = (\psi, A\psi) = (\psi, A^\dagger\psi) = (A\psi, \psi) = a^*(\psi, \psi)$
  - orthogonality of states with different eigenstates_{HOMEWORK}
  - completeness of eigenstates. (proof ...)
- **hypothesis II: Hermitian operator is a candidate of physics quantity.**
- why??

Examples

- position state $\mid x\rangle$: position operator: $\hat{X}$

$$\hat{X} \mid x\rangle = x \mid x\rangle$$

- spin state $\mid +\rangle, \mid -\rangle$: spin operator: $\hat{S}$

$$\hat{S} \mid \pm\rangle = \pm\frac{\hbar}{2} \mid \pm\rangle$$

(the eigenvalues of spin are $\pm\hbar/2$)

- pages of a book $\mid page = x\rangle$: page operator: $\hat{G}$

$$\hat{G} \mid page = x\rangle = x \mid pate = x\rangle$$

- a classmate in our lesson
  - $\mid name\rangle$: name operator

  - $\mid ID\rangle$: ID operator
  - $\mid name\rangle \longleftrightarrow \mid ID\rangle$ (?)
  - $\mid column, row\rangle$: column operator and row operator (?)

- momentum operator $\hat{p} \equiv -i\hbar\nabla$ and eigenstate in position space...

commutator: $\hat{p}$ and $\hat{x}$

- $\hat{p}\hat{x} = ?$
- $\hat{x}\hat{p} = ?$
- commutator $[\hat{p}, \hat{x}] = ...$
- what do we know from commutator? after the section Hypothesis III

experiment and statistical hypothesis

- what can we measure? in classical physics and in QM
- determinacy vs. probability: physical results in classical physics vs. ones in QM–diffraction
- measurement in QM
- how to describe measurement in QM?

  hypothesis III: modular of vector in Hilbert corresponds to probability of measurement

- an example: expand a random state (vector) in a set of complete eigenstates $\{\psi_i\}$(base vector)

$$\Psi = \sum_i c_i \psi_i$$

- inner product and physics mean:

$$(\psi_i, \Phi) = (\psi_i, \sum_j c_j \psi_j) = c_i \quad \longrightarrow \quad p_i = |c_i|^2$$
$$(\psi_1 + \psi_2, \Phi) = (\psi_1 + \psi_2, \sum_j c_j \psi_j) = c_1 + c_2 \quad \longrightarrow \quad p_i = |c_1 + c_2|^2$$
$$(\chi, \Phi) = (\sum_i c'_i \psi_i, \Phi) \qquad \cdots$$

  inner product (χ, ϕ) "represents" a probability of finding state χ from ϕ, or a probability of a system with initial state ψ and final state χ

matrix mechanism

- a set of complete Hermitian eigenstates $\rightarrow$ a set of base vectors in linear space

- right-vector, ket, corresponds to column vector and left-vector, bra, corresponds to row vector

- operator corresponds to a $n \times n$ matrix

  a operator transform a state to another state: $A \mid \Psi\rangle \rightarrow A \mid i\rangle = \sum_j c_j^i \mid j\rangle$; coefficient c_j^i includes all information

$$c_j^i = \langle j \mid A \mid i\rangle \equiv A_{ji}$$

- a operator representation under itself eigenstates: diagonal matrix with eigenvalues diagonal matrix element

representation

- why we can choose different representation in math and in physics?

  In math, a set of eigenstates of Hermitian operator is complete. In physics, a "measurement" result is always gotten when acting a Hermitian operator on random state.

- concept: representation transformation

$$|\ \Psi\rangle = \sum_i \eta_i \mid a_i\rangle = \sum_j \zeta_i \mid b_i\rangle \tag{1.4}$$

$$\rightarrow \quad |\ a_i\rangle = \sum_j F_{ij} \mid b_j\rangle \tag{1.5}$$

$$\rightarrow \quad F_{ij} = \langle a_i \mid b_j\rangle \tag{1.6}$$

F_{ij} transforms $\{|\ b_j\rangle\}$ into $\{|\ a_i\rangle\}$

- how to transform $\{|\ a_i\rangle\}$ into $\{|\ b_i\rangle\}$? $F^\dagger_{ij}$
- unitarity of F_{ij} HOMEWORK
- representation transformation and physical measurement
- inner product, eigenvalues, $det(A)$ and $tr(A)$ are invariant under representation transformation HOMEWORK

$$\begin{aligned}\langle\Psi \mid \Phi\rangle &= \sum_{i,j}\Big(\eta_i \mid i\rangle\Big)^\dagger \zeta_j \mid j\rangle = \sum_i \eta_i^* \zeta_i \\ &= \sum_{i,j}\Big(\eta_i(\sum_k F_{ki} \mid i\rangle)\Big)^\dagger \zeta_j(\sum_l F_{lj} \mid j\rangle) \\ &= \sum_{i,j} \eta_i^* \zeta_j \sum_{k,l}(F_{ki})^\dagger F_{lj}\delta_{i,j} = \sum_i \eta_i^* \zeta_i \sum_{k,l} F^*_{ik} F_{li} = \sum_i \eta_i^* \zeta_i\end{aligned}$$

ket and wave function

- what's the meaning of $|\ \alpha\rangle$?
- what's the meaning of $|\ x\rangle$?
- what's the meaning of $\langle\beta \mid \alpha\rangle$?
- what's the meaning of $\langle x \mid \alpha\rangle$?
- what's the meaning of $\phi_\alpha(x)$?
- now, we have $\phi_\alpha(x) = \langle x \mid \alpha\rangle$

on Schrodinger' Eq

- Schrodinger's Eq

$$i\hbar\frac{\partial}{\partial t}\phi(x) = ((-i\hbar\nabla)^2 + V(x))\psi(x)$$

- $V(x)$ a function? or an operator?
- express it into Dirac sign...(exercise in classs)
- time independent form:

$$\hat{H}\phi(x) = E\phi(x)$$

- time evolution relation: ...
- now, we know Schrodinger's EQ, i.e. energy eigenstate equation + evolution relation

on hypothesis

- three hypothesis=three questions
  - how to *describe* a physical system in QM? a state (vector) in Hilbert space
  - how to get *physical information* of a state? to act a Hermitian operator on the state
  - how to *explain* results? probability
- measurement:

  act a Hermitian operator on a state = measurement?

two hypothesis in SR:

- M-M experiment: velocity of light: natural law
- relativity principle: from Galilean's relativity to SR relativity
- invariant interval of spacetime
- Lorentz transformation

1.3 Lecture 3: Notations and Lagrangian Formula

Natural Units

- interactional unit: CGS with 3 independent unit
- in particle physics, velocity(c) and action($\hbar$) are taken unit, i.e. $c = \hbar = 1$.
- the left unit is chosen as energy. eV

$$[S](\hbar) = 1 = [E][T], [V](c) = 1 = [L]/[T], [E](GeV) = [M][c]^2 = [M]$$

so, time and energy have a converse unit; $\cdots$

- relation: $[V], [S], [E]$ in natural unit and $[T], [L], [M]$ in CGS

$$[T] = [S][E]^{-1}; [L] = [V][T] = [V][S][E]^{-1}; [M] = [E][V]^{-2};$$
$$[V] = [L][T]^{-1}; [S] = [M][L]^2[T]^{-1}; [E] = [M][V]^2 = [M][L]^2[T]^{-2}.$$

- number transformation: $((c) = 3 \times 10^{10}, (\hbar) = 1.05 \times 10^{-27}, (\epsilon) = 1.6 \times 10^{-3}$ in CGS)

$$1cm = (c)^{-1}(\hbar)^{-1}(\epsilon)GeV^{-1} \cdot c \cdot \hbar = 5.08 \times 10^{13} GeV^{-1}$$
$$1sec = (\hbar)^{-1}(\epsilon)GeV^{-1}\hbar = 1.5 \times 10^{24} GeV^{-1}$$
$$1g = (c)^2(\epsilon)^{-1}GeV \cdot c^2 = 0.56 \times 10^{24} GeV$$

or

$$1GeV^{-1}(\cdot c \cdot \hbar) = 0.198 \times 10^{-13} cm$$
$$1GeV^{-1}(\cdot \hbar) = 0.658 \times 10^{-24} sec$$
$$1GeV(\cdot c^2) = 1.78 \times 10^{-24} g$$

- physical formulas in natural unit:....
- electromagnetism formulas

$$F = \frac{QQ}{r^2}$$

in Gaussian system and

$$F = \frac{qq}{4\pi r^2}$$

in Lorentz-Heaviside system (chosen)

in LH system, electric charge

$$e = \sqrt{4\pi\hbar c\alpha}$$

with fine-structure constant $\alpha \simeq 137^{-1}$.

4D space-time

- review of 3-D Euclidean space:

$$\begin{aligned} \vec{x} &= (x_1, x_2, x_3) \\ \vec{A} &= (A_1, A_2, A_3) \\ \vec{A}\cdot\vec{B} &= \sum_{i=1,2,3} A_i B_i = A_i B_i = \delta_{ij} A_i B_j = \delta^{ij} A_i B_j \\ \delta_{ij} &= \delta^{ij} = \begin{cases} 1 & for\ i=j \\ 0 & other \end{cases} = diag(1,1,1,1) \\ \vec{A}^2 &\geq 0 \end{aligned}$$

- 4-D Minkowski space: invariant $ds^2 = c^2t^2 - x^2 - y^2 - z^2$

$$\begin{aligned} x^\mu &= (ct, \vec{x}) \\ x_\mu &= (ct, -\vec{x}) = g_{\mu\nu} x^\nu = g_\mu^\nu x_\nu \\ g_{\mu\nu} &\equiv (g^{\mu\nu})^{-1} = g^{\mu\nu} = diag(1,-1,-1,-1) = g^{\nu\mu} = g_{\nu\mu} \\ g_\mu^\nu &= g^{\nu\rho} g_{\mu\rho} = diag(1,1,1,1) \\ g^{\mu\nu} g_{\nu\rho} &= \delta_\rho^\mu \end{aligned}$$

$g_{\mu\nu}$: gauge tensor. what is the means of "gauge"? the role of $g_{\mu\nu}$: represent the construct of space-time, rise subscript and down superscript

$$
\begin{aligned}
A^\mu &= (A^0, \vec{A}) \\
A_\mu &= g_{\mu\nu} A^\nu = \cdots \\
A_\mu B^\mu &= g_{\mu\nu} A^\mu B^\nu = A^0 B^0 - \vec{A}\cdot\vec{B} = A^0 B^0 - A^i B^i \\
x^2 &= x_\mu x^\mu = g_{\mu\nu} x^\mu x^\nu = c^2 t^2 - \vec{x}^2
\end{aligned}
$$

- Lorentz invariant length

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - d\vec{x}^2$$

- time-like region of space-time: $ds^2 > 0$;
  space-like region of space-time: $ds^2 < 0$;
  $x^2 = 0$ defines trajectories for light-like particle, called light-like region.
- plot $t - x$ figure to show three regions.
- future light cone and past light cone.
- derivative

$$
\begin{aligned}
\partial^\mu &= \frac{\partial}{\partial x_\mu} = (\frac{\partial}{\partial t}, -\nabla) \\
\partial_\mu &= \cdots \\
\partial^2 &= \cdots
\end{aligned}
$$

$$
\begin{aligned}
p^\mu &= (E, \vec{p}) \\
p^2 &= \cdots = m^2
\end{aligned}
$$

$$
\begin{aligned}
\epsilon^{ijk} &= \cdots \\
\epsilon^{\mu\nu\rho\sigma} &= \cdots
\end{aligned}
$$

exercise

- $A \cdot A$

- $\partial_\mu A^2$
- $\frac{\partial}{\partial(\partial_\mu A_\nu)} F^2$ with $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$

principle of least action

- the action: $I = \int_{t_1}^{t_2} dt L(q(t), \dot{q}(t))$
- Lagrange depends on positions $q(t)$, velocities $\dot{q}(t)$, and sometimes also explicitly on time for open systems.
- Lagrange only depends on the first derivation of the position. why?
- another principle to building action: Lagrange should be relativistic invariant
- equation of motion (action I depends on path connected $t = 0$ to $t = T$

$$
\begin{aligned}
0 &= \delta I = \int_0^t dt \frac{\delta I}{\delta q(t)} \delta q(t) \\
&= \int_0^t dt \Big\{ \frac{\partial I}{\partial q(t)} \delta q(t) + \frac{\partial I}{\partial \dot{q}(t)} \delta \dot{q}(t) \Big\} \\
&= \int_0^t dt \Big\{ \frac{\partial I}{\partial q(t)} \delta q(t) + \frac{\partial}{\partial t} \Big[\frac{\partial I}{\partial \dot{q}(t)} \delta q(t) \Big] - \frac{\partial}{\partial t} \frac{\partial I}{\partial \dot{q}(t)} \delta q(t) \Big\} \\
&= \int_0^t dt \Big\{ \frac{\partial I}{\partial q(t)} - \frac{\partial}{\partial t} \frac{\partial I}{\partial \dot{q}(t)} \Big\} \delta q(t) + \Big[\frac{\partial I}{\partial \dot{q}(t)} \delta q(t) \Big] \Big|_{t=0}^{t=T} \\
&\Rightarrow \frac{\partial I}{\partial q(t)} - \frac{\partial}{\partial t} \frac{\partial I}{\partial \dot{q}(t)} = 0
\end{aligned}
$$

  due to $\delta q(0) = \delta q(T) = 0$. why?

Hamilton's function

- $L(q, \dot{q}) \Rightarrow H(p, q)$
- Legendre transformation

$$H(p, q) = p_i \dot{q}_i - L$$

  or more precise

$$H(p, q) = p_i \dot{q}_i(p, q) - L(q, \dot{q}(p, q))$$

  with conjugate momenta $p_i = \frac{\partial}{\partial \dot{q}} L(q, \dot{q})$

- derive Hamilton equation of motion

$$
\begin{aligned}
dH(p,q) &= \Big\{\dot{q}_i(p,q) + p_j\frac{\partial}{\partial p_i}\dot{q}_j(p,q) - \frac{\partial}{\partial \dot{q}_j}L(q,\dot{q}(p,q))\frac{\partial}{\partial p_i}\dot{q}_j(p,q)\Big\}dp_i \\
&\quad + \Big\{p_j\frac{\partial}{\partial q_i}\dot{q}_j(p,q) - \frac{\partial}{\partial q_i}L(q,\dot{q}(p,q)) - \frac{\partial}{\partial \dot{q}_j}L(q,\dot{q}(p,q))\frac{\partial}{\partial q_i}\dot{q}_j(p,q)\Big\}dq_i \\
&= \Big\{\dot{q}_i(p,q) + \Big[p_j - \frac{\partial}{\partial \dot{q}_j}L\Big]\frac{\partial}{\partial p_i}\dot{q}_j(p,q)\Big\}dp_i \\
&\quad + \Big\{-\frac{\partial}{\partial q_i}L(q,\dot{q}(p,q)) + \Big[p_j - \frac{\partial}{\partial \dot{q}_j}L\Big]\frac{\partial}{\partial q_i}\dot{q}_j(p,q)\Big\}dq_i \\
&= \dot{q}_i(p,q)dp_i - \frac{\partial}{\partial q_i}L(q,\dot{q}(p,q))dq_i \\
&\Rightarrow \begin{cases} \frac{\partial H}{\partial p_i} = \dot{q}_i, \\ \frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i} = -\dot{p}_i \end{cases}
\end{aligned}
$$

Infinite degree of freedom system - field

- the action for many degree of freedom system:

$$L = \sum_i \frac{1}{2}m_i\dot{q}_i^2 - V(q_1, q_2..q_i..q_N)$$

V includes interaction energy between particles $v(q_i - q_j)$ and the energy due to an external potential $w(q_i)$

$$V = \sum_{i,j}\frac{1}{2}k_{ij}(q_i - q_j)^2 + \cdots$$

- when the interval l ($l = q_{i+1} - q_i$) tending to zero, we obtain a infinite degree of freedom system: field

the subscript $_i$ in $q_i(t)$ means that $q_i(t)$ depends on position, so $q_i(t) \to \phi(t, x)$.

$$
\begin{aligned}
m_i\dot{q}_i^2 &\to m_i(\frac{\partial}{\partial t}\phi)^2 \\
(q_{i+1} - q_i)^2 &\to \left(\frac{\partial}{\partial x}\phi\right)^2 + \left(\frac{\partial}{\partial y}\phi\right)^2 + \left(\frac{\partial}{\partial z}\phi\right)^2 + \cdots \\
\sum_i &\to \int dxdydz = \int d^3x
\end{aligned}
$$

so

$$L = \int d^3x \left\{ \sigma \left(\frac{\partial}{\partial t} \phi \right)^2 - \rho \nabla^2 \phi \right\} - \tau \phi^2 - \xi \phi^4 \cdots$$

after re-defined parameters and ϕ (but why?), we have

$$S = \int dt d^3x (\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4 + \cdots) = \int d^4x \mathcal{L}$$

- Lorentz covariance:

$$\begin{aligned} \delta S &= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial \nabla \phi} \delta \nabla \phi + \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \delta \dot{\phi} \right) \\ &= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \nabla \frac{\partial \mathcal{L}}{\partial \nabla \phi} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) \delta \phi \end{aligned}$$

eom:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \nabla \phi} = 0$$

, which means that $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$

- position x is not dynamical variable any longer. ϕ becomes dynamical variable in QFT.

an example: electromagnetic field

- Maxwell's equation

$$\nabla \vec{E} = \rho$$
$$\nabla \vec{B} = 0$$
$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j}$$
$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

- charge conservation is

$$\frac{\partial \rho}{\partial t} + \nabla \vec{j} = 0$$

- it is a set of equation of motion on electromagnetic field, which Lagrange does correspond to the equations?

- Clues: symmetries on $\vec{B}$ and $\vec{E}$, and on time t and position x
- notation for 4-D vector, tensor, etc, to exhibit relativistic covriance

$$\begin{aligned} x^\mu &\equiv (t, \vec{x}) \\ j^\mu &\equiv (\rho, \vec{j}) \\ F^{\mu\nu} &\equiv -F^{\nu\mu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \end{aligned}$$

for μ, ν taking $0, 1, 2, 3$

- Maxwell's eq.

$$\partial_\mu F^{\mu\nu} = j^\nu$$

i.e.

$$\frac{\partial}{\partial t} F^{0\nu} + \frac{\partial}{\partial x^i} F^{i\nu} = j^\nu$$

for $\nu = 0$,

$$\nabla \vec{E} = \rho$$

for $\nu = i$,

$$\frac{\partial}{\partial t} F^{0k} + \frac{\partial}{\partial x^i} F^{ik} = j^k \Rightarrow -\frac{\partial}{\partial t} \vec{E} + \nabla \times \vec{B} = \vec{j}$$

(using

$$\frac{\partial}{\partial x^i} F^{i1} = \frac{\partial}{\partial x^2} B^3 - \frac{\partial}{\partial x^3} B^2 = \epsilon^{ijk} \nabla_i B^j \; (when\; k = 1)$$

so, $\frac{\partial}{\partial x^i} F^{ik} = \nabla \times \vec{B}$.)

- where are other two Maxwell's eq.?
- define antisymmetric tensor

$$\tilde{F}^{\mu\nu} \equiv -\tilde{F}^{\nu\mu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{pmatrix}$$

other two Maxwell's eq. are

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$

- current conservation becomes

$$\partial_\mu j^\mu = 0$$

up to now, it's only a simply fashion to express Maxwell's eq. by $F^{\mu\nu}$. The physics quantities are still electromagnetic field strength $\vec{E}$ and $\vec{B}$. How to express them in QFT?

- define electromagnetic field in QFT: $A^\mu = (\phi, \vec{A})$
- $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$
- $\vec{E}, \vec{B} = \cdots$
- eom: $\partial^2 A^\mu - \partial^\mu \partial^\nu A_\nu = j^\mu$
- the eom is gauge invariant under $A^\mu \to A^\mu + \partial^\mu \phi$

The above clue: Maxwell' eq. $\to F^{\mu\nu} \to$ eom on A^μ. But how to write EM field theory directly?

- Lagrangian equation on EM:

$$\frac{\partial}{\partial \phi_i}\mathcal{L} - \partial_\mu \frac{\partial}{\partial \partial_\mu \phi_i}\mathcal{L} = 0$$

- $\mathcal{L} = \mathcal{L}(A_\mu, \partial_\mu A_\nu)$
- Lorentz invariance

$$\mathcal{L} = \frac{1}{2}\left\{a\partial_\mu A^\nu \partial^\mu A_\nu + b\partial_\mu A^\nu \partial_\nu A^\mu + c(\partial_\mu A^\mu)^2 + dA^2 + eA_\mu j^\mu\right\}$$

- coincide with eom

$$\mathcal{L} = -\frac{1}{4}(F^{\mu\nu})^2 - j_\mu A^\mu + c\left((\partial_\mu A^\mu)^2 - \partial_\mu A^\nu \partial_\nu A^\mu\right)$$

- the last term is a total derivation

$$(\partial_\mu A^\mu)^2 - \partial_\mu A^\nu \partial_\nu A^\mu = \partial_\mu \Big\{A_\nu(g^{\mu\nu}(\partial_\rho A^\rho) - \partial^\nu A^\mu)\Big\}$$

1.4 Lecture 4: Symmetries

why symmetry

- symmetry appears in physics
- symmetry provides a tool to describe the beauty of the nature
- symmetry corresponds conservation law
- so, symmetry is physics

examples

- snowflake: rotation and reflact
- Higgs potential: continuous rotation
- charge conjugation: $W^- \to e^- \nu_e$
- isospin: intro.,

classification

- space-time vs. internal
- continuous vs. discrete
- exact vs. approximate

conservations in physics

- space translation
- time translation
- space rotation
- parity

group theory

- a closed set with defined operating (addition or product)
- an example: parity(intro., define, time table)
- unity, invert element
- an example: group elements in snowflake (elements, product table, generator)
- Lie group: infinity elements (an example: rotation), generators, Abelian and non-abelian, Lie algebra

spacetime transformation and conservation

- assuming a space-time transformation $x'_\mu = x_\mu + \delta x_\mu$ or a inner transformation, the action is invariant.
- the action:

$$\begin{aligned}
0 &= \delta S = \int_{R'} d^4x'\mathcal{L}'(x') - \int_R d^4x\mathcal{L}(x) \\
&= \int_{R'} d^4x'\mathcal{L}'(x') - \int_{R'} d^4x'\mathcal{L}(x) + \int_{R'} d^4x'\mathcal{L}(x) - \int_R d^4x\mathcal{L}(x) \\
&= \int_{R'} d^4x'\delta\mathcal{L} + \int_{R'} d^4x'\mathcal{L}(x) - \int_R d^4x\mathcal{L}(x)
\end{aligned}$$

using

$$d^4x' = J(\frac{x'}{x})d^4x = Det(\frac{\partial x'^\mu}{\partial x^\nu})d^4x = \left(1 + \frac{\partial}{\partial x^\mu}\delta x^\mu\right) d^4x$$

,

$$
\begin{aligned}
\delta S &= \int_{R'} d^4x'\delta\mathcal{L} + \int_{R'} d^4x'\mathcal{L}(x) - \int_R d^4x\mathcal{L}(x) \\
&= \int_{R'} d^4x'\delta\mathcal{L} + \int \left(1 + \frac{\partial}{\partial x^\mu}\delta x^\mu\right) d^4x\mathcal{L}(x) - \int_R d^4x\mathcal{L}(x) \\
&= \int_{R'} d^4x'\delta\mathcal{L} + \int d^4x\mathcal{L}(x)\frac{\partial}{\partial x^\mu}\delta x^\mu \\
&= \int_R d^4x\left(\partial_\mu\mathcal{L}\delta^\mu + \delta_0\mathcal{L}\right) + \int d^4x\mathcal{L}(x)\frac{\partial}{\partial x^\mu}\delta x^\mu \\
&= \int d^4x(\partial_\mu\mathcal{L}\delta x^\mu + \frac{\partial\mathcal{L}}{\partial\phi}\delta_0\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_0(\partial_\mu\phi)) + \int d^4x\frac{\partial}{\partial x^\mu}\delta x^\mu\mathcal{L}(x) \\
&= \int d^4x\left(\frac{\partial\mathcal{L}}{\partial\phi}\delta_0\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\delta_0\phi\right) + \int d^4x\partial_\mu\left(\mathcal{L}\delta x^\mu\right) \\
&= \int d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi}\delta_0\phi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_0\phi\right) - \left(\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta_0\phi\right] + \int d^4x\partial_\mu\left(\mathcal{L}\delta x^\mu\right) \\
&= \int d^4x\partial_\mu\left(\frac{\partial}{\partial(\partial_\mu\phi)}\mathcal{L}\delta_0\phi + \mathcal{L}\delta x^\mu\right) + e.o.f.
\end{aligned}
$$

- δ_0 denote a change of function form
- $\delta\phi$ means a total change of ϕ under all transformations

$$\delta\phi = \partial_\mu\phi\delta x^\mu + \delta_0\phi, \quad (\delta\phi \equiv iT^i\phi\delta\epsilon^i)$$

- independent variation $\delta\epsilon^i$ and δx. WHY??

•

$$
\begin{aligned}
\delta S &= \int d^4x\partial_\mu\left(\frac{\partial}{\partial(\partial_\mu\phi)}\mathcal{L}(\delta\phi - \partial_\nu\phi\delta x^\nu) + \mathcal{L}\delta x^\mu\right) \\
&= \int d^4x\partial_\mu\left(i\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}T^i\phi\delta\epsilon^i + \left(\mathcal{L}g^{\mu\nu} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi\right)\delta x^\nu\right) \\
&= \int d^4x\partial^\mu j_\mu\delta\alpha
\end{aligned}
$$

with current $j_\mu = \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\delta\phi - \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\partial_\nu\phi\delta x^\nu + \mathcal{L}\delta x_\mu$.

Here, we can obviously see independent variations, $\delta\epsilon^i$ and δx^ν.

- charge:

$$
\begin{aligned}
0 &= \partial_\mu j^\mu \\
&= \partial_t j^0 - \nabla j^i \\
Q &\equiv \int d^3x j^0 \\
\partial_t Q &= \partial_t \int d^3x j^0 = \int d^3x \nabla j^i = \vec{j}
\end{aligned}
$$

some example on spacetime transformation

- time translation:
  - transformation

$$
\begin{aligned}
x^0 &\to x^0 + \delta t \\
\delta x^0 &\to \delta t \\
\delta\phi &\to 0
\end{aligned}
$$

  - conservation current and charge:

$$j^\mu = \mathcal{L} g^{\mu 0} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_0 \phi$$

$$\int d^3x \Theta^{00} = \int d^3x (\mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\dot{\phi})} \dot{\phi}) = -\int d^3x \mathcal{H} = -H$$

- position translation:
  - transformation

$$
\begin{aligned}
x^i &\to x^i + \delta x^i \\
\delta x^\mu &\to (0, \delta x^i) \\
\delta\phi &\to 0
\end{aligned}
$$

  - conservation current and charge:

$$j^\mu = \mathcal{L} g^{\mu i} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \nabla \phi$$

$$\int d^3x \Theta^{0i} = \int d^3x \frac{\partial \mathcal{L}}{\partial(\dot{\phi})} \nabla \phi = \int d^3x p \nabla \phi$$

- spacetime translation in 4-dim
- spacetime rotation

Noether' Theorem and conservation laws

- if the action is invariant under a continuous transformation, there is a conservation.
- Proof:
  - transformation:

$$
\begin{aligned}
\phi &\rightarrow \phi' = e^{iT^\alpha \epsilon^\alpha}\phi \\
\phi' &= (1 + iT^\alpha \epsilon^\alpha)\phi \\
\delta\phi_i &= i(T^\alpha)_{ij}\epsilon^\alpha \phi_j \\
\delta\partial_\mu\phi_i &= i(T^\alpha)_{ij}\partial_\mu(\epsilon^\alpha\phi_j) \\
&= i(T^\alpha)_{ij}[(\partial_\mu\epsilon^\alpha)\phi_j + \epsilon^\alpha\partial_\mu(\phi_j)]
\end{aligned}
$$

    the first term stand for the case of ϵ depending on spacetime, i.e. $\epsilon(x)$.

  - change of the action:

$$
\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\delta(\partial_\mu\phi^i)
$$

  - in a simple case: $\epsilon(x) = \epsilon$

$$
\begin{aligned}
\delta\mathcal{L} &= i\left[\frac{\partial\mathcal{L}}{\partial\phi_i}(T^\alpha)_{ij}\phi_j + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}(T^\alpha)_{ij}\partial_\mu(\phi_j)\right]\epsilon^\alpha \\
&= i\left[\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}(T^\alpha)_{ij}\phi_j + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}(T^\alpha)_{ij}\partial_\mu(\phi_j)\right]\epsilon^\alpha \\
&= i\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}(T^\alpha)_{ij}\phi_j\right]\epsilon^\alpha \\
&= i\partial_\mu j^{\mu,\alpha}\epsilon^\alpha
\end{aligned}
$$

- the number of conversation charges = the number of generator elements of group
  - spacetime translation: 4

- space rotation: 3
- spacetime rotation: ?thinking?

1.5 Lecture 6: Lorentz Symmetry

Lorentz group definition

- Einstein's SRT:

$$g_{\mu\nu}dx^\mu dx^\nu = g_{\mu\nu}dx'^\mu dx'^\nu, \tag{1.7}$$

- A general coordinate transformation $x^\mu \to x'^\mu$

$$x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu \tag{1.8}$$

  - Lorentz transformation
  - Transformation operator $T(\Lambda, a)$ induced on physical state corresponding to above transformation.
  - $T(\Lambda, a)$ form a group:

$$T(\Lambda, a)T(1, 0) = T(1, 0)T(\Lambda, a) = T(\Lambda, a), \tag{1.9}$$

$$T(\bar{\Lambda}, \bar{a})T(\lambda, a) = T(\bar{\Lambda}\Lambda, \bar{\Lambda}a + \bar{a}), exce. \tag{1.10}$$

$$T(\Lambda, a)T(\Lambda^{-1}, -\Lambda^{-1}a) = T(1, 0). \tag{1.11}$$

  This whole group is called inhomogeneous Lorentz group or Poincare group.

properties and subgroup

- transformation in homogeneous Lorentz group

$$g_{\mu\nu}\Lambda^\mu_\rho\Lambda^\nu_\lambda = g_{\rho\lambda}. \tag{1.12}$$

- it gives

$$(\mathrm{Det}\Lambda)^2 = 1, \tag{1.13}$$

$$(\Lambda^0_0)^2 = 1 + \Lambda^i_0\Lambda^i_0 = 1 + \Lambda^0_i\Lambda^0_i \Rightarrow |\Lambda^0_0| \geq 1. \tag{1.14}$$

- $\mathrm{Det}\Lambda$ and Λ^0_0 can be used to classify the group

- important subgroup of Poincare group:
  - homogeneous Lorentz group $T(\Lambda, 0)$ form homogeneous Lorentz group.
  - $\text{Det}\Lambda = +1$ subgroup $\text{Det}\Lambda = +1$ form a subgroup.
  - $\Lambda^0_0 \geq 1$ subgroup $\Lambda^0_0 \geq 1$ form a subgroup,

$$\begin{aligned} (\bar{\Lambda}\Lambda)^0_0 &= \bar{\Lambda}^0_0\Lambda^0_0 + \bar{\Lambda}^0_1\Lambda^1_0 + \bar{\Lambda}^0_2\Lambda^2_0 + \bar{\Lambda}^0_3\Lambda^3_0 \qquad (1.15) \\ &= \bar{\Lambda}^0_0\Lambda^0_0 + (\bar{\Lambda}^1_0, \bar{\Lambda}^2_0, \bar{\Lambda}^3_0)(\Lambda^1_0, \Lambda^2_0, \Lambda^3_0) \qquad (1.16) \\ &\geq \bar{\Lambda}^0_0\Lambda^0_0 - \sqrt{(\bar{\Lambda}^0_0)^2 - 1}\sqrt{(\Lambda^0_0)^2 - 1} \geq 1 \qquad (1.17) \end{aligned}$$

  - The subgroup of Lorentz transformation with $\text{Det}\Lambda = +1$ and $\Lambda^0_0 \geq 1$ is know as the proper orthochronous Lorentz group.
  - Any Lorentz group transformation can be written as the produce of an element of the proper orthochronous Lorentz group with one of the discrete transformations P or T or PT.

The Poincare Algebra

- unity element: $\Lambda^\mu_\nu = \delta^\mu_\nu$ and $a_\mu = 0$.
- the infinitesimal transformation:

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu, \qquad a_\mu = \epsilon_\mu. \qquad (1.18)$$

- from formula $g_{\mu\nu}\Lambda^\mu_\rho\Lambda^\nu_\lambda = g_{\rho\lambda}$, ω must satisfy

$$\omega_{\mu\nu} = -\omega_{\nu\mu}$$

- closing to identity $U(1,0)$, $U(1+\omega,\epsilon)$ must be equal to

$$U(1+\omega,\epsilon) = 1 + \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} - i\epsilon_\mu P^\mu + \cdot$$

- $J^{\mu\nu}$ and P_μ are Hermitian operators Why? , $J^{\mu\nu\dagger} = J^{\mu\nu}$, $P^\dagger_\mu = P^\mu$.
- $J^{\mu\nu}$ also antisymmetic

$$J^{\mu\nu} = -J^{\nu\mu}$$

- Under Lorentz transformation

$$U(\Lambda,a)U(1+\omega,\epsilon)U^{-1}(\Lambda,a) = U(\Lambda(1+\omega)\Lambda^{-1}, \Lambda\epsilon - \Lambda\omega\Lambda^{-1}a) \tag{1.19}$$

$$\Rightarrow \quad U(\Lambda,a)(\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} - \epsilon_\mu P^\mu)U^{-1}(\Lambda,a) = (\Lambda(1+\omega)\Lambda^{-1})_{\mu\nu}J^{\mu\nu} - (\Lambda\epsilon - \Lambda\omega\Lambda^{-1}a)_\mu P^\mu \tag{1.20}$$

$$\Rightarrow \quad U(\lambda,a)J^{\rho\lambda}U^{-1}(\Lambda,a) = \Lambda^\rho_\mu \Lambda^\lambda_\nu (J^{\mu\nu} - a^\mu P^\nu + a^\nu P^\mu) \tag{1.21}$$

$$\& \quad U(\lambda,a)P^\mu U^{-1}(\Lambda,a) = \Lambda^\mu_\nu P^\nu \tag{1.22}$$

- if $U(\Lambda,a)$ also be infinitesimal transformation, i.e. $\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu$ and $a_\mu = \epsilon_\mu$, above formula become

$$i[\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} - \epsilon_\mu P^\mu, J^{\rho\lambda}] = \omega^\rho_\mu J^{\mu\lambda} + \omega^\lambda_\nu J^{\rho\nu} - \epsilon^\rho P^\lambda + \epsilon^\lambda P^\rho \tag{1.23}$$

$$i[\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} - \epsilon_\mu P^\mu, P^\lambda] = \omega^\lambda_\mu P^\mu \tag{1.24}$$

and we find the commutation rules

$$i[J^{\mu\nu}, J^{\rho\lambda}] = \eta^{\nu\rho}J^{\mu\lambda} - \eta^{\mu\rho}J^{\nu\lambda} - \eta^{\lambda\mu}J^{\rho\nu} + \eta^{\lambda\nu}J^{\lambda\mu} \tag{1.25}$$

$$i[P^\mu, J^{\rho\lambda}] = \eta^{\mu\rho}P^\lambda - \eta^{\mu\lambda}P^\rho \tag{1.26}$$

$$[P^\mu, P^\nu] = 0 \tag{1.27}$$

This is the Lie algebra of the Poincare group.

- In physics, the conserved operator play a special role. Definition of special operators:

  - energy operator $H = P^0$
  - momentum operator $\vec{P} = (P^1, P^2, P^3)$
  - angular-momentum operator $\vec{J} = (J^{23}, J^{31}, J^{12})$
  - Lorentz 'boost' vector operator $\vec{K} = (J^{01}, J^{02}, J^{03})$

- These operators have following communication relation:

$$
\begin{aligned}
[J_i, J_j] &= i\epsilon_{ijk} J_k, && (1.28)\\
[J_i, K_j] &= i\epsilon_{ijk} K_k, && (1.29)\\
[K_i, K_j] &= -i\epsilon_{ijk} J_k, && (1.30)\\
[J_i, P_j] &= i\epsilon_{ijk} P_k, && (1.31)\\
[K_i, P_j] &= -iH\delta_{ij}, && (1.32)\\
[J_i, H] &= [P_i, H] = [H, H] = 0, && (1.33)\\
[K_i, H] &= -iP_i. && (1.34)
\end{aligned}
$$

Lorentz group: $SU(2) \otimes SU(2)$

- Lorentz boost does not form a group: $[K_i, K_j] = -i\epsilon_{ijk} J_k$
- define a new linear operator

$$N_i \equiv \frac{1}{2}(J_i + iK_i)$$

- commutation relations

$$[N_i, N_j^\dagger] = 0, \quad [N_i, N_j] = i\epsilon_{ijk} N_k, \quad [N_i^\dagger, N_j^\dagger] = i\epsilon_{ijk} N_k^\dagger$$

- N_i: $SU(2)$; $N_j^\dagger$: another $SU(2)$
- Lorentz group: $SU(2) \otimes SU(2)$
- review of angular momentum ($SU(2)$):

$$J^2 \mid lm\rangle = ..., J_z \mid lm\rangle$$

  l, m denote eigenvalues of operators J^2 and J_z

- similarly,
  - $N_i N_i$ eigenvalues $n(n+1)$
  - $N_j^\dagger N_j^\dagger$ eigenvalues $m(m+1)$

- Lorentz group representation (m, n)
  - $(0, 0)$
  - $(1/2, 0)$
  - $(0, 1/2)$
  - $(1/2, 0) \otimes (0, 1/2) = (1/2, 1/2)$
  - $(1/2, 0) \otimes (1/2, 0) = (1, 0) \oplus (0, 0)$

particle classification

- review of Noether theorem: the number of conservation quantities
- 6 rotation $+$ 4 translation $=$ 10 conservation quantities
- Lorentz boost is not a conservation quantities: Why? ($[K_i, H] = -iP_i$)
- 1 energy H + 3 momentum $\vec{P}$ + 3 angular momentum $\vec{J}$ = 7, where are other 3?
- Pauli-Lubanski four-vector

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma} = -\frac{i}{2}\epsilon^{\mu\nu\rho\sigma} S_{\rho\sigma}\partial_\nu$$

- Casimir operator: intro.
- Casimir operator of Lorentz group: $P_\mu P^\mu$ (with eigenvalue m^2) and $W_\mu W^\mu$ with eigenvalue $-m^2 s(s+1)$
- physical states (particle) are classified in terms of Lorentz group
  - $P^2 = m^2 > 0$: $s = 0, 1/2, 1, \cdots$
  - $P^2 = m^2 = 0$: $(P \cdot W = 0)$ $s = 0, 1/2, 1, \cdots$
  - $P^2 = m^2 = 0$: spin is continuous (an infinite number of polarization states). Not found in nature
  - $P^2 = m^2 < 0$: tachyon

1.5.1 elementary particles and their interactions

- atom ⟶nuclear ⟶quarks
- elementary particles introduction
- four kinds of interactions
- electromagnetic interaction
  - between electronic charged particle, like $e^{\pm}, \mu^{\pm}, \tau^{\pm}, u, d, c, s, t, b$
  - coupling: e
  - mediating by massless photon
- strong interaction
  - between quarks and gluons which have color charges
  - mediating by gluons
  - massless gluons with color charges can interact with themselves. (but photon don't interact with itself)
- weak interaction
  - between all particle of SM
  - mediating massive $W^{\pm}$ and Z
- all matter is constructed by fermions, leptons and quarks. interaction is mediated by vector gauge bosons, $\gamma, g, W^{\pm}, Z$

1.5.2 particles and their fields

- photon: a massless vector gauge boson A_μ
- electron/positron: fermion field ψ in EM interaction; weak isospin (fermion field): $(\nu_e, e^-)^T$
- proton/neutron: isospin doublet $(p, n)^T$

- pi meson: pesudo-scalar mesons, $\pi^{\pm}$ form a isospin doublet
- $\cdots$