西安交通大学电子与信息工程学院研究生课程

《等离子体电子学》

第五章 玻尔兹曼方程和带电粒子输运方程(1)

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玻尔兹曼方程和带电粒子输运方程

- 玻尔兹曼方程

  - 相空间输运与玻尔兹曼方程推导

$$dn = g(v, r, t)\, dv\, dr.$$

↓

$$n = n(r, t).$$

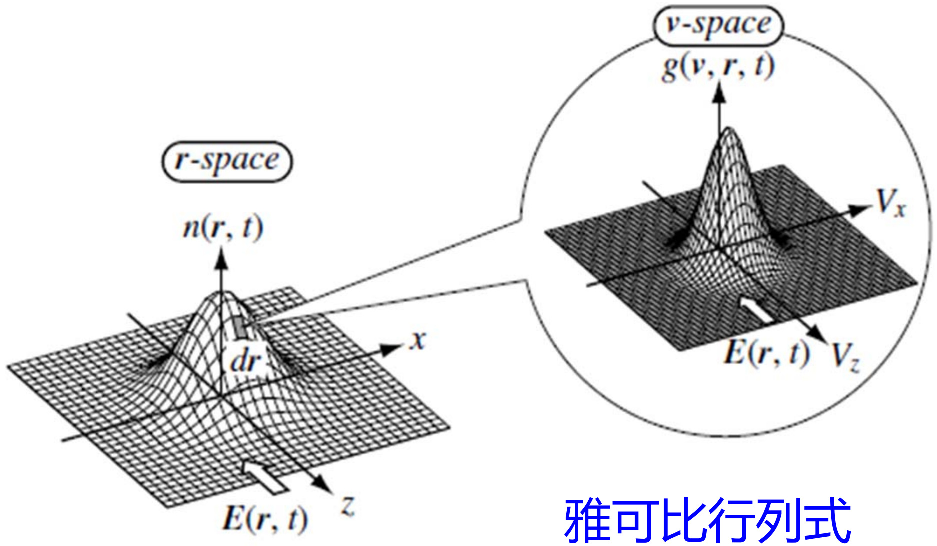
$$dn' = g(v', r', t + dt) dv' dr'.$$

↓

$$dv' dr' = \frac{\partial(r', v')}{\partial(r, v)}\, dv\, dr,$$

↓

$$dn' - dn = [g(v + \alpha dt, r + v dt, t + dt) - g(v, r, t)]\, dv\, dr$$
$$= \left[\frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial r} + \alpha \cdot \frac{\partial}{\partial v}\right] g(v, r, t)\, dv\, dr\, dt,$$



雅可比行列式

$$\frac{\partial(r', v')}{\partial(r, v)} = \begin{vmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \cdots & \frac{\partial x'}{\partial v_y} & \frac{\partial x'}{\partial v_z} \\ \frac{\partial y'}{\partial x} & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \frac{\partial v'_y}{\partial x} & \cdot & & \cdot & \cdot \\ \frac{\partial v'_z}{\partial x} & \cdot & & \cdot & \frac{\partial v'_z}{\partial v_z} \end{vmatrix} = 1.$$

玻尔兹曼方程和带电粒子输运方程

- 玻尔兹曼方程(续)

  - 相空间输运与玻尔兹曼方程推导(续)

$$
\begin{aligned}
dn' - dn &= [g(v + \alpha dt, r + vdt, t + dt) - g(v, r, t)]\, dv\, dr \\
&= \left[\frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial r} + \alpha \cdot \frac{\partial}{\partial v}\right] g(v, r, t)\, dv\, dr\, dt,
\end{aligned}
$$

关于v和r做泰勒级数展开，并保留一阶项

$$dn' - dn = J(g, F) dv\, dr\, dt.$$

$$\frac{\partial}{\partial t} g(v, r, t) + v \cdot \frac{\partial}{\partial r} g(v, r, t) + \alpha \cdot \frac{\partial}{\partial v} g(v, r, t) = J(g, F).$$

玻尔兹曼方程/Fokker-Planck方程

扩散　外场作用　碰撞项

$$F = m\alpha(r, t) = eE(r, t) + ev \times B(r, t),$$

$$\frac{\partial}{\partial t} g(v, r, t) + v \cdot \frac{\partial}{\partial r} g(v, r, t) + \alpha \cdot \frac{\partial}{\partial v} g(v, r, t) = 0.$$

无碰撞玻尔兹曼方程/弗拉索夫方程

玻尔兹曼方程和带电粒子输运方程

- 输运系数

□ 宏观量

■ 平均速度

$$n(\mathbf{r}, t) = \int g(\mathbf{v}, \mathbf{r}, t) d\mathbf{v}.$$

平均量

$$\langle \mathbf{A}(\mathbf{r}, t) \rangle = \frac{\int \mathbf{A}(\mathbf{v}, \mathbf{r}, t) g(\mathbf{v}, \mathbf{r}, t) d\mathbf{v}}{\int g(\mathbf{v}, \mathbf{r}, t) d\mathbf{v}}.$$

$$\langle \mathbf{v}(\mathbf{r}, t) \rangle = \frac{1}{n} \int \mathbf{v} g(\mathbf{v}, t) d\mathbf{v},$$

$$\mathbf{v}(\mathbf{r}, t) = \mathbf{v}_d(\mathbf{r}, t) + \mathbf{v}_r(\mathbf{v}, \mathbf{r}, t),$$

定向运动运动　随机热运动

$$\langle \mathbf{v}_r \rangle = 0.$$

■ 平均动能

$$\langle \varepsilon(\mathbf{r}, t) \rangle = \frac{1}{n} \int \frac{1}{2} m v^2 g(\mathbf{v}, \mathbf{r}, t) d\mathbf{v} = \frac{1}{2} m \left(v_d^2 + \langle v_r^2 \rangle \right).$$

玻尔兹曼方程和带电粒子输运方程

- 输运系数（续）

  - 宏观量（续）

    - 流体速度

$$g(v, r, t) = g^0(v, t)n(r, t) + \mathbf{g}^1(v, t) \cdot \nabla_r n(r, t) + \mathbf{g}^2(v, t) \odot \nabla_r^2 n(r, t) + \cdots .$$

$$\int \mathbf{g}^k(v, t)dv = \begin{cases} 1; & k = 0 \\ 0; & k \neq 0 \end{cases} .$$

$$\langle v(r, t)\rangle = \frac{1}{n}\int v g^0(v, t)dv + \frac{1}{n}\int v\mathbf{g}^1(v, t)dv \cdot \nabla_r n(r, t).$$ 流体速度

    - 迁移速度 $v_d(t) = \frac{1}{n}\int v g^0(v, t)dv$

    - 扩散速度 $\mathbf{D}(t) = \frac{1}{n}\int v\mathbf{g}^1(v, t)dv.$

$$D_T(t) = \frac{1}{n}\int v_x g_x^1(v, t)dv = \frac{1}{n}\int v_y g_y^1(v, t)dv,$$ 横向扩散系数

$$D_L(t) = \frac{1}{n}\int v_z g_z^1(v, t)dv.$$ 纵向扩散系数

玻尔兹曼方程和带电粒子输运方程

- 输运系数（续）

  - 宏观量（续）

    - 压力张量

$$
\begin{aligned}
\mathbf{P}_T(\boldsymbol{r}, t) &= m \int \boldsymbol{v}\boldsymbol{v} g(\boldsymbol{v}, \boldsymbol{r}, t) d\boldsymbol{v}, \\
&= m \int \boldsymbol{v}_r \boldsymbol{v}_r g(\boldsymbol{v}, \boldsymbol{r}, t) d\boldsymbol{v} + mn\boldsymbol{v}_d \boldsymbol{v}_d, \\
&= mn(\boldsymbol{r}, t)\langle \boldsymbol{v}_r \boldsymbol{v}_r \rangle + mn(\boldsymbol{r}, t)\boldsymbol{v}_d \boldsymbol{v}_d.
\end{aligned}
$$

$$
\mathbf{P} = \begin{vmatrix} P_x & 0 & 0 \\ 0 & P_y & 0 \\ 0 & 0 & P_z \end{vmatrix}, \quad p = p_x = p_y = p_z = \frac{1}{3} mn \langle v_r^2 \rangle.
$$

随机运动速度一般远大于定向运动速度，则近似各向同性

玻尔兹曼方程和带电粒子输运方程

- 输运系数（续）

  - 宏观量（续）

    - 能流矢量

$$Q(r,t)=\int \frac{1}{2}mv^2 v g(v,r,t)dv.$$

$$\begin{aligned}\langle v^2 v\rangle &= \langle[(v_d+v_r)\cdot(v_d+v_r)](v_d+v_r)\rangle \\ &= v_d^2 v_d + v_d^2\langle v_r\rangle + 2v_d v_d\cdot\langle v_r\rangle + 2v_d\cdot\langle v_r v_r\rangle + \langle v_r^2\rangle v_d + \langle v_r^2 v_r\rangle \\ &= v_d^2 v_d + 2v_d\cdot\langle v_r v_r\rangle + \langle v_r^2\rangle v_d + \langle v_r^2 v_r\rangle.\end{aligned}$$

$$\begin{aligned}Q(r,t) &= \frac{1}{2}mn\langle v^2 v\rangle \\ &= \frac{1}{2}mn\left(v_d^2+\langle v_r^2\rangle\right)v_d + mn\langle v_r v_r\rangle\cdot v_d + \frac{1}{2}mn\langle v_r^2 v_r\rangle \\ &= n(r,t)\langle\varepsilon(r,t)\rangle v_d + \mathbf{P}\cdot v_d + \frac{1}{2}mn(r,t)\langle v_r^2 v_r\rangle.\end{aligned}$$

$$q(r,t)=\frac{1}{2}mn(r,t)\langle v_r^2 v_r\rangle.$$

定向运动造成的能量输运　　外场做功　　随机运动产生的能量输运（热流）

玻尔兹曼方程和带电粒子输运方程

● 输运系数（续）

☐ 宏观量（续）

■ 扩散系数：可根据粒子的运动轨迹统计获得，与前面的计算公式物理意义一致

$$D_L(t) = \frac{1}{2}\frac{d}{dt}\left\langle (z(t) - \langle z(t)\rangle)^2 \right\rangle$$
$$= \langle z(t)v_z(t)\rangle - \langle z(t)\rangle\langle v_z(t)\rangle.$$
$$D_T(t) = \frac{1}{4}\frac{d}{dt}\langle x(t)^2 + y(t)^2\rangle$$
$$= \frac{1}{2}(\langle x(t)v_x(t)\rangle + \langle y(t)v_y(t)\rangle).$$

即空间二阶矩

$$\mathbf{M}_\mathbf{k}(t) = <(\boldsymbol{r}(t) - <\boldsymbol{r}(t)>)^\mathbf{k}>. \quad (\mathbf{k} = \mathbf{3}, \mathbf{4}..)$$

$$\mathbf{D}_3 = \frac{1}{3!}\frac{d}{dt}\mathbf{M}_3(t),$$

$$\mathbf{D}_4 = \frac{1}{4!}\frac{d}{dt}\left(\mathbf{M}_4(t) - \mathbf{3}(\mathbf{M}_2(t))^2\right).$$

空间高阶矩与3、4阶输运系数
空间三阶矩：斜度
空间四阶矩：峰度

玻尔兹曼方程和带电粒子输运方程

- 输运方程

□ 任意函数作用于玻尔兹曼方程

A(v,r,t): 任意函数

$$\frac{\partial}{\partial t}g(v,r,t)+v\cdot\frac{\partial}{\partial r}g(v,r,t)+\alpha\cdot\frac{\partial}{\partial v}g(v,r,t)=J(g,F).$$

$$\int \mathbf{A}(v,r,t)\frac{\partial g(v,r,t)}{\partial t}dv+\int \mathbf{A}(v,r,t)v\cdot\frac{\partial g(v,r,t)}{\partial r}dv$$
$$+\int \mathbf{A}(v,r,t)\alpha\cdot\frac{\partial g(v,r,t)}{\partial v}dv=\int \mathbf{A}(v,r,t)J(g,F)dv.$$

$$\int \mathbf{A}\frac{\partial g}{\partial t}dv=\frac{\partial}{\partial t}\int \mathbf{A}g dv-\int\frac{\partial \mathbf{A}}{\partial t}g dv$$
$$=\frac{\partial}{\partial t}(n(r,t)\langle\mathbf{A}\rangle)-n(r,t)\left\langle\frac{\partial \mathbf{A}}{\partial t}\right\rangle,$$

$$\int \mathbf{A}\alpha\cdot\frac{\partial g}{\partial v}dv=\frac{\partial}{\partial v}\cdot\int\alpha\mathbf{A}g dv-\int\left(\frac{\partial}{\partial v}\cdot\alpha\mathbf{A}\right)g dv$$
$$=\frac{\partial}{\partial v}\cdot(n(r,t)\langle\alpha\mathbf{A}\rangle)-n(r,t)\left\langle\frac{\partial}{\partial v}\cdot\alpha\mathbf{A}\right\rangle$$
$$=-n(r,t)\left\langle\alpha\cdot\frac{\partial}{\partial v}\mathbf{A}\right\rangle.$$

$$\int \mathbf{A}v\cdot\frac{\partial g}{\partial r}dv=\int\frac{\partial}{\partial r}\cdot(v\mathbf{A}g)dv-\int\left(\frac{\partial}{\partial r}\cdot v\mathbf{A}\right)g dv,$$
$$=\frac{\partial}{\partial r}\cdot(n(r,t)\langle v\mathbf{A}\rangle)-n(r,t)\left\langle\frac{\partial}{\partial r}\cdot v\mathbf{A}\right\rangle.$$

玻尔兹曼方程和带电粒子输运方程

- 输运方程（续）

□ 数密度守恒方程

$$\frac{\partial}{\partial t}(n(\boldsymbol{r},t)\langle \mathbf{A}(v,\boldsymbol{r},t)\rangle) - n(\boldsymbol{r},t)\left\langle \frac{\partial \mathbf{A}(v,\boldsymbol{r},t)}{\partial t}\right\rangle + \frac{\partial}{\partial \boldsymbol{r}}\cdot(n(\boldsymbol{r},t)\langle v\mathbf{A}(v,\boldsymbol{r},t)\rangle)$$
$$- n(\boldsymbol{r},t)\left\langle \frac{\partial}{\partial \boldsymbol{r}}\cdot v\mathbf{A}(v,\boldsymbol{r},t)\right\rangle - n(\boldsymbol{r},t)\left\langle \alpha\cdot\frac{\partial \mathbf{A}(v,\boldsymbol{r},t)}{\partial v}\right\rangle$$
$$= \int \mathbf{A}(v,\boldsymbol{r},t)J(g,F)dv. \tag{5.34}$$

$\mathbf{A} = 1$

$$\frac{\partial}{\partial t}n(\boldsymbol{r},t) + \frac{\partial}{\partial \boldsymbol{r}}\cdot(n(\boldsymbol{r},t)\langle v\rangle) = n_e(\boldsymbol{r},t)R_0(\boldsymbol{r},t).$$

$$R_0(\boldsymbol{r},t) = R_i(\boldsymbol{r},t) - R_a(\boldsymbol{r},t).$$

R_0: 粒子生成速率

玻尔兹曼方程和带电粒子输运方程

● 输运方程（续）

□ 动量守恒方程

v, r, t相互独立无关

$$\frac{\partial}{\partial t}(n(\mathbf{r},t)\langle\mathbf{A}(\mathbf{v},\mathbf{r},t)\rangle) - n(\mathbf{r},t)\left\langle\frac{\partial\mathbf{A}(\mathbf{v},\mathbf{r},t)}{\partial t}\right\rangle + \frac{\partial}{\partial\mathbf{r}}\cdot(n(\mathbf{r},t)\langle\mathbf{v}\mathbf{A}(\mathbf{v},\mathbf{r},t)\rangle) - n(\mathbf{r},t)\left\langle\frac{\partial}{\partial\mathbf{r}}\cdot\mathbf{v}\mathbf{A}(\mathbf{v},\mathbf{r},t)\right\rangle - n(\mathbf{r},t)\left\langle\boldsymbol{\alpha}\cdot\frac{\partial\mathbf{A}(\mathbf{v},\mathbf{r},t)}{\partial\mathbf{v}}\right\rangle = \int\mathbf{A}(\mathbf{v},\mathbf{r},t)J(g,F)d\mathbf{v}. \quad (5.34)$$

$\mathbf{A} = m\mathbf{v}$

$$\frac{\partial}{\partial t}(mn(\mathbf{r},t)\langle\mathbf{v}\rangle) + \frac{\partial}{\partial\mathbf{r}}\cdot(mn(\mathbf{r},t)\langle\mathbf{v}\mathbf{v}\rangle) - mn(\mathbf{r},t)\left\langle\boldsymbol{\alpha}\cdot\frac{\partial}{\partial\mathbf{v}}\mathbf{v}\right\rangle = m\int\mathbf{v}J\,d\mathbf{v}.$$

$\langle\mathbf{v}\rangle = \mathbf{v}_d$　　代入数密度守恒方程

$$mn(\mathbf{r},t)\frac{\partial}{\partial t}\mathbf{v}_d + m\mathbf{v}_d\frac{\partial}{\partial t}n(\mathbf{r},t) = mn(\mathbf{r},t)\frac{\partial}{\partial t}\mathbf{v}_d - m\mathbf{v}_d\frac{\partial}{\partial\mathbf{r}}\cdot(n(\mathbf{r},t)\mathbf{v}_d) + mn_e(\mathbf{r},t)\mathbf{v}_d R_0,$$

玻尔兹曼方程和带电粒子输运方程

- 输运方程（续）

  - 动量守恒方程（续）

$$mn(\boldsymbol{r},t)\frac{\partial}{\partial t}\boldsymbol{v}_d + m\boldsymbol{v}_d\frac{\partial}{\partial t}n(\boldsymbol{r},t) = mn(\boldsymbol{r},t)\frac{\partial}{\partial t}\boldsymbol{v}_d - m\boldsymbol{v}_d\frac{\partial}{\partial \boldsymbol{r}}\cdot(n(\boldsymbol{r},t)\boldsymbol{v}_d) + mn_e(\boldsymbol{r},t)\boldsymbol{v}_d R_0,$$

$$\langle \boldsymbol{v}\rangle = \boldsymbol{v}_d$$

$$\frac{\partial}{\partial t}(mn(\boldsymbol{r},t)\langle \boldsymbol{v}\rangle) + \frac{\partial}{\partial \boldsymbol{r}}\cdot(mn(\boldsymbol{r},t)\langle \boldsymbol{v}\boldsymbol{v}\rangle) - mn(\boldsymbol{r},t)\left\langle \boldsymbol{\alpha}\cdot\frac{\partial}{\partial \boldsymbol{v}}\boldsymbol{v}\right\rangle = m\int \boldsymbol{v}J\,d\boldsymbol{v}.$$

$$\frac{\partial}{\partial \boldsymbol{r}}\cdot(mn(\boldsymbol{r},t)\boldsymbol{v}_d\boldsymbol{v}_d) + \frac{\partial}{\partial \boldsymbol{r}}\cdot(mn(\boldsymbol{r},t)\langle \boldsymbol{v}_r\boldsymbol{v}_r\rangle) = \boldsymbol{v}_d\frac{\partial}{\partial \boldsymbol{r}}\cdot(mn(\boldsymbol{r},t)\boldsymbol{v}_d) + mn(\boldsymbol{r},t)\left(\boldsymbol{v}_d\cdot\frac{\partial}{\partial \boldsymbol{r}}\right)\boldsymbol{v}_d + \frac{\partial}{\partial \boldsymbol{r}}\cdot(mn(\boldsymbol{r},t)\langle \boldsymbol{v}_r\boldsymbol{v}_r\rangle),$$

$$-mn(\boldsymbol{r},t)\left\langle \boldsymbol{\alpha}\cdot\frac{\partial}{\partial \boldsymbol{v}}\boldsymbol{v}\right\rangle = -n(\boldsymbol{r},t)e(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}).$$

$$mn(\boldsymbol{r},t)\frac{\partial}{\partial t}\boldsymbol{v}_d + mn(\boldsymbol{r},t)\boldsymbol{v}_d\left(\frac{\partial}{\partial \boldsymbol{r}}\cdot\boldsymbol{v}_d\right) + m\frac{\partial}{\partial \boldsymbol{r}}\cdot(n(\boldsymbol{r},t)\langle \boldsymbol{v}_r\boldsymbol{v}_r\rangle) = n(\boldsymbol{r},t)e(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}) - mn_e(\boldsymbol{r},t)\boldsymbol{v}_d R_0 + m\langle \boldsymbol{v}J\rangle.$$

玻尔兹曼方程和带电粒子输运方程

- 输运方程（续）

  - 动量守恒方程（续）

$$mn(\mathbf{r},t)\frac{\partial}{\partial t}\mathbf{v}_d + mn(\mathbf{r},t)\mathbf{v}_d\left(\frac{\partial}{\partial \mathbf{r}}\cdot\mathbf{v}_d\right) + m\frac{\partial}{\partial \mathbf{r}}\cdot\left(n(\mathbf{r},t)\langle\mathbf{v}_r\mathbf{v}_r\rangle\right)$$
$$= n(\mathbf{r},t)e(\mathbf{E}+\mathbf{v}\times\mathbf{B}) - mn_e(\mathbf{r},t)\mathbf{v}_d R_0 + m\langle\mathbf{v}J\rangle.$$

$$m\langle\mathbf{v}J\rangle = -mn(\mathbf{r},t)\mathbf{v}_d R_m.$$

R_m: 动量转移碰撞速率

若E为常数，B=0，并假设定向运动速度v_d为不随时间和空间变化

$$R_m \gg R_0$$

$$mn\mathbf{v}_d R_m = en\mathbf{E} - m\frac{\partial}{\partial \mathbf{r}}\cdot\left(n\langle\mathbf{v}_r\mathbf{v}_r\rangle\right).$$

玻尔兹曼方程和带电粒子输运方程

- 输运方程（续）

□ 能量守恒方程

$$\frac{\partial}{\partial t}(n(r,t)\langle \mathbf{A}(v,r,t)\rangle) - n(r,t)\left\langle \frac{\partial \mathbf{A}(v,r,t)}{\partial t}\right\rangle + \frac{\partial}{\partial r}\cdot(n(r,t)\langle v\mathbf{A}(v,r,t)\rangle)$$
$$- n(r,t)\left\langle \frac{\partial}{\partial r}\cdot v\mathbf{A}(v,r,t)\right\rangle - n(r,t)\left\langle \alpha\cdot\frac{\partial \mathbf{A}(v,r,t)}{\partial v}\right\rangle$$
$$= \int \mathbf{A}(v,r,t) J(g,F)dv. \qquad (5.34)$$

replace **A** with $mv^2/2$

replace **A** with $mv^2/2$

$$-n(r,t)\left\langle \frac{1}{m}[eE(r,t) + ev\times B(r,t)]\cdot\frac{\partial}{\partial v}\frac{mv^2}{2}\right\rangle = -n(r,t)\langle eE\cdot v\rangle,$$

磁场不对带电粒子做功

$$\frac{\partial}{\partial t}\left(\frac{mn(r,t)}{2}\langle v^2\rangle\right) + \frac{\partial}{\partial r}\cdot\left(\frac{mn(r,t)}{2}\langle v^2 v\rangle\right) - n(r,t)eE\cdot v_d$$
$$= \int \frac{mv^2}{2} J\, dv.$$

玻尔兹曼方程和带电粒子输运方程

- 输运方程（续）

  - 能量守恒方程（续）

$$\frac{\partial}{\partial t}\left(\frac{mn(\boldsymbol{r},t)}{2}\langle v^2\rangle\right)+\frac{\partial}{\partial \boldsymbol{r}}\cdot\left(\frac{mn(\boldsymbol{r},t)}{2}\langle v^2\boldsymbol{v}\rangle\right)-n(\boldsymbol{r},t)e\mathbf{E}\cdot\boldsymbol{v}_d$$
$$=\int\frac{mv^2}{2}J\,d\boldsymbol{v}.$$

$$\begin{aligned}Q(r,t)&=\frac{1}{2}mn\langle v^2\boldsymbol{v}\rangle\\&=\frac{1}{2}mn\left(v_d^2+\langle v_r^2\rangle\right)\boldsymbol{v}_d+mn\langle\boldsymbol{v}_r\boldsymbol{v}_r\rangle\cdot\boldsymbol{v}_d+\frac{1}{2}mn\left\langle v_r^2\boldsymbol{v}_r\right\rangle\\&=n(r,t)\langle\varepsilon(r,t)\rangle\boldsymbol{v}_d+\mathbf{P}\cdot\boldsymbol{v}_d+\frac{1}{2}mn(r,t)\left\langle v_r^2\boldsymbol{v}_r\right\rangle.\end{aligned}$$

代到第二项

$$\frac{\partial}{\partial t}(n(\boldsymbol{r},t)\langle\varepsilon(\boldsymbol{r},t)\rangle)+\frac{\partial}{\partial \boldsymbol{r}}\cdot(n(\boldsymbol{r},t)\langle\varepsilon(\boldsymbol{r},t)\rangle\boldsymbol{v}_d)+\frac{\partial}{\partial \boldsymbol{r}}\cdot(\mathbf{P}\cdot\boldsymbol{v}_d)$$
$$+\frac{\partial}{\partial \boldsymbol{r}}\cdot\boldsymbol{q}-n(\boldsymbol{r},t)e\mathbf{E}\cdot\boldsymbol{v}_d=\int\frac{mv^2}{2}J\,d\boldsymbol{v}.$$

玻尔兹曼方程和带电粒子输运方程

- 输运方程（续）

□ 方程汇总

$$\frac{\partial}{\partial t}n(\boldsymbol{r},t)+\frac{\partial}{\partial \boldsymbol{r}}\cdot(n(\boldsymbol{r},t)\langle\boldsymbol{v}\rangle)=n_e(\boldsymbol{r},t)R_0(\boldsymbol{r},t)$$

$$R_0(\boldsymbol{r},t)=R_i(\boldsymbol{r},t)-R_a(\boldsymbol{r},t).$$

$$mn(\boldsymbol{r},t)\frac{\partial}{\partial t}\boldsymbol{v}_d+mn(\boldsymbol{r},t)\boldsymbol{v}_d\left(\frac{\partial}{\partial \boldsymbol{r}}\cdot\boldsymbol{v}_d\right)+m\frac{\partial}{\partial \boldsymbol{r}}\cdot(n(\boldsymbol{r},t)\langle\boldsymbol{v}_r\boldsymbol{v}_r\rangle)$$
$$=n(\boldsymbol{r},t)e(\mathbf{E}+\boldsymbol{v}\times\mathbf{B})-mn_e(\boldsymbol{r},t)\boldsymbol{v}_dR_0+m\langle\boldsymbol{v}J\rangle.$$

$$m\langle\boldsymbol{v}J\rangle=-mn(\boldsymbol{r},t)\boldsymbol{v}_dR_m.$$

$$\frac{\partial}{\partial t}(n(\boldsymbol{r},t)\langle\varepsilon(\boldsymbol{r},t)\rangle)+\frac{\partial}{\partial \boldsymbol{r}}\cdot(n(\boldsymbol{r},t)\langle\varepsilon(\boldsymbol{r},t)\rangle\boldsymbol{v}_d)+\frac{\partial}{\partial \boldsymbol{r}}\cdot(\mathbf{P}\cdot\boldsymbol{v}_d)$$
$$+\frac{\partial}{\partial \boldsymbol{r}}\cdot\boldsymbol{q}-n(\boldsymbol{r},t)e\mathbf{E}\cdot\boldsymbol{v}_d=\int\frac{mv^2}{2}J\,d\boldsymbol{v}.$$

$$\int\frac{mv^2}{2}J\,d\boldsymbol{v}=\frac{2m}{M}\langle\varepsilon(\boldsymbol{r},t)\rangle R_mn_e(\boldsymbol{r},t)$$
$$+\left(\Sigma\varepsilon_jR_j+\varepsilon_iR_i-\left\langle\varepsilon NQ_a(\varepsilon)\left(\frac{2\varepsilon}{m}\right)^{1/2}\right\rangle\right)n_e(\boldsymbol{r},t),$$

玻尔兹曼方程和带电粒子输运方程

- 玻尔兹曼方程的碰撞项

□ 积分碰撞

properties by the differential cross section $\sigma(\theta, \phi; \varepsilon)$ and the integral cross sections $Q(\varepsilon)$. The velocity distribution of neutral gas molecules is represented by $F(V', r, t)$, where gas molecules have a mass M and velocity V' before collision. The charged particles of mass m and velocity v' before collision are described by the corresponding velocity distribution function $g(v', r, t)$. When we consider a small element of the phase space $dv\, dr$, then the number of the charged particles that enter this element during a short time dt is equal to

进入相空间单元dvdr中的带电粒子数(碰撞前)

$$J_{in} = \int_{\Omega} \int_{V'} F(V', r, t) dV' g(v', r, t) dv' v'_{\gamma} \sigma(\theta', \phi'; v'_{\gamma}) d\Omega' dr\, dt.$$

离开相空间单元dvdr的带电粒子数（碰撞后）

$$J_{out} = \int_{\Omega} \int_{V} F(V, r, t) dV g(v, r, t) dv v_{\gamma} \sigma(\theta, \phi; v_{\gamma}) d\Omega dr\, dt,$$

↓

dt时间内在dvdr单元内粒子数的改变量

$$J\, dv\, dr\, dt = \{J_{in} - J_{out}\}.$$

玻尔兹曼方程和带电粒子输运方程

● 玻尔兹曼方程的碰撞项（续）

□ 电子-气体分子碰撞积分

- 定向运动速度远小于热运动速度，故电子速度分布接近于球对称

- 可在速度空间用用球谐函数展开

$$g(v) = \sum_{mn} g_{mn}(v) Y_{mn}^{e}(\theta, \varphi),$$

$$Y_{mn}^{e}(\theta, \varphi) = P_n^m(\cos\theta)\cos m\varphi$$

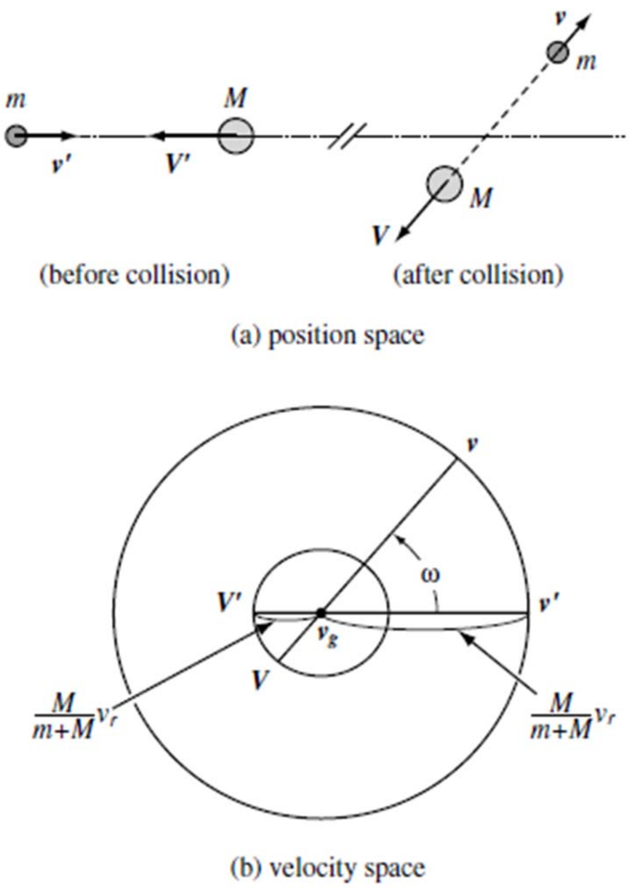


FIGURE 5.2
The definition of a collision in (a) a laboratory frame of reference (real space) and (b) a center-of-mass frame of reference (velocity space).

玻尔兹曼方程和带电粒子输运方程

- 玻尔兹曼方程的碰撞项（续）

  - 弹性碰撞项J_{elas}

    - 简化条件

i. The relative velocity between the electron and the gas molecule is not changed after the collision;

碰撞前后相对速度不变

ii. The velocity of gas molecules is much less than the velocity of electrons, $|V| \ll |v|$, and we represent the velocity distribution of gas molecules with the number density N by $F(V) = N\delta(V)$, where δ is Dirac's delta function; and

简化的中性粒子速度分布函数

iii. From the momentum and energy conservation equations before and after collisions, v' and v, we have the relation

$$\frac{v'^2 - v^2}{v^2} = \frac{2m}{M+m}(1 - \cos\omega), \qquad (5.46)$$

能量转移系数

玻尔兹曼方程和带电粒子输运方程

- 玻尔兹曼方程的碰撞项（续）

  - 弹性碰撞项J_{elas}（续）

$$\frac{v'^2 - v^2}{v^2} = \frac{2m}{M+m}(1-\cos\omega),$$

$$dv'/dv = (v'/v)^3,$$

将dv'变换为dv

$$\begin{aligned} J_{elas}dv = & \int_{\Omega'}\int_{V'} N\delta(V')g(v')v_\gamma\sigma(v_\gamma,\omega)d\Omega' dv' dV' \\ & - \int_{\Omega}\int_{V} N\delta(V)g(v)v_\gamma\sigma(v_\gamma,\omega)d\Omega dv\, dV \\ = & N\left(\int_{\Omega'}\frac{v'^3}{v^3}g(v')v'\sigma(v',\omega)d\Omega' - \int_{\Omega}g(v)v\sigma(v,\omega)d\Omega\right)dv. \end{aligned}$$

$$g(v) = \sum_{mn} g_{mn}(v)Y^e_{mn}(\theta,\varphi),$$

$$\sigma(v,\omega) = \sum_{n'}\sigma_{n'}(v)P_{n'}(\cos\omega).$$

$$\begin{aligned} & N\frac{v'^4}{v^3}\sum_{mn}\sum_{n'} g_{mn}(v')\sigma_{n'}(v')\left[\int Y^e_{mn}(\theta',\varphi')P_{n'}(\cos\theta')P_n(\cos\theta)d\Omega'\right. \\ & + 2\sum_{m=1}^{n'}\frac{(n-m)!}{(n+m)!}\int_{\Omega} Y^e_{mn}(\theta',\varphi')\left\{Y^e_{mn}(\theta',\varphi')Y^e_{mn}(\theta,\varphi)\right. \\ & \left.\left. + Y^0_{mn}(\theta',\varphi')Y^0_{mn}(\theta,\varphi)\right\}d\Omega'\right] = N\frac{v'^4}{v^3}\left[\sum_n g_{0n}(v')\sigma_n(v')\frac{4\pi}{2n+1}P_n(\cos\theta)\right. \\ & \left. + \sum_{m=1}^{n}\sum_{n} g_{mn}(v')\sigma(v')\frac{4\pi}{2n+1}Y^e_{mn}(\theta,\varphi)\right]. \end{aligned}$$

利用球谐函数性质，将θ'和φ'变换为θ和φ

玻尔兹曼方程和带电粒子输运方程

- 玻尔兹曼方程的碰撞项（续）

  - 弹性碰撞项J_{elas}（续）

$$\int_0^{\pi} P_n(\cos\theta) P_{n'}(\cos\theta)\sin\theta d\theta = \begin{cases} \dfrac{2}{2n+1}, & n = n' \\ 0, & n \neq n' \end{cases}.$$

$$\begin{aligned} J_{elas} &= N\sum_{mn}\int_{\Omega} Y^e_{mn}(\theta,\varphi)\frac{1}{v^3}\left[v'^4 g_{mn}(v')\sigma(v',\omega)P_n(\cos\omega) - v^4 g_{mn}(v)\sigma(v,\omega)\right]d\Omega \\ &= N\sum_{mn}\int_{\Omega} Y^e_{mn}(\theta,\varphi)\frac{1}{v^3}\left[v'^4 g_{mn}(v')\sigma(v',\omega) - v^4 g_{mn}(v)\sigma(v,\omega)\right]P_n(\cos\omega)d\Omega \\ &\quad -N\sum_{mn}\int_{\Omega} Y^e_{mn}(\theta,\varphi) g_{mn}(v) v\sigma(v,\omega)\{1 - P_n(\cos\omega)\}d\Omega. \end{aligned} \tag{5.50}$$

$$\frac{v'^2 - v^2}{v^2} = \frac{2m}{M+m}(1-\cos\omega),$$

$$\Delta v^2 = v'^2 - v^2.$$

将v'变换为v

$$\begin{aligned} J_{elas} &\cong N\sum_{mn}\int_{\Omega} Y^e_{mn}(\theta,\varphi)\frac{1}{v}\frac{2m}{M+m}(1-\cos\omega)\frac{\partial[v^4 g_{mn}(v)\sigma(v,\omega)]}{\partial(v^2)}P_n(\cos\omega)d\Omega \\ &\quad - N\sum_{mn} Y^e_{mn}(\theta,\varphi) g_{mn}(v) v\sigma(v,\omega)\{1 - P_n(\cos\omega)\}d\Omega. \end{aligned} \tag{5.51}$$

玻尔兹曼方程和带电粒子输运方程

- 玻尔兹曼方程的碰撞项（续）

  - 弹性碰撞项J_{elas}（续）

$$J_{elas} \cong N\sum_{mn}\int_{\Omega} Y^{e}_{mn}(\theta,\varphi)\frac{1}{v}\frac{2m}{M+m}(1-\cos\omega)\frac{\partial[v^4 g_{mn}(v)\sigma(v,\omega)]}{\partial(v^2)}P_n(\cos\omega)d\Omega$$
$$-N\sum_{mn} Y^{e}_{mn}(\theta,\varphi)g_{mn}(v)v\sigma(v,\omega)\{1-P_n(\cos\omega)\}d\Omega. \qquad (5.51)$$

$(m=0, n=0)$, $P_0(\cos\omega)=1$, 取各向同性特例情况

$$J^{00}_{elas} = N\frac{2m}{M+m}\frac{1}{2v^2}\frac{\partial}{\partial v}\left\{v^4 g_{00}(v)Q_m(v)\right\}.$$

$$Q_m(v)=\int_{\Omega}(1-\cos\omega)\sigma(v,\omega)d\Omega.$$

考虑带电粒子与气体分子碰撞对速度造成的影响

$v=v_0+O(V).$

$$O(V^2) = -g_{00}(v_0^2)\Big/\frac{\partial g_{00}(v_0^2)}{\partial(v^2)} = \frac{2kT_g}{m}$$

$$J^{00}_{elas} = N\frac{m}{M+m}\frac{1}{v^2}\frac{\partial}{\partial v}\left[v^4 Q_m(v)\left(g_{00}(v)+\frac{kT_g}{mv}\frac{\partial}{\partial v}g_{00}(v)\right)\right]$$

玻尔兹曼方程和带电粒子输运方程

- 玻尔兹曼方程的碰撞项（续）

  - 激发碰撞项J_{ex}

i. The change of kinetic energy in jth inelastic collisions ε_j usually satisfies the relation $\varepsilon_j \gg kT_g$ with gas molecule $F(V) = N\delta(V)$; and

ii. From the energy conservation we have

$$\frac{1}{2}mv'^2 = \frac{1}{2}mv^2 + \varepsilon_j. \tag{5.55}$$

$$v'dv' = vdv,$$

将dv'变换为dv

$$\begin{aligned} J_{exj}dv &= N\int_{\Omega'}\int_V g(v')v'\sigma_j(v',\omega)\delta(V')d\Omega' dv' dV' \\ &\quad -N\int_{\Omega}\delta(V)g(v)v\sigma_j(v,\omega)d\Omega dv\, dV \\ &= N\frac{1}{v}\left[\int g(v')v'^2\sigma_j(v',\omega)d\Omega' - g(v)v^2\int \sigma_j(v,\omega)d\Omega\right]dv, \end{aligned}$$

$$Q_j(v) = \int_{\Omega}\sigma_j(v,\omega)d\Omega.$$

利用球谐函数展开，将θ'和φ'变换为θ和φ

$$J_{exj} = N\frac{1}{v}\sum Y_{mn}(\theta,\varphi)\left[g_{mn}(v')v'^2\int_{\Omega}\sigma_j(v',\omega)P_n(\cos\omega)d\Omega - g_{mn}(v)v^2Q_j(v)\right], \tag{5.57}$$

玻尔兹曼方程和带电粒子输运方程

- 玻尔兹曼方程的碰撞项（续）

  - 电离碰撞项J_{ion}

i. The change of kinetic energy in ionization ε_i usually satisfies the relation $\varepsilon_i \gg kT_g$ with gas molecule $F(V) = N\delta(V)$.

ii. In the principle of indistinguishability, the incoming and newly produced electrons at ionization cannot be distinguished after collision, but experimentally we know that there exist a pair of electrons with high and low energy. We assume that the rest of the kinetic energy is shared by two electrons according to the ratio $(1-\Delta):\Delta$. Then, the velocity element can be expressed as

两个电子分享动能的比例

$$dv' = \frac{1}{\Delta}\frac{v'}{v}dv_2 \quad \text{or} \quad dv' = \frac{1}{(1-\Delta)}\frac{v'}{v}dv_1. \tag{5.58}$$

iii. From the energy conservation, we have

$$\frac{1}{2}mv'^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 + \varepsilon_i. \tag{5.59}$$

玻尔兹曼方程和带电粒子输运方程

- 玻尔兹曼方程的碰撞项（续）

  - 电离碰撞项J_{ion}(续)

$$
\begin{aligned}
J_{ion} = & N\frac{1}{(1-\Delta)v}\int_{\Omega} g(v')v'^2\sigma_i(v',\omega)d\Omega + N\frac{1}{\Delta v}\int_{\Omega} g(v'')v''^2\sigma_i(v'',\omega)d\Omega \\
& -N\frac{1}{v}g(v)v^2\int_{\Omega}\sigma_i(v,\omega)d\Omega \\
= & N\sum_{mn} Y_{mn}(\theta,\varphi)\frac{1}{v}\left\{\frac{1}{(1-\Delta)}\int_{\Omega} g_{mn}(v')v'^2\sigma_i(v',\omega)P_n(\cos\omega)d\Omega\right. \\
& \left. +\frac{1}{\Delta}\int_{\Omega} g_{mn}(v'')v''^2\sigma_i(v'',\omega)P_n(\cos\omega)d\Omega - Ng_{mn}v^2Q_i(v)\right\}, \qquad (5.60)
\end{aligned}
$$

$$
Q_i(v) = \int \sigma_i(v,\omega)d\Omega.
$$

玻尔兹曼方程和带电粒子输运方程

- 玻尔兹曼方程的碰撞项（续）

  - 附着碰撞项J_{att}

Electron attachment is a very specific and nonconservative process wherein an electron with energy ε is lost in collision with threshold energy ε_a. The collision operator is simplified under $J_{in} = 0$ as

不新产生电子

$$J_{atta} = -Ng(v)v\int_{\Omega}\sigma_a(v,\omega)d\Omega$$
$$= -N\sum_{mn}Y^e_{mn}(\theta,\varphi)g_{mn}(v)vQ_a(v),$$

where the integrated attachment cross section is given by

$$Q_a(v) = \int_{\Omega}\sigma_a(v,\omega)d\Omega.$$

《等离子体电子学》

第五章 玻尔兹曼方程和带电粒子输运方程

本章待续

下一节：第六章 气体中带电粒子输运的一般性质

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