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Research paper

# Uniqueness theorem for negative solutions of fully nonlinear elliptic equations in a ball

Zhenghuan Gao

*Department of Mathematics, Shanghai University, Shanghai, 200444, China*

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ABSTRACT

In this paper, we prove the uniqueness of negative radial solution to a Dirichlet problem of $(m,k)$-Hessian equation in a finite ball of $\mathbb{R}^n$ for $1<k<\frac{n}{m}$. Our proof is based on a Pohozaev identity and the monotone separation techniques.

## 1. Introduction

Let $B_R = B_R(0)$ be a ball with radius $R$ in $\mathbb{R}^n$, $n \geq 3$. Let $1 \leq k \leq n$ be an integer, $m > 1$ and $km \leq n$, $f(t)$ be a real continuous function defined for all $t \geq 0$. We consider the uniqueness of negative radial solutions to the fully nonlinear elliptic equation

$$\begin{cases} \sigma_k(D(|Du|^{m-2}Du)) = f(-u) & \text{in } B_R, \\ u = 0 & \text{on } \partial B_R. \end{cases} \tag{1.1}$$

Denote by $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, $\sigma_k(\lambda)$ is the $k$th elementary symmetric function of $\lambda$ as follows:

$$\sigma_k(\lambda) = \sum_{1\leq i_1<i_2<\cdots<i_k\leq n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}.$$

For a (non-symmetric) matrix $B$, $\sigma_k(B)$ is the sum of $k\times k$ principal minors of $B$. Then the $(m,k)$–Hessian operator is defined by

$$S_{m,k}[u] = \sigma_k(D(|Du|^{m-2}Du)) = \sigma_k(\lambda(D(|Du|^{m-2}Du))),$$

where $\lambda(D(|Du|^{m-2}Du))$ are the eigenvalues of $D(|Du|^{m-2}Du)$. Recall that the Gårding cone is defined as

$$\Gamma_k = \{\lambda \in \mathbb{R}^n \mid \sigma_k(\lambda) \geq 0,\ j = 1, \dots, k\}.$$

Let

$$\Phi(B_R) = \{u \in C^0(\overline{B}_R) \cap C^1(B_R) \mid D(|Du|^{m-2}Du) \in C^0(B_R)\}.$$

A function $u \in \Phi(B_R)$ is called $m$-$k$-admissible if $\lambda(D(|Du(x)|^{m-2}Du(x))) \in \Gamma_k$, $\forall\, x \in B_R$. In particular, 2-$k$-admissible is also called $k$-admissible. We denote the set of $m$-$k$-admissible functions which vanish on the boundary by $\Phi_0^{m,k}(B_R)$. Taking restriction on admissible functions makes the $(m,k)$-Hessian operator elliptic.

*E-mail address:* gzh@shu.edu.cn.

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The $(m, k)$-Hessian operator is firstly introduced by Trudinger–Wang in [37], where they use the $(m, k)$-Hessian operator to give a local gradient estimate for $k$-admissible function. When $m = 2$ and $k = 1$, (1.1) reduces to the well-known semilinear elliptic equation

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } B_R, \\ u = 0 & \text{on } \partial B_R. \end{cases} \tag{1.2}$$

The existence of nontrivial solutions to (1.2) has been researched by many authors (see [6] and reference therein). The symmetry of solutions to problem (1.2) was studied in an excellent paper [33] of Serrin. Using the powerful Alexandroff–Serrin method developed in [33], Gidas–Ni–Nirenberg [16] proved that if $f$ is $C^1$, then the positive solution to (1.2) is radially symmetric. Under further condition on $f$, there are many interesting works about the existence of radially symmetric solution, we refer to [3,5,11,19,35]. When $f(u) = u^p$ with $1 < p < \frac{n+2}{n-2}$, Gidas–Ni–Nirenberg proved that (1.2) has a unique solution. While for $p \geq \frac{n+2}{n-2}$, there is no solutions to (1.2) by Pohozaev identity [30]. For $f = \lambda u^p + u_q$, Ni-Nussbaum [25] proved the uniqueness of solutions provided $1 < p < q < \frac{n}{n-2}$. Zhang [45] proved the uniqueness for $\frac{q-1}{p+1} \leq \frac{2}{n}$. When $f = \lambda u + u^q$, the uniqueness was studied by Kwong-Li [22] for $1 < q < \frac{n+2}{n-2}$ and by Zhang [44], Srikanth [34] and Adimurthi–Yadava [2] for $1 < q \leq \frac{n+2}{n-2}$. Finally, for $n \geq 6$, Erbe-Tang completely solved the uniqueness for $f = \lambda u^p + u^q$ with $1 < p < q \leq \frac{n+2}{n-2}$, and gave some partial uniqueness results for the cases $n = 3, 4, 5$. On the other hand, the work of Brezis–Nirenberg [6] revealed that the uniqueness of solutions does not hold for all $n \geq 3$ and $1 < p \leq \frac{n+2}{n-2}$. More uniqueness results on further condition on $f$ can be found in [11,18–21,23,24,28,29,43].

When $k = 1$, $1 < m \leq n$, (1.1) becomes the following quasilinear elliptic equation

$$\begin{cases} \operatorname{div}(|Du|^{m-2} Du) + f(u) = 0 & \text{in } B_R, \\ u = 0 & \text{on } \partial B_R \end{cases} \tag{1.3}$$

Eq. (1.3) was by studied by Franchi–Lanconelli–Serrin in [15], where they proved the uniqueness of positive radial solution for sublinear case. Using the separation technique in [15], Citti [10] get some uniqueness results to (1.3) in the case $1 < m \leq 2$ and $n \geq 2 - \frac{1}{m}$ by applying the method in Kwong-Zhang [23]. For $f(u) = \lambda u + u^q$ with $m - 1 < q < \frac{nm-n+m}{n-m}$, a uniqueness result was given by Adimurthi–Yadava [2]. Later, Erbe-Tang [13] established a new Pohozaev-type identity and used it to prove the uniqueness when $f(u) = u^p$ with $m - 1 < p < \frac{mn-n+m}{n-m}$ and $f(u) = \lambda u^p + u^q$ with $m - 1 \leq p < q \leq \frac{mn-n+m}{n-m}$, $n \geq m + m^2$. Uniqueness of positive radial solutions of (1.3) with exponential nonlinearities was studied by Pucci–Serrin [32], Adimurthi [1]. In [36], Tang studied the uniqueness for $n$-laplace case.

When $m = 2$, $2 \leq k \leq n$, (1.1) becomes a type of fully nonlinear elliptic equations, the $k$-Hessian equation.

$$\begin{cases} \sigma_k(D^2 u) = f(-u) & \text{in } B_R, \\ u = 0 & \text{on } \partial B_R \end{cases} \tag{1.4}$$

The $k$-Hessian equation has been studied extensively in past decades, one can refer to Caffarelli, Nirenberg and Spruck [7] and Wang [40]. Let $\gamma(k)$ be the critical exponent for the $k$-Hessian, that is $\gamma(k) = \frac{(n+2)k}{n-2k}$ for $1 \leq k < \frac{n}{2}$ and $\gamma(k) = \infty$ for $\frac{n}{2} \leq k \leq n$. Tso [38] proved that (1.4) with $f(-u) = (-u)^p$ is solvable if and only if $p < \gamma(k)$. In [9,39,40], Wang proved that (1.4) has a $C^2$ solution in the superlinear case, sublinear case and critical case. Chou-Geng-Yan [8] got existence and nonexistence results of radial solution to (1.4) with $f(-u) = (-u)^{\gamma(k)} + \lambda(-u)^k$. Suppose $f \in C^2([0, \infty))$, $f(0) = 0$, $f(s) > 0$ on $(0, \infty)$. Recently, Wei [42] proved that (1.4) admits at least two solutions for $f(-u) = \lambda(-u)^p + (-u)^q$ with small enough $\lambda$ and $0 < p < k < q < \gamma(k)$. In [41], Wei proved that (1.4) has at most one negative radial solution in either of the following two cases

- $-sf'(-s) > kf(-s)$ and $H_k'(s) < 0$ for $s < 0$,
- $-sf'(-s) < kf(-s)$,

where $H_k(s) = \frac{(n-2k)f(-s)s-(k+1)nF(s)}{f(-s)}$ and $F(s) = \int_0^s f(-r)\mathrm{d}r$.

It is natural to consider the uniqueness of radial solution to (1.1) for general $m$ and general $k$. In the following of this paper, we always assume $k \geq 1$, $m > 1$, $mk < n$, $f \in C^2([0, \infty))$, $f(0) = 0$, $f(s) > 0$ on $(0, \infty)$ throughout this paper. Set

$$\gamma(m, k) = \frac{((m-1)n + m)k}{n - mk}.$$

and

$$H_{m,k}(s) = \begin{cases} \frac{(n-mk)f(-s)s-n((m-1)k+1)F(s)}{f(-s)}, & s < 0, \\ 0, & s = 0, \end{cases} \tag{1.5}$$

where $F(s) = \int_0^s f(-r)\mathrm{d}r$. Moreover, we suppose $f$ satisfies one of the following conditions:

(C1) $-sf'(-s) > (m-1)kf(-s)$ and $H'_{m,k}(s) < 0$ for $s < 0$,

(C2) $-sf'(-s) < (m-1)kf(-s)$ for $s < 0$.

We will apply the argument of Erbe and Tang [12,13,36] which was used by Wei [41] to deal with $k$-Hessian problem, to the $(m, k)$-Hessian equation. The main theorem is as follows:

**Theorem 1.1.** *If $f$ satisfies (C1) or (C2), then Eq. (1.1) has at most one negative radial solution in $\Phi_0^{m,k}(B_R)$.*

The corollaries follow naturally:

**Corollary 1.2.** *If $f(-u) = (-u)^p$, $(m-1)k < p \le \gamma(m,k)$ or $0 < p < (m-1)k$, then (1.1) has at most one negative radial solution.*

**Corollary 1.3.** *If $f(-u) = \lambda(-u)^p + (-u)^q$, $(m-1)k < p, q \le \gamma(m,k)$, $\lambda > 0$, then (1.1) has at most one negative radial solution.*

There are several results about $(m,k)$-Hessian equations. In [37], Trudinger–Wang proved that any $C^2$ $k$-admissible function is $m$-$l$-admissible for any $l = 1, \dots, k-1$ and $m \le \frac{n(k-l)}{n-k}$. Very recently, Bao-Feng [4] considered the entire solution to the $m$-$k$-Hessian equation, they gave a necessary and sufficient condition to the global solvability for the $m$-$k$-Hessian inequalities. Feng-Zhang studied the existence of infinitely many boundary blow-solutions in [14], later Zhang-Yang [46] generalized it to a more general $m$-$k$-Hessian type equation via a new technique and Kan-Zhang [17] extended it to $m$-$k$-Hessian systems.

In the next section, we give some preliminary results about a Cauchy problem about the ODE related to (1.1). In Section 3, we prove the uniqueness when $f$ satisfies (C2). In Section 4, we obtain a Pohozaev type identity. In Section 5, we use the Pohozaev type identity and separation technique to prove the uniqueness when $f$ satisfies (C1).

## 2. Preliminary results

In this section, we give some preliminary results and reduce the problem to an ODE problem.

Assume $u \in \Phi(B_R)$ be a radial solution of (1.1). Let $r = |x|$, $x \in B_R(0)$, $\theta = \frac{x}{|x|} \in S^n$, $x \in B_R(0)\backslash\{0\}$. Then we have

$$|Du|^{m-2}Du = |u'|^{m-2}u'\theta,$$

and

$$\begin{aligned} D(|Du|^{m-2}Du) =&(|u'|^{m-2}u')'\theta\otimes\theta + |u'|^{m-2}u'D\theta = (|u'|^{m-2}u')'\theta\otimes\theta + |u'|^{m-2}u'r^{-1}(I-\theta\otimes\theta) \\ =& r^{-1}|u'|^{m-2}u'I + ((|u'|^{m-2}u')' - r^{-1}|u'|^{m-2}u')\theta\otimes\theta, \end{aligned}$$

By direct calculation,

$$\begin{aligned} S_{m,k}[u] =& \sigma_k(r^{-1}|u'|^{m-2}u'I + ((|u'|^{m-2}u')' - r^{-1}|u'|^{m-2}u')\theta\otimes\theta) \\ =& C_n^k(r^{-1}|u'|^{m-2}u')^k + \frac{k}{n}C_n^k(r^{-1}|u'|^{m-2}u')^{k-1}((|u'|^{m-2}u')' - r^{-1}|u'|^{m-2}u') \\ =& \frac{C_n^k}{n}r^{1-n}(r^n(r^{-1}|u'|^{m-2}u')^k)', \end{aligned}$$

where $C_n^k = \frac{n!}{k!(n-k)!}$. It follows that a negative radial symmetric ground state $u = u(r)$ to Eq. (1.1) can be consider as a solution to

$$\begin{cases} \frac{C_n^k}{n}(r^n(r^{-1}|u'|^{m-2}u')^k)' = r^{n-1}f(-u) & \text{in } (0,R), \\ u(R) = 0, \end{cases} \tag{2.1}$$

By Maclaurin inequality, for any $1 \le l \le k$, the radial solution $u$ also satisfies

$$\frac{C_n^l}{n}(r^n(r^{-1}|u'|^{m-2}u')^l)' \ge r^{n-1}C_n^l(C_n^k)^{-\frac{l}{k}}f^{\frac{l}{k}}(-u) \quad \text{in } (0,R), \tag{2.2}$$

Note that there may occur singularity to $|u'|^{m-2}$ when $u' = 0$, it is necessary concern the precise meaning of solution to

$$\frac{C_n^k}{n}(r^n(r^{-1}|u'|^{m-2}u')^k)' = r^{n-1}f(-u) \quad \text{in } (0,R). \tag{2.3}$$

Here will shall treat the classical solutions, with the precise meaning that $u \in C^1([0,R))$ with $u'(0) = 0$ and $v = |u'|^{m-2}u' \in C^1((0,R))$.

**Proposition 2.1.** *If $u$ is a classical solution to (2.3) for odd $k$, then $v = |u'|^{m-2}u' \in C^1([0,R))$ and*

$$v(r) = r\left(\frac{n}{C_n^k}\int_0^r \frac{t^{n-1}}{r^n}f(-u(t))\mathrm{d}t\right)^{\frac{1}{k}}, \quad v(0) = 0, \quad v'(0) = \left(\frac{1}{C_n^k}f(-u(0))\right)^{\frac{1}{k}}. \tag{2.4}$$

**Remark 2.2.** If $u$ is a negative solution to (1.1), with $u \in C^0(\overline{B}_R \setminus \{0\}) \cap C^1(B_R \setminus \{0\})$ and $D(|Du|^{m-2}Du) \in C^0(B_R \setminus \{0\})$. Then this proposition implies that $D(|Du|^{m-2}Du) \in C^0(B_R)$.

**Proof.** By direct calculation, we obtain

$$(r^n(r^{-1}v)^k)' = \frac{n}{C_n^k}r^{-1}f(-u).$$

Integrate it on $(0, r)$, we get

$$r^n(r^{-1}v)^k = \frac{n}{C_n^k}\int_0^r t^{n-1} f(-u(t))\mathrm{d}t.$$

So

$$v = r\left(\frac{n}{C_n^k}\int_0^r \frac{t^{n-1}}{r^n} f(-u(t))\mathrm{d}t\right)^{\frac{1}{k}}.$$

It follows that

$$\begin{aligned} v' =&\left(\frac{n}{C_n^k}\int_0^r \frac{t^{n-1}}{r^n} f(-u(t))\mathrm{d}t\right)^{\frac{1}{k}} + \frac{1}{k} r\left(\frac{n}{C_n^k}\int_0^r \frac{t^{n-1}}{r^n} f(-u(t))\mathrm{d}t\right)^{\frac{1}{k}-1}\left(\frac{n}{C_n^k}\int_0^r \frac{t^{n-1}}{r^n} f(-u(t))\mathrm{d}t\right)' \\ =&\left(\frac{n}{C_n^k}\int_0^r \frac{t^{n-1}}{r^n} f(-u(t))\mathrm{d}t\right)^{\frac{1}{k}} \\ &+ \frac{1}{k}\left(\frac{n}{C_n^k}\int_0^r \frac{t^{n-1}}{r^n} f(-u(t))\mathrm{d}t\right)^{\frac{1}{k}-1}\left(\left(\frac{n}{C_n^k}\int_0^r \frac{t^{n-1}}{r^{n-1}} f(-u(t))\mathrm{d}t\right)' - \left(\frac{n}{C_n^k}\int_0^r \frac{t^{n-1}}{r^n} f(-u(t))\mathrm{d}t\right)\right) \\ =&(1-\frac{1}{k})\left(\frac{n}{C_n^k}\int_0^r \frac{t^{n-1}}{r^n} f(-u(t))\mathrm{d}t\right)^{\frac{1}{k}} + \frac{1}{k}\left(\frac{n}{C_n^k}\int_0^r \frac{t^{n-1}}{r^n} f(-u(t))\mathrm{d}t\right)^{\frac{1}{k}-1}\left(\frac{n}{C_n^k}\int_0^r \frac{t^{n-1}}{r^{n-1}} f(-u(t))\mathrm{d}t\right)'. \end{aligned}$$

By computation, we have

$$\left(\int_0^r \frac{t^{n-1}}{r^{n-1}} f(-u(t))\mathrm{d}t\right)' = f(-u(r)) - (n-1)\int_0^r \frac{t^{n-1}}{r^n} f(-u(t))\mathrm{d}t.$$

By l'Hopital's rule

$$\lim_{r\to 0}\int_0^r \frac{t^{n-1}}{r^n} f(-u(t)) = \frac{1}{n} f(-u(0)).$$

So

$$v(0) = 0 \quad \text{and} \quad v'(0) = \left(\frac{1}{C_n^k} f(-u(0))\right)^{\frac{1}{k}}. \qquad \square$$

According to Proposition 2.1, the solution to (2.1) is also a solution to the following problem

$$\begin{cases} \frac{C_n^k}{n}(r^n(r^{-1}|u'|^{m-2}u')^k)' = r^{n-1} f(-u) \qquad \text{in } (0, R), \\ u(0) = -\alpha < 0, u'(0) = 0. \end{cases} \tag{2.5}$$

It follows from Propositions A1–A4 Franchi–Lanconelli–Serrin [15], (2.5) has a unique solution depending continuous on the initial data when $k = 1$. For odd $k$, the proof are the same as the case $k = 1$ in [15]. Suppose that $u$ is the unique solution to (2.5), which depends continuously on the initial data. We denote this solution by $u(t, \alpha)$. Let

$$b(\alpha, u) = \sup\{T \in \mathbb{R} \mid u < 0 \text{ on } [0, T]\}.$$

We only consider the case $b(\alpha, u) < 0$, since in fact we concern the solution to (2.1). Since $u < 0$ and $f(-u) > 0$ in $[0, b(\alpha, u))$, we obtain see from Eq. (2.2) that $(|u'|^{m-2}u')^l$ is positive in $[0, b(\alpha, u))$ for $1 \le l \le k$. So $u'$ is positive. The solution to (2.5) in fact satisfies

$$\begin{cases} \frac{C_n^k}{n}(r^{n-k}(u')^{(m-1)k})' = r^{n-1} f(-u), \quad u' > 0 \qquad \text{in } (0, R), \\ u(0) = -\alpha < 0, u'(0) = 0. \end{cases} \tag{2.6}$$

## 3. Sublinear case

We say $f$ is superlinear with respect to the $(m, k)$-Hessian operator if

$$-sf'(-s) > (m-1)kf(-s), \quad s < 0$$

and $f$ is sublinear with respect to the $(m, k)$-Hessian operator if

$$-sf'(-s) < (m-1)kf(-s), \quad s < 0.$$

In this section, we will deal with the sublinear case.

The following type of monotonicity proposition for quasilinear equation was proved by Tang [36], Wei [41] proved it for $k$-Hessian equation.

**Proposition 3.1.** *If $f$ is superlinear with respect to the $(m,k)$-Hessian, $u_1$ and $u_2$ are two solutions of (2.6) with $u_1(0)=\alpha_1$ and $u_2(0)=\alpha_2$ ($\alpha_1<\alpha_2$) such that*

$$u_1<u_2<0 \quad \text{on } [0,t_0),$$

*then $\frac{u_1}{u_2}$ is strictly decreasing on $[0,t_0)$.*

**Proof.** Write

$$w_i(t)=t^{1-k}u_i^{(m-1)k}.$$

By (2.6), we find that

$$w_i'+\frac{n-1}{t}w_i=\frac{n}{C_n^k}f(-u_i). \tag{3.1}$$

Then

$$\lim_{t\to0^+}w_i(t)=0. \tag{3.2}$$

In fact, on $[0,t_0)$, $u_i$ is bounded, so is $f(-u_i)$. By (3.1),

$$w_i(t)=\frac{n}{C_n^k}t^{1-n}\int_0^t s^{n-1}f(-u_i(s))\mathrm{d}s\le Ct.$$

So (3.2) holds.

Letting $t\to0$ in (3.1) and using L'Hospital's rule, we get

$$\lim_{t\to0^+}w_i'(t)=\frac{f(-\alpha_i)}{C_n^k}.$$

By using L'Hospital's rule, we obtain

$$\lim_{t\to0^+}\frac{(u_1')^{(m-1)k}}{(u_2')^{(m-1)k}}=\lim_{t\to0^+}\frac{w_1(t)}{w_2(t)}=\lim_{t\to0^+}\frac{w_1'(t)}{w_2'(t)}=\frac{f(-\alpha_1)}{f(-\alpha_2)}>\frac{(-\alpha_1)^{(m-2)k}}{(-\alpha_2)^{(m-2)k}},$$

where the last inequality is due to $(C1)$. Then

$$\lim_{t\to0^+}\frac{u_2(t)}{u_2'(t)}\Big(\frac{u_1(t)}{u_2(t)}\Big)'=\lim_{t\to0^+}\Big(\frac{u_1'(t)}{u_2'(t)}-\frac{u_1(t)}{u_2(t)}\Big)=\Big(\frac{f(-\alpha_1)}{f(-\alpha_2)}\Big)^{\frac{1}{(m-2)k}}-\frac{\alpha_1}{\alpha_2}>0.$$

So $\big(\frac{u_1(t)}{u_2(t)}\big)'<0$ for sufficient small $t$. If the assertion does not hold, there exists $t_1\in(0,t_0)$ such that

$$\Big(\frac{u_1}{u_2}\Big)'(t_1)=0 \quad\text{and}\quad \Big(\frac{u_1}{u_2}\Big)''(t_1)\ge0.$$

Thus $u_1'(t_1)u_2(t_1)=u_1(t_1)u_2'(t_1)$. Use the fact $u_1(t_1)<u_2(t_1)$ and $(C1)$, we find

$$\begin{aligned}\Big(\frac{u_1}{u_2}\Big)''&=\Big(\frac{u_1'u_2-u_1u_2'}{u_2^2}\Big)'\\&=\frac{1}{m-1}\frac{(u_1')^{2-m}\big((u_1')^{(m-1)k}\big)'u_2-u_1(u_2')^{2-m}\big((u_2')^{(m-1)k}\big)'}{u_2^2}\\&=\frac{1}{m-1}\frac{1}{u_2^2}u_2(u_1')^{2-m}\Big(\frac{n}{kC_n^k}\frac{t^{k-1}f(-u_1)}{(u_1')^{(k-1)(m-1)}}-\frac{n-k}{k}\frac{(u_1')^{m-1}}{t}\Big)\\&\quad-\frac{1}{m-1}\frac{1}{u_2^2}u_1(u_2')^{2-m}\Big(\frac{n}{kC_n^k}\frac{t^{k-1}f(-u_2)}{(u_2')^{(k-1)(m-1)}}-\frac{n-k}{k}\frac{(u_2')^{m-1}}{t}\Big)\\&=\frac{n}{(m-1)kC_n^k}\frac{t^{k-1}}{u_2^2}\frac{f(-u_1)u_1}{((u_2'))^{(m-1)k-1}}\Big(\frac{u_2^{(m-1)k}}{u_1^{(m-1)k}}-\frac{f(-u_2)}{f(-u_1)}\Big).\end{aligned}$$

The last line is negative due to the superlinear assumption. This leads a contradiction to $\big(\frac{u_1}{u_2}\big)''(t_1)\ge0$. So $\frac{u_1(t)}{u_2(t)}<0$ on $[0,t_0)$. □

By the same discussion, we obtain the following

**Proposition 3.2.** *If $f$ is sublinear with respect to the $(m,k)$-Hessian, $u_1$ and $u_2$ are two solutions of (2.5) with $u_1(0)=\alpha_1$ and $u_2(0)=\alpha_2$ ($\alpha_1<\alpha_2$) such that*

$$u_1<u_2<0 \quad \text{on } [0,t_0),$$

*then $\frac{u_1}{u_2}$ is strictly increasing on $[0,t_0)$.*

**Theorem 3.3.** *If $f$ satisfies (C2), then Eq. (1.1) has at most one negative radial solution.*

**Proof.** Suppose $u_1$ and $u_2$ are two different radial solutions to (1.1) with $u_1(0) = \alpha_1$, $u_2(0) = \alpha_2$. Assume that $\alpha_1 < \alpha_2$, then

$$u_1 < u_2 \quad \text{in } [0, R).$$

In fact, if there exists $r_0 \in [0, R)$ satisfying

$$u_1(r_0) = u_2(r_0) \quad \text{and} \quad u_1(r) < u_2(r) \text{ in } [0, r_0)$$

Then

$$\frac{u_1(r_0)}{u_2(r_0)} = 1 < \frac{\alpha_1}{\alpha_2} = \frac{u_1(0)}{u_2(0)}.$$

This contradicts with $\frac{u_1}{u_2}$ is strictly increasing on $[0, r_0)$. So $u_1 < u_2$ on $[0, R)$.

Using Proposition 3.2 and L'Hospital's rule, we find that $\forall\, r \in [0, R)$,

$$\frac{u_1(r)}{u_2(r)} < \lim_{t\to R} \frac{u_1(t)}{u_2(t)} = \frac{u_1'(R)}{u_2'(R)}. \tag{3.3}$$

Since $u_1$ and $u_2$ satisfy (2.1), we get

$$\begin{aligned}\int_0^R r^{n-1} f(-u_i(r))\mathrm{d}r =& \frac{C_{n-1}^{k-1}}{k}\int_0^R \left(r^{n-k}\left(u_i'(r)\right)^{(m-1)k}\right)'\mathrm{d}r\\ =& \frac{C_{n-1}^{k-1}}{k} R^{n-k}\left(u_i'(R)\right)^{(m-1)k}.\end{aligned} \tag{3.4}$$

It follows from (3.3), (3.4) and assumption (C1) that

$$\begin{aligned}0 =& \left(u_1'(R)\right)^{(m-1)k}\int_0^R r^{n-k} f(-u_2(r))\mathrm{d}r - \left(u_2'(R)\right)^{(m-1)k}\int_0^R r^{n-k} f(-u_1(r))\mathrm{d}r\\ =& \int_0^R r^{n-1} f(-u_2(r))\left(u_2'(R)\right)^{(m-1)k}\left(\frac{\left(u_1'(R)\right)^{(m-1)k}}{\left(u_2'(R)\right)^{(m-1)k}} - \frac{f(-u_1(r))}{f(-u_2(r))}\right)\\ \geq& \int_0^R r^{n-1} f(-u_2(r))\left(u_2'(R)\right)^{(m-1)k}\left(\frac{(u_1(R))^{(m-1)k}}{(u_1(R))^{(m-1)k}} - \frac{f(-u_1(r))}{f(-u_2(r))}\right)\\ >& 0.\end{aligned}$$

It is a contradiction! So we have $u_1 = u_2$ in $[0, R]$. □

## 4. A Pohozaev type identity for radial solutions

Pohozaev identity has been shown to be very useful in studying the existence and uniqueness of Laplacian equation and quasilinear equation. The Pohozaev type identity for $m$-Laplace equation can be found in [26,27,31]. In [38] Tso prove a Pohozaev type identity for $k$-Hessian on a general domain, Wei [41] used the radial solution version on a ball to prove the uniqueness of radial solution to $k$-Hessian equation. In this section, we establish a Pohozaev type identity for $(m, k)$-Hessian equation on a ball.

**Proposition 4.1.** *Let $u = u(t, \alpha)$ be a solution of (2.6), then for each $a \in \mathbb{R}$, $e \in \mathbb{R}$,*

$$\begin{aligned}&\frac{C_n^k}{(m-1)n} a(u')^{(m-1)k} u r^{e-1} + \frac{kC_n^k}{n((m-1)k+1)}(u')^{(m-1)k+1} r^e - \frac{1}{m-1}F(u) r^{e+k-1}\\ =& \frac{a}{m-1}\int_0^r u f(-u) t^{e+k-2} - \frac{e+k-1}{m-1}F(u)t^{e+k-2}\mathrm{d}t\\ &+ \frac{a(e-1-n+k)C_n^k}{(m-1)n}\int_0^r (u')^{(m-1)k} u\mathrm{d}t\\ &+ \left(\frac{a}{(m-1)k} + \frac{e}{(m-1)k+1} - \frac{n-k}{(m-1)k}\right)\frac{k}{n}C_n^k\int_0^r (u')^{(m-1)k+1} t^{e-1}\mathrm{d}t,\end{aligned} \tag{4.1}$$

*where $F(s) = \int_0^s f(-t)\mathrm{d}t$.*

*In particular, let $a = \frac{n-mk}{(m-1)k+1}$, $e = n-k+1$, we have*

$$\begin{aligned}&\frac{n-mk}{n} C_n^k (u')^{(m-1)k} u r^{n-k} + (m-1)\frac{k}{n}C_n^k (u')^{(m-1)k+1} r^{n-k+1} - ((m-1)k+1)F(u) r^n\\ =& (n-mk)\int_0^r u f(-u) t^{n-1}\mathrm{d}t - n((m-1)k+1)\int_0^r F(u) t^{n-1}\mathrm{d}t.\end{aligned} \tag{4.2}$$

**Remark 4.2.** If we denote

$$\bar{H}_{m,k}(s) = (n - mk)f(-s)s - n((m-1)k+1)F(s),$$

and

$$\begin{aligned}\bar{P}_{m,k}(u(r)) =&(n-mk)\frac{C_n^k}{n}(u')^{(m-1)k}ur^{n-k}\\ &+ (m-1)\frac{kC_n^k}{n}(u')^{(m-1)k+1}r^{n-k+1} - ((m-1)k+1)F(u)r^n,\end{aligned} \tag{4.3}$$

then (4.2) is equivalent to

$$\bar{P}_{m,k}(u(r)) = \int_0^r \bar{H}_{m,k}(u(t))t^{n-1}\mathrm{d}t. \tag{4.4}$$

**Proof.** By (2.1), we have

$$r^{n-1}f(-u(r)) = \frac{n-k}{n}C_n^k r^{n-k-1}(u'(r))^{(m-1)k} + \frac{k}{n}C_n^k r^{n-k}(u'(r))^{(m-1)(k-1)}((u'(r))^{m-1})'. \tag{4.5}$$

Differentiating two sides of (4.1), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}r}\Big(\frac{aC_n^k}{(m-1)n}(u')^{(m-1)k}ur^{e-1}\Big)\\ =&\frac{aC_n^k}{(m-1)n}\Big((e-1-n+k)(u')^{(m-1)k}ur^{e-2} + (u')^{(m-1)k+1}r^{e-1}\Big) + \frac{a}{m-1}ufr^{e+k-2},\end{aligned} \tag{4.6}$$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}r}\Big(\frac{kC_n^k}{n((m-1)k+1)}(u')^{(m-1)k+1}r^e\Big)\\ =&\Big(\frac{e}{(m-1)k+1} - \frac{n-k}{(m-1)k}\Big)\frac{k}{n}C_n^k(u')^{(m-1)k+1}r^{e-1} + \frac{1}{m-1}u'r^{e+k-1}f(-u),\end{aligned} \tag{4.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(\frac{1}{m-1}F(u)r^{e+k-1}\Big) = \frac{1}{m-1}f(-u)u'r^{e+k-1} + \frac{e+k-1}{m-1}F(u)r^{e+k-2}. \tag{4.8}$$

Putting (4.6), (4.7) and (4.8) together and integrating on $(0, r)$, we obtain (4.1). □

For a fixed $\alpha < 0$, as $u'(r) > 0$, we have the inverse function $r = r(u)$, which satisfies $r'(u) > 0$ in $(0, \alpha)$. and

$$r'(u) = \frac{1}{u'(r)} = ((u')^{m-1})^{-\frac{1}{m-1}}, \quad r''(u) = -\frac{1}{m-1}((u')^{m-1})^{-\frac{m+1}{m-1}}((u')^{m-1})'. \tag{4.9}$$

It follows by (2.5) that

$$\begin{cases}-(m-1)r'' + \frac{n-k}{k}\frac{(r')^2}{r} = \frac{n}{kC_n^k}(r')^{(m-1)k+2}r^{k-1}f & \text{in } (0,\alpha),\\ r(0) = b(\alpha), \ r(\alpha) = 0.\end{cases} \tag{4.10}$$

Denote

$$\begin{aligned}P_{m,k}(u(r)) =&\frac{C_n^k}{n}(r(u))^{n-k}(r'(u))^{-(m-1)k}\Big(H_{m,k}(u) - (n-mk)u\Big)\\ &- (m-1)\frac{kC_n^k}{n}(r(u))^{n-k+1}(r'(u))^{-(m-1)k-1} + ((m-1)k+1)F(u)(r(u))^n.\end{aligned} \tag{4.11}$$

**Proposition 4.3.** *Let $u = u(t,\alpha)$ be the solution to (2.6), $t = t(u,\alpha)$ be the inverse function of $u$. Then*

$$P_{m,k}(u(r)) = \frac{C_n^k}{n}\int_\alpha^u H'_{m,k}(s)(r(s))^{n-k}(r'(s))^{-(m-1)k}\mathrm{d}s. \tag{4.12}$$

**Proof.** By (4.3), and (4.9), we have

$$\begin{aligned}\bar{P}_{m,k}(u(r)) =&(n-mk)\frac{C_n^k}{n}(u')^{(m-1)k}ur^{n-k} + (m-1)\frac{kCn^k}{n}(u')^{(m-1)k+1}r^{n-k+1} - ((m-1)k+1)F(u)r^n\\ =&(n-mk)\frac{C_n^k}{n}(r')^{-(m-1)k}ur^{n-k} + (m-1)\frac{kC_n^k}{n}(r')^{-(m-1)k-1}r^{n-k+1} - ((m-1)k+1)F(u)r^n.\end{aligned} \tag{4.13}$$

From (4.10), we obtain

$$f = \frac{kC_n^k}{n}r^{1-k}(r')^{-(m-1)k-2}\Big(-(m-1)r'' + \frac{n-k}{k}r^{-1}(r')^2\Big).$$

Recall that for $s < 0$,

$$H_{m,k}(s) = \frac{(n-mk)f(-s)s - n((m-1)k+1)F(s)}{f(-s)} = \frac{\bar{H}_{m,k}(s)}{f(-s)},$$

we have

$$\begin{aligned}
&\int_0^r \bar{H}_{m,k}(u(t))t^{n-1}\mathrm{d}t \\
=&\int_\alpha^u \bar{H}_{m,k}(s)(r(s))^{n-1}r'(s)\mathrm{d}s \\
=&\int_\alpha^u H_{m,k}(s)f(-s)(r(s))^{n-1}r'(s)\mathrm{d}s \\
=&\frac{kC_n^k}{n}\int_\alpha^u H_{m,k}(s)(r(s))^{n-k}(r'(s))^{-(m-1)k-1}\Big(-(m-1)r''(s) \\
&+\frac{n-k}{k}(r(s))^{-1}(r'(s))^2\Big)\mathrm{d}s \\
=&-(m-1)\frac{kC_n^k}{n}\int_\alpha^u H_{m,k}(s)(r(s))^{n-k}(r'(s))^{-(m-1)k-1}r''(s)\mathrm{d}s \\
&+\frac{n-k}{n}C_n^k\int_\alpha^u H_{m,k}(s)(r(s))^{n-k-1}(r'(s))^{-(m-1)k+1}\mathrm{d}s \\
=&\frac{C_n^k}{n}\int_\alpha^u H_{m,k}(s)(r(s))^{n-k}\mathrm{d}(r'(s))^{-(m-1)k} \\
&+\frac{n-k}{n}C_n^k\int_\alpha^u H_{m,k}(s)(r(s))^{n-k-1}(r'(s))^{-(m-1)k+1}\mathrm{d}s \\
=&\frac{C_n^k}{n}H_{m,k}(u)(r(u))^{n-k}(r'(u))^{-(m-1)k} - \frac{C_n^k}{n}\int_\alpha^u (r'(s))^{-(m-1)k}\left(H_{m,k}(s)(r(s))^{n-k}\right)'\mathrm{d}s \\
&+\frac{n-k}{n}C_n^k\int_\alpha^u H_{m,k}(s)(r(s))^{n-k-1}(r'(s))^{-(m-1)k+1}\mathrm{d}s \\
=&\frac{C_n^k}{n}H_{m,k}(u)(r(u))^{n-k}(r'(u))^{-(m-1)k} - \frac{C_n^k}{n}\int_\alpha^u (r'(s))^{-(m-1)k}H'_{m,k}(s)(r(s))^{n-k}\mathrm{d}s.
\end{aligned} \tag{4.14}$$

By (4.4), (4.13) and (4.14), we have

$$\begin{aligned}
&(n-mk)\frac{C_n^k}{n}(r')^{-(m-1)k}ur^{n-k} + (m-1)\frac{kC_n^k}{n}(r')^{-(m-1)k-1}r^{n-k+1} - ((m-1)k+1)F(u)r^n \\
=&\frac{C_n^k}{n}H_{m,k}(u)(r(u))^{n-k}(r'(u))^{-(m-1)k} - \frac{C_n^k}{n}\int_\alpha^u H'_{m,k}(s)(r'(s))^{-(m-1)k}(r(s))^{n-k}\mathrm{d}s.
\end{aligned}$$

This finishes the proof. □

## 5. Superlinear case

In this section, we use the Pohozaev identity and monotone separation technique to prove the uniqueness of radial solution to (1.1) in the superlinear case under assumption $H'_{m,k}(s) < 0$.

**Theorem 5.1.** *If $f$ satisfies (C1), then Eq. (1.1) has at most one negative radial solution.*

This theorem is an immediate consequence of the following two lemma.

**Lemma 5.2.** *Assume $f$ is superlinear with respect to the $(m,k)$-Hessian, $u_1(r) = u_1(r,\alpha_1)$ and $u_2(r) = u_2(r,\alpha_2)$ are two solutions of (2.5) with $b(\alpha_1) = b(\alpha_2)$ such that $u_1 \le u_2$ in $[0, b(\alpha_1)]$, then $u_1 = u_2$.*

**Lemma 5.3.** *Assume $f$ satisfies (C1), $u_1(r) = u_1(r,\alpha_1)$ and $u_2(r) = u_2(r,\alpha_2)$ are two solutions of (2.5) with $\alpha_1 < \alpha_2 < 0$ and $b(\alpha_1) = b(\alpha_2)$. Then $u_1 \le u_2$ in $[0, b(\alpha_1)]$.*

Lemma 5.2 was proved by Adimurthi–Yadava [2] and Erbe-Tang [13] in different way for quasilinear equations. We use the method of Tang [36] and Wei [41]. In [13], Erbe-Tang proved Lemma 5.3 for $p$-Laplacian equation by dividing the interval into two part and using the monotone separation technique. Here we divide it into three part as Wei did in [41]. To prove Lemmas 5.2 and 5.3, we first prove a monotonicity lemma.

**Lemma 5.4.** *Let $u_1(t) = u(t,\alpha_1)$ and $u_2(t) = u(t,\alpha_2)$ be two solutions of (2.5), $t_1(u) = t(u,\alpha_1)$ and $t_2(u) = t(u,\alpha_2)$ be the inverse functions of $u_1$ and $u_2$ respectively. Set*

$$S(u) = \frac{(t_1(u))^{n-k}}{(t_1'(u))^{(m-1)k}} \Big/ \frac{(t_2(u))^{n-k}}{(t_2'(u))^{(m-1)k}}.$$

*Then*

$$S'(u) < 0 \quad \textit{if and only if} \quad t_1^{k-1}(t_1')^{(m-1)k+1} < t_2^{k-1}(t_2')^{(m-1)k+1}. \tag{5.1}$$

**Proof.** Since

$$\begin{aligned}\left(\frac{t_1^{n-k}}{(t_1')^{(m-1)k}}\right)' =&(n-k)t_1^{n-k-1}(t_1')^{1-(m-1)k} - (m-1)kt_1^{n-k}(t_1')^{-(m-1)k-1}t_1'' \\ =&kt_1^{n-k}(t_1')^{-(m-1)k-1}\left(\frac{n-1}{k}\frac{(t_1')^2}{t_1} - (m-1)t_1''\right) \\ =&\frac{n}{C_n^k}f(-u)(t_1')^{-(m-1)k-1}(t_1')^{(m-1)k+2}t_1^{k-1} \\ =&\frac{n}{C_n^k}f(-u)t_1^{n-1}t_1',\end{aligned}$$

we have

$$S'(u) = \frac{n}{C_n^k}\frac{t_1^{n-k}(t_2')^{(m-1)k}}{t_2^{n-k}(t_1')^{(m-1)k}}\left(t_1^{k-1}(t_1')^{(m-1)k+1} - t_2^{k-1}(t_2')^{(m-1)k+1}\right)f(-u).$$

So (5.1) holds. □

**Proof of Lemma 5.2.** Denote $b = b(\alpha_1) = b(\alpha_2)$. If $\alpha_1 = \alpha_2$, then $u_1 = u_2$. Now we suppose $\alpha_1 < \alpha_2$. Then we have

$$u_1 < u_2 \quad \text{for} \quad r \in [0, b).$$

In fact, if there is $r_0 \in (0, b)$ such that $u_1 < u_2$ in $(0, r_1)$, $u_1(r_1) = u_2(r_1)$, then by Proposition 3.1, $\frac{u_1}{u_2}$ is strictly decreasing. This contradicts with $u_1(r_1) = u_2(r_1)$. So $u_1 < u_2$ holds on $[0, b)$.

Applying Proposition 3.1 and L'Hostipals rule, we obtain

$$\frac{u_1(r)}{u_2(r)} > \lim_{t\to b}\frac{u_1(t)}{u_2(t)} = \frac{u_1'(b)}{u_2'(b)}. \tag{5.2}$$

Note that $u_1(b) = u_2(b) = 0$, by (2.1), we have

$$b^{n-k}(u_i'(b))^{(m-1)k} = \frac{n}{C_n^k}\int_0^b r^{n-1}f(-u_i(r))\mathrm{d}r.$$

So by (5.2), we obtain

$$\begin{aligned}0 =&(u_1'(b))^{(m-1)k}\int_0^b r^{n-1}f(-u_2(r))\mathrm{d}r - (u_2'(b))^{(m-1)k}\int_0^b r^{n-1}f(-u_1(r))\mathrm{d}r \\ =&\int_0^b r^{n-1}f(-u_2(r))(u_2'(b))^{(m-1)k}\left(\frac{(u_1'(b))^{(m-1)k}}{(u_2'(b))^{(m-1)k}} - \frac{f(-u_1(r))}{f(-u_2(r))}\right)\mathrm{d}r \\ <&\int_0^b r^{n-1}f(-u_2(r))(u_2'(b))^{(m-1)k}\left(\frac{(-u_1(r))^{(m-1)k}}{(-u_2(r))^{(m-1)k}} - \frac{f(-u_1(r))}{f(-u_2(r))}\right)\mathrm{d}r \\ <&0,\end{aligned}$$

where the last inequality holds provided that $\frac{f(-s)}{(-s)^{(m-1)k}}$ is decreasing for $s < 0$. This is due to $-sf'(-s) > (m-1)kf(-s)$ for $s < 0$. So we get a contradiction and complete the proof.

**Remark 5.5.** The proof of Lemma 5.2 is similar to that of Theorem 3.3 except that we need $u_1 \leq u_2$, which is not easy to get. This is why we need Lemma 5.3.

**Proof of Lemma 5.3.** Let $t_1(u) = t(u, \alpha_1)$, $t_2(u) = t(u, \alpha_2)$ be the inverses of $u_1$ and $u_2$. If the assertion of this lemma is not valid, then the graph of $t_1$ and $t_2$ must intersect in $(\alpha_2, 0)$. It follows that there is a point $v_1 \in (\alpha_2, 0)$ such that

$$t_1(v_1) = t_2(v_1), \quad t_1'(v_1) < t_2'(v_1), \quad t_1(u) > t_2(u), \text{ in } (\alpha_2, v_1). \tag{5.3}$$

Note that $t_1(0) = t_2(0) = b(\alpha_1)$, there exists a point $v_2 \in (v_1, 0]$ such that

$$t_1(v_2) = t_2(v_2), \quad t_1'(v_2) > t_2'(v_2), \quad t_1(u) < t_2(u), \text{ in } (v_1, v_2). \tag{5.4}$$

Thus there is a point $v_c \in (v_1, v_2)$ such that

$$t_1'(v_c) = t_2'(v_c), \quad t_1'(u) < t_2'(u), \text{ in } [v_1, v_c). \tag{5.5}$$

Moreover, we have

$$t_1'(u) < t_2'(u), \text{ in } (\alpha_2, v_1]. \tag{5.6}$$

In fact, if (5.6) does not hold, there exists $u_0 \in (\alpha_2, v_1)$, such that

$$t_1'(u_0) = t_2'(u_0).$$

By (4.10), we have

$$\begin{aligned}(m-1)(t_1''(u_0) - t_2''(u_0)) =& \frac{n-k}{k}(t_1'(u_0))^2\Big(\frac{1}{t_1(u_0)} - \frac{1}{t_2(u_0)}\Big)\\ &- \frac{n}{kC_n^k}(t_1'(u_0))^{(m-1)k+2}\big((t_1(u_0))^{k-1} - (t_2(u_0))^{k-1}\big)f(-u_0).\end{aligned}$$

Since $t_1(u) > t_2(u)$ in $[\alpha_2, v_1)$, we have $(t_1 - t_2)''(u_0) < 0$. So $t_1 - t_2$ has at most one critical point in $(\alpha_2, v_1)$. However, because of $t_1'(v_1) < t_2'(v_1)$ and $t_1' < t_2'$ near $\alpha_2$, $t_1 - t_2$ has no less than two critical points in $(\alpha_2, v_1)$. So we have proved (5.6) holds.

Let $S_c = S(v_c)$, where $S(u)$ is defined in Lemma 5.4. Then we have

$$S_c = \frac{(t_1(v_c))^{n-k}}{(t_1'(v_c))^{(m-1)k}} \Big/ \frac{(t_2(v_c))^{n-k}}{(t_2'(v_c))^{(m-1)k}} = \frac{(t_1(v_c))^{n-k}}{(t_2(v_c))^{n-k}} < 1, \tag{5.7}$$

and

$$S_c = \frac{(t_1(v_c))^{n-k}}{(t_2(v_c))^{n-k}} < \frac{(t_1(v_c))^{n-k+1}}{(t_2(v_c))^{n-k+1}} \le \frac{(t_1(v_c))^{n}}{(t_2(v_c))^{n}}. \tag{5.8}$$

For $i = 1, 2$, set

$$\begin{aligned}P_{m,k}^i(u) =& \frac{C_n^k}{n}\frac{(t_i(u))^{n-k}}{(t_i'(u))^{(m-1)k}}\big(H_{m,k}(u) - (n-mk)u\big)\\ &- (m-1)\frac{kC_n^k}{n}\frac{(t_i(u))^{n-k+1}}{(t_i'(u))^{(m-1)k+1}} + ((m-1)k+1)F(u)(t_i(u))^n.\end{aligned}$$

By Proposition 4.3, we have

$$\begin{aligned}P_{m,k}^1(v_c) - P_{m,k}^2(v_c) =& \frac{C_n^k}{n}\int_{\alpha_1}^{v_c} H_{m,k}'(s)\frac{(t_1(s))^{n-k}}{(t_1'(s))^{(m-1)k}}ds - S_c\frac{C_n^k}{n}\int_{\alpha_2}^{v_c} H_{m,k}'(s)\frac{(t_2(s))^{n-k}}{(t_2'(s))^{(m-1)k}}ds\\ =& \frac{C_n^k}{n}\int_{\alpha_1}^{\alpha_2} H_{m,k}'(s)\frac{(t_1(s))^{n-k}}{(t_1'(s))^{(m-1)k}}ds\\ &+ \frac{C_n^k}{n}\int_{\alpha_2}^{v_1} H_{m,k}'(s)\Big(\frac{(t_1(s))^{n-k}}{(t_1'(s))^{(m-1)k}} - S_c\frac{(t_2(s))^{n-k}}{(t_2'(s))^{(m-1)k}}\Big)ds\\ &+ \frac{C_n^k}{n}\int_{v_1}^{v_c} H_{m,k}'(s)\Big(\frac{(t_1(s))^{n-k}}{(t_1'(s))^{(m-1)k}} - S_c\frac{(t_2(s))^{n-k}}{(t_2'(s))^{(m-1)k}}\Big)ds\\ =:& \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned} \tag{5.9}$$

It is easy to see

$$\mathrm{I} := \frac{C_n^k}{n}\int_{\alpha_1}^{\alpha_2} H_{m,k}'(s)\frac{(t_1(s))^{n-k}}{(t_1'(s))^{(m-1)k}}ds \le 0, \tag{5.10}$$

since we have assumed $H_{m,k}'(s) \le 0$ for $s < 0$.

In $(\alpha_2, v_1)$, by (5.3), (5.6) and (5.7), we obtain that

$$\frac{(t_1(s))^{n-k}}{(t_1'(s))^{(m-1)k}} - S_c\frac{(t_2(s))^{n-k}}{(t_2'(s))^{(m-1)k}} > 0.$$

So we get

$$\mathrm{II} := \frac{C_n^k}{n}\int_{\alpha_2}^{v_1} H_{m,k}'(s)\Big(\frac{(t_1(s))^{n-k}}{(t_1'(s))^{(m-1)k}} - S_c\frac{(t_2(s))^{n-k}}{(t_2'(s))^{(m-1)k}}\Big)ds \le 0. \tag{5.11}$$

In $(v_1, v_c)$, by (5.4) and (5.5), we have

$$t_1^{k-1}(t_1')^{(m-1)k+1} < t_2^{k-1}(t_2')^{(m-1)k+1} \quad \text{in } (v_1, v_c).$$

It follows by Lemma 5.4 that $S'(u) < 0$ in $(v_1, v_c)$. So

$$\frac{(t_1(s))^{n-k}}{(t_1'(s))^{(m-1)k}} - S_c\frac{(t_2(s))^{n-k}}{(t_2'(s))^{(m-1)k}} = \frac{(t_2(s))^{n-k}}{(t_2'(s))^{(m-1)k}}(S(s)-S_c) > 0,\ \ s\in(v_1,v_c).$$

Hence we obtain

$$\mathrm{III} := \frac{C_n^k}{n}\int_{v_1}^{v_c} H'_{m,k}(s)\Big(\frac{(t_1(s))^{n-k}}{(t_1'(s))^{(m-1)k}} - S_c\frac{(t_2(s))^{n-k}}{(t_2'(s))^{(m-1)k}}\Big)\mathrm{d}s \le 0. \tag{5.12}$$

Substituting (5.10), (5.11) and (5.12) into (5.9), we get

$$P^1_{m,k}(v_c) - P^2_{m,k}(v_c) \le 0. \tag{5.13}$$

On the other hand, by (4.11),

$$\begin{aligned}
&P^1_{m,k}(v_c) - P^2_{m,k}(v_c)\\
=&\frac{C_n^k}{n}\big(H_{m,k}(v_c)-(n-mk)v_c\big)\Big(\frac{(t_1(v_c))^{n-k}}{(t_1'(v_c))^{(m-1)k}} - S_c\frac{(t_2(v_c))^{n-k}}{(t_2'(v_c))^{(m-1)k}}\Big)\\
&-(m-1)\frac{kC_n^k}{n}\Big(\frac{(t_1(v_c))^{n-k+1}}{(t_1'(v_c))^{(m-1)k-1}} - S_c\frac{(t_2(v_c))^{n-k+1}}{(t_2'(v_c))^{(m-1)k-1}}\Big)\\
&+((m-1)k+1)F(v_c)\big((t_1(v_c))^n - S_c(t_2(v_c))^n\big)\\
<&0.
\end{aligned} \tag{5.14}$$

The inequality holds because of (5.8) and $F(v_c) < 0$. It contradicts with (5.13). The proof is completed. □

**Proof of Corollary 1.2.** For $f(t) = t^p$, if $0 < p < (m-1)k$, then $f$ is sublinear with respect to $(m,k)$-Hessian. So (C2) holds.

If $p > (m-1)k$, $f$ is superlinear with respect to $(m,k)$-Hessian. We have

$$F(s) = \int_0^s f(-t)\mathrm{d}t = -\frac{1}{p+1}(-s)^{p+1}.$$

By (1.5),

$$H_{m,k}(s) = (n - mk - \frac{n((m-1)k+1)}{p+1})s.$$

So if we assume $p \le \gamma_{m,k} = \frac{(mn+m-n)k}{n-mk}$.

$$H'_{m,k}(s) = n - mk - \frac{n((m-1)k+1)}{p+1} \le 0.$$

So (C1) holds. By Theorem 1.1, we finish the proof. □

**Remark 5.6.** By direction calculation, $u = -c|x|^\beta$ is a solution to $\sigma_k(D(|Du|^{m-2}Du)) = (-u)^p$ for some $c = c(m,k,p)$, $\beta = \frac{mk}{(m-1)k-p}$. It is clear the $u \in C^1(\mathbb{R}^n)$ for $p < (m-1)k$ and $D(|Du|^{m-2}Du) \in C^0(\mathbb{R}^n)$. When $p > (m-1)k$, $(-u)^{-\frac{p}{k}}D(|Du|^{m-2}Du) \in C^0(\mathbb{R}^n)$.

**Proof of Corollary 1.3.** For $f = \lambda t^p + t^q$ with $\lambda > 0$, we have

$$F(s) = -\lambda\frac{(-s)^{p+1}}{p+1} - \frac{(-s)^{q+1}}{q+1}.$$

Then

$$H_{m,k}(s) = \frac{\lambda sA - (-s)^{q-p+1}B}{\lambda + (-s)^{q-p}}.$$

$$H'_{m,k}(s) = \frac{\lambda^2 A + (-s)^{2(q-p)}B - (q-p-1)A + (q-p+1)\lambda(-s)^{q-p}B}{(\lambda + (-s)^{q-p})^2},$$

where

$$A = n - mk - \frac{n((m-1)k+1)}{p+1}, \quad B = n - mk - \frac{n((m-1)k+1)}{q+1}.$$

If $p \le \gamma(m,k)$, we have

$$A \ge 0 \quad \text{and} \quad B \ge 0.$$

If $n \ge (m-1)mk^2 + 2mk$, we have

$$\begin{aligned}
q-p-1 \le& \frac{((m-1)n+m)k}{n-mk} - (m-1)k - 1\\
=&\frac{(m-1)nk + mk - (m-1)nk + (m-1)mk^2 - n + mk}{n-mk}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2mk + (m-1)mk^2 - n}{n - mk} \\
&\le 0,
\end{aligned}
$$

and

$$
\begin{aligned}
q - p + 1 &\ge (m-1)k - \frac{((m-1)n + m)k}{n - mk} + 1 \\
&= \frac{(m-1)nk - (m-1)mk^2 - (m-1)nk - mk + n - mk}{n - mk} \\
&= \frac{-(m-1)mk^2 - 2mk + n}{n - mk} \\
&\ge 0.
\end{aligned}
$$

Hence $f$ satisfies (C1). By Theorem 1.1, we finish the proof. □

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