

Serrin-Type Overdetermined Problem in \mathbb{H}^n

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Abstract. In this paper, we prove the symmetry of the solution to overdetermined problem for the equation $\sigma_k(D^2u - uI) = C_n^k$ in hyperbolic space. Our approach is based on establishing a Rellich-Pohozaev type identity and using a P function. Our result generalizes the overdetermined problem for Hessian equation in Euclidean space.

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1 Introduction

In the seminal paper [1] Serrin established the symmetry of the solution to

$$\Delta u = n \tag{1.1}$$

in a bounded C^2 domain $\Omega \subset \mathbb{R}^n$ with

$$u = 0 \quad \text{and} \quad u_\gamma = 1 \quad \text{on} \quad \partial\Omega, \tag{1.2}$$

where γ is the unit outer normal to $\partial\Omega$. If $u \in C^2(\overline{\Omega})$ is a solution to (1.1) and (1.2), then $u = \frac{|x|^2 - 1}{2}$ up to a translation and Ω is the unitary ball. The proof is based on the method of *moving planes* and it can be applied to more general uniformly elliptic equations. In [2] Weinberger provided an alternative proof by using maximum principle for P function and a Rellich-Pohozaev type identity.

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There have been many generalizations of Serrin and Weinberger’s work to quasilinear elliptic equations (see, e.g., [3–5] and reference therein) and fully nonlinear equations such as Hessian equation and Weingarten curvature equation (see, e.g., [6–8]). In Euclidean space, the overdetermined boundary problem for $\sigma_k(D^2u) = C_n^k$ was studied in [6] by using a Rellich-Pohozaev type identity and some geometric inequalities and was also dealt in [8] by using method of moving planes. Using the P function $P = |Du|^2 - 2u$ as mentioned in [2,9] we can give an alternative proof which is parallel to Weinberger’s.

Theorem 1.1 ([6]). *Suppose $\Omega \subset \mathbb{R}^n$ is a C^2 bounded domain and $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ is a solution to the following problem*

$$\begin{cases} \sigma_k(D^2u) = C_n^k & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u_\gamma = c_0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

with $k \in \{1, \dots, n\}$ and c_0 a positive constant. Then upto a translation, $u = \frac{|x|^2 - 1}{2}$ and Ω is a ball of radius c_0 .

In space forms, a few work has been done to generalize the Serrin’s symmetry to equation $\Delta u + nKu = c$ using the method of moving planes or P functions and Rellich-Pohozaev type identities (see [10–14] and reference therein).

The hyperbolic space \mathbb{H}^n can be described as the warped product space $[0, \infty) \times S^{n-1}$ equipped with the rotationally symmetric metric

$$g = dr^2 + h^2 g_{S^{n-1}}, \tag{1.4}$$

where $h = \sinh r$, $g_{S^{n-1}}$ is the round metric on the $n - 1$ dimensional sphere.

In the present paper, we consider the overdetermined problem below in hyperbolic space,

$$\begin{cases} \sigma_k(D^2u - uI) = C_n^k & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u_\gamma = c_0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded C^2 domain of \mathbb{H}^n . Our result is the following:

Theorem 1.2. *Let $\Omega \subset \mathbb{H}^n$ be a C^2 bounded domain and $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ be a solution to (1.5) with $k \in \{1, \dots, n\}$ and c_0 a positive constant. Then Ω is a geodesic ball B_R , and u is radially symmetric.*

By maximum principle, $u < 0$ in Ω , and the solution to Dirichlet problem of $\sigma_k(D^2u - uI) = C_n^k$ is unique. In Theorem 1.2, if we assume the center of B_R is the origin, then $u(r) = \frac{\cosh r}{\cosh R} - 1$ is the unique solution to (1.5), where r is the distance from 0, R and c_0 satisfy the relationship $\frac{\sinh R}{\cosh R} = c_0$.

In the next section, we recall some notations concerning the Hessian equation in Euclidean space and hyperbolic space. Some known facts about curvature of level sets and Minkowskian integral formulas are also introduced there. In the third section, we use a P function and a Rellich-Pohozaev type identity for problem (1.3) to give an alternative proof of Theorem 1.1. In the last section, we derive the Rellich-Pohozaev type identity for problem (1.5), and combine with a P function to present the proof of Theorem 1.2.

2 Preliminary

2.1 Elementary symmetric functions

Let \mathcal{S}^n be the space of real symmetric $n \times n$ matrices. We denote by $A = (a_{ij})$ a matrix in \mathcal{S}^n , and by $\lambda_1, \dots, \lambda_n$ the eigenvalues of A . For $1 \leq k \leq n$, recall the definition of k -th elementary symmetric functions of A ,

$$\sigma_k(A) = \sigma_k(\lambda_1, \dots, \lambda_n) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}. \tag{2.1}$$

Denote by

$$\sigma_k^{ij}(A) := \frac{\partial \sigma_k(A)}{\partial a_{ij}}, \tag{2.2}$$

then it is easy to see from the definition above that

$$\sigma_k(A) = \frac{1}{k} \sum_{i,j=1}^n \sigma_k^{ij}(A) a_{ij} \quad \text{and} \quad (n-k+1)\sigma_{k-1}(A) = \sum_{i=1}^n \sigma_k^{ii}(A). \tag{2.3}$$

For $1 \leq k \leq n-1$, Newton's inequalities say that

$$(n-k+1)(k+1)\sigma_{k-1}(A)\sigma_{k+1}(A) \leq k(n-k)\sigma_k^2(A). \tag{2.4}$$

For $1 \leq k \leq n$, recall that the Garding cone is defined as

$$\Gamma_k = \{A \in \mathcal{S}^n : \sigma_1(A) > 0, \dots, \sigma_k(A) > 0\}. \tag{2.5}$$

Assume that $A \in \Gamma_k$, the following inequalities, known as MacLaurin inequalities hold,

$$\left(\frac{\sigma_k(A)}{C_n^k}\right)^{\frac{1}{k}} \leq \left(\frac{\sigma_l(A)}{C_n^l}\right)^{\frac{1}{l}}, \quad \forall k \geq l \geq 1. \tag{2.6}$$

The equalities hold if and only if the eigenvalues $\lambda_1, \dots, \lambda_n$ of A are equal to each other.

Assume that $A \in \Gamma_k$, the following inequalities also holds,

$$\frac{\sigma_k(A)/C_n^k}{\sigma_{k-1}(A)/C_n^{k-1}} \leq \frac{\sigma_l(A)/C_n^l}{\sigma_{l-1}(A)/C_n^{l-1}}, \quad \forall k \geq l \geq 1. \tag{2.7}$$

All the formulas are well known. For example, see [15] for a proof of (2.4), (2.6) and (2.7). The following proposition can be found in [16], see also [17] for non-symmetric matrices.

Proposition 2.1. *For any $n \times n$ matrix A , we have*

$$\sigma_k^{ij}(A) = \sigma_{k-1}(A)\delta_{ij} - \sum_{l=1}^n \sigma_{k-1}^{il}(A)a_{jl}. \quad (2.8)$$

In the following we write D, D^2 and Δ for the gradient, Hessian and Laplacian on hyperbolic space \mathbb{H}^n . For simplicity, we will use u_i, u_{ij}, \dots and u_γ to denote covariant derivatives and normal derivative of function u with respect to the metric on \mathbb{H}^n . We write $X \cdot Y$ instead of $g(X, Y)$ for vector fields X, Y . We also follow Einstein's summation convention.

2.2 Hessian operators

2.2.1 Hessian operators in Euclidean space

Let Ω be an open subset of \mathbb{R}^n and let $u \in C^2(\Omega)$. The k -Hessian operator $S_k[u]$ is defined as the k -th elementary symmetric function $\sigma_k(D^2u)$ of D^2u . Notice that

$$S_1[u] = \Delta u \quad \text{and} \quad S_n[u] = \det D^2u. \quad (2.9)$$

A function u is called k -convex in Ω , if $D^2u(x) \in \Gamma_k$ for any $x \in \Omega$. A direct computation yields that $(\sigma_k^{1j}(D^2u), \dots, \sigma_k^{nj}(D^2u))$ is divergence free, that is

$$\frac{\partial}{\partial x_i} \sigma_k^{ij}(D^2u) = 0. \quad (2.10)$$

By using (2.8), it is easy to see that

$$\sigma_k^{ij}(D^2u)u_{il} = \sigma_k^{il}(D^2u)u_{ij}. \quad (2.11)$$

2.2.2 Hessian operators in hyperbolic space

Let Ω be an open subset of \mathbb{H}^n and let $u \in C^2(\Omega)$. The k -Hessian operator $S_k[u]$ is defined as the k -th elementary symmetric function $\sigma_k(D^2u - uI)$ of $D^2u - uI$. Notice that

$$S_1[u] = \Delta u - nu \quad \text{and} \quad S_n[u] = \det(D^2u - uI). \quad (2.12)$$

A function u is called k -admissible in Ω if $D^2u(x) - u(x)I \in \Gamma_k$ for any $x \in \Omega$. A function $u \in C^2(\Omega)$ is called an *admissible solution* of $S_k[u] = f$ in Ω , if u solves the equation and u is an k -admissible function.

We list the following propositions with proofs, which will be used in the following sections.

Proposition 2.2. *Suppose $u \in C^3(\Omega)$, then*

$$D_i(\sigma_k^{ij}(D^2u - uI)) = 0. \quad (2.13)$$

Proof. We prove it by induction. For $k = 1$, (2.13) holds obviously. Suppose that (2.13) holds for $k - 1$, since $u_{ilj} = u_{ijl} - u_l\delta_{ij} + u_j\delta_{il}$, we have

$$\begin{aligned} D_j(\sigma_k^{ij}(D^2u - uI)) &= D_j(\sigma_{k-1}(D^2u - uI)\delta_{ij} - \sigma_{k-1}^{js}(D^2u - uI)(u_{si} - u\delta_{si})) \\ &= D_i\sigma_{k-1}(D^2u - uI) - \sigma_{k-1}^{js}(D^2u - uI)(u_{si} - u\delta_{si})_j \\ &= \sigma_{k-1}^{js}(D^2u - uI)((u_{js} - u\delta_{js})_i - (u_{si} - u\delta_{si})_j) \\ &= 0. \end{aligned} \tag{2.14}$$

This completes the proof. □

Proposition 2.3. *Let $u \in C^2(\Omega)$, then*

$$\sigma_k^{ij}(D^2u - uI)u_{il} = \sigma_k^{il}(D^2u - uI)u_{ij}. \tag{2.15}$$

Proof. By Proposition 2.1, we obtain

$$\begin{aligned} \sigma_k^{ij}(D^2u - uI)(u_{il} - u\delta_{il}) &= (\sigma_{k-1}(D^2u - uI)\delta_{ij} - \sigma_{k-1}^{is}(D^2u - uI)(u_{js} - u\delta_{js}))(u_{il} - u\delta_{il}) \\ &= (\sigma_{k-1}(D^2u - uI)\delta_{sl} - \sigma_{k-1}^{is}(D^2u - uI)(u_{il} - u\delta_{il}))(u_{js} - u\delta_{js}) \\ &= \sigma_k^{sl}(D^2u - uI)(u_{js} - u\delta_{js}). \end{aligned} \tag{2.16}$$

Thus,

$$\begin{aligned} \sigma_k^{ij}(D^2u - uI)u_{il} &= \sigma_k^{ij}(D^2u - uI)(u_{il} - u\delta_{il}) + \sigma_k^{lj}(D^2u - uI)u \\ &= \sigma_k^{il}(D^2u - uI)(u_{ij} - u\delta_{ij}) + \sigma_k^{lj}(D^2u - uI)u \\ &= \sigma_k^{il}(D^2u - uI)u_{ij}. \end{aligned} \tag{2.17}$$

This completes the proof. □

2.3 Minkowskian integral formulas

Let Ω be a C^2 bounded domain, and $\partial\Omega$ be the boundary of Ω . Denote the principal curvatures of $\partial\Omega$ by $\kappa = (\kappa_1, \dots, \kappa_{n-1})$. The k -th curvature of $\partial\Omega$ is defined as

$$H_k := \sigma_k(\kappa), \quad k = 1, \dots, n - 1. \tag{2.18}$$

Ω is called k -convex with $k \in \{1, \dots, n - 1\}$, if $H_i > 0$ for $i = 1, \dots, k$. In particular, $n - 1$ -convex is strictly convex, 1-convex is also called mean convex.

In the theory of convex bodies and differential geometry, Minkowskian integral formula (see [18–20]) is widely known. Suppose Ω is a domain of \mathbb{R}^n , then the Minkowskian integral formula says

$$\int_{\partial\Omega} \frac{H_k}{C_{n-1}^k} x \cdot \gamma d\sigma = \int_{\partial\Omega} \frac{H_{k-1}}{C_{n-1}^{k-1}} d\sigma. \tag{2.19}$$

Suppose Ω is a domain of \mathbb{H}^n . Let $p \in \Omega$, r be the distance from p . Let $V(x) = \cosh(r(x))$, then $D^2V = VI$ and $\Delta V = nV$. Then the Minkowskian integral formula (see [19]) says

$$\int_{\partial\Omega} \frac{H_k}{C_{n-1}^k} V_\gamma d\sigma = \int_{\partial\Omega} \frac{H_{k-1}}{C_{n-1}^{k-1}} V d\sigma. \tag{2.20}$$

2.4 Curvatures of level sets

Let u be a smooth function in space form $\mathbb{R}^n, \mathbb{S}^n$, or \mathbb{H}^n , for any regular $c \in \mathbb{R}$ of u (that is, $Du(x) \neq 0$ for any $x \in \mathbb{H}^n$ such that $u(x) = c$), the level set $\Sigma_c := u^{-1}(c)$ is a smooth hypersurface by the implicit function theorem. The k -th order curvature H_k of the level set Σ_c is given by

$$H_{k-1} = \frac{\sigma_k^{ij}(D^2u)u_iu_j}{|Du|^{k+1}}, \tag{2.21}$$

which can be found in [16, 21].

3 Overdetermined problem in \mathbb{R}^n

In this section, we present a Rellich-Pohozaev type identity for $\sigma_k(D^2u) = C_n^k$ with zero Dirichlet boundary condition, and use a P function to give a proof of theorem 1.1.

The following lemma is proven in [6], which implies the solution to (1.3) is k -convex. This ensures MacLaurin inequalities (2.6) can be applied.

Lemma 3.1 ([6]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2 domain and $u \in C^2(\overline{\Omega})$ is a solution to (1.3), then u is k -convex in Ω .*

P functions have been extensively investigated and inspired several effective works in the context of elliptic partial differential equations. For fully nonlinear equations, the P function for 2-dimensional Monge-Ampère equation was given by Ma [9], and the P functions for k -Hessian equations and k -curvature equations were given by Philippin and Safoui [22]. Based on Lemma 3.1, we are able to derive the lemma below, which was prove in [22]. So we can apply the maximum principle on the P function. For completeness, we present the proof.

Lemma 3.2 ([22]). *Let $u \in C^3(\Omega)$ be an admissible solution of $\sigma_k(D^2u) = C_n^k$ in $\Omega \subset \mathbb{R}^n$. Then the following P function*

$$P := |Du|^2 - 2u \tag{3.1}$$

satisfies

$$\sigma_k^{ij}(D^2u)P_{ij} \geq 0. \tag{3.2}$$

Proof. By direct computation, we have

$$\begin{aligned} \sigma_k^{ij}(D^2u)P_{ij} &= 2\sigma_k^{ij}(D^2u)(u_{li}u_{lj} + u_lu_{lij} - u_{ij}) = 2\sigma_k^{ij}(D^2u)u_{li}u_{lj} - 2k\sigma_k(D^2u) \\ &= 2(\sigma_1(D^2u)\sigma_k(D^2u) - (k+1)\sigma_{k+1}(D^2u) - k\sigma_k(D^2u)). \end{aligned} \tag{3.3}$$

If $\sigma_{k+1}(D^2u) > 0$, then $D^2u \in \Gamma_{k+1}$. By (2.7), we have

$$(k+1)\sigma_{k+1}(D^2u) \leq \frac{n-k}{n}\sigma_1(D^2u)\sigma_k(D^2u). \tag{3.4}$$

If $\sigma_{k+1} \leq 0$, (3.4) holds naturally. So by using (2.6), we get

$$\sigma_k^{ij}(D^2u)P_{ij} \geq 2k\sigma_k \left(\frac{\sigma_1(D^2u)}{n} - \frac{\sigma_k(D^2u)}{C_n^k} \right) \geq 0. \tag{3.5}$$

This completes the proof. □

The Rellich-Pohozaev type identity for Hessian equation has already been found [6, 23]. Brandolini, Nitsch, Salani and Trombetti [6] gave the Rellich-Pohozaev type identity for $\sigma_k(D^2u) = f(u)$ in Ω , with $u = 0$ on $\partial\Omega$,

$$\frac{n-2k}{k(k+1)} \int_{\Omega} \sigma_k^{ij}(D^2u)u_iu_j dx + \frac{1}{k+1} \int_{\partial\Omega} x \cdot \gamma |Du|^{k+1} H_{k-1} d\sigma = n \int_{\Omega} F(u) dx, \tag{3.6}$$

where $F(u) = \int_u^0 f(s) ds$. In their proof, they need to deal with the fourth order term $\frac{\partial^2}{\partial x_i \partial x_j} \sigma_k^{ij}(D^2u)$, which may be difficult in space forms. To overcome it, we come up with a different Rellich-Pohozaev type identity.

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2 domain. If $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a solution to the problem*

$$\begin{cases} \sigma_k(D^2u) = C_n^k & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Then

$$\frac{1}{2} \int_{\partial\Omega} \sigma_k^{ij}(D^2u)x_j\gamma_i |Du|^2 d\sigma = -kC_n^k \int_{\Omega} u dx + \frac{n-k+1}{2} \int_{\Omega} \sigma_{k-1}(D^2u) |Du|^2 dx. \tag{3.8}$$

Proof. Multiplying the equation (3.7) by u , we obtain

$$kC_n^k u = \sigma_k^{ij}(D^2u)u_{ij}u = \frac{1}{2}\sigma_k^{ij}(D^2u)u_{il}u(|x|^2)_{lj}. \tag{3.9}$$

By (2.10), we have

$$\begin{aligned} \sigma_k^{ij}(D^2u)u_{il}u(|x|^2)_{ij} &= (\sigma_k^{ij}(D^2u)u_{il}u(|x|^2)_i)_j - \sigma_k^{ij}(D^2u)u_{ilj}u(|x|^2)_l \\ &\quad - \sigma_k^{ij}(D^2u)u_{il}u_j(|x|^2)_i. \end{aligned} \tag{3.10}$$

It follows from (3.7) that

$$\sigma_k^{ij}(D^2u)u_{ijl}=0. \quad (3.11)$$

By (2.11), we have

$$\sigma_k^{ij}(D^2u)u_{il}u_j(|x|^2)_l=2\sigma_k^{il}(D^2u)u_{ij}u_jx_l=\sigma_k^{ij}(D^2u)(|Du|^2)_ix_j. \quad (3.12)$$

Putting (3.9)–(3.12) together, and integrating it on Ω , we find

$$kC_n^k \int_{\Omega} u dx = \frac{1}{2} \int_{\Omega} (\sigma_k^{ij}(D^2u)u_{il}u_j(|x|^2)_l)_j dx - \frac{1}{2} \int_{\Omega} \sigma_k^{ij}(D^2u)(|Du|^2)_ix_j dx. \quad (3.13)$$

It follows from divergence theorem and $u=0$ on $\partial\Omega$ that

$$kC_n^k \int_{\Omega} u dx = -\frac{1}{2} \int_{\Omega} \sigma_k^{ij}(D^2u)(|Du|^2)_ix_j dx. \quad (3.14)$$

Using (2.10) again, we get

$$\sigma_k^{ij}(D^2u)(|Du|^2)_ix_j = (\sigma_k^{ij}(D^2u)|Du|^2x_j)_i - (n-k+1)\sigma_{k-1}(D^2u)|Du|^2. \quad (3.15)$$

Substituting (3.15) into (3.14) and using divergence theorem again yield (3.8). \square

The following lemma help us to deal with the term of boundary integral in (3.8).

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a C^2 bounded domain, $u \in C^2(\overline{\Omega})$ be a solution to Eq. (1.3), then*

$$\sigma_k^{ij}(D^2u)x_j\gamma_i|Du|^2 = \sigma_k^{ij}(D^2u)u_iu_jx \cdot \gamma \quad \text{on } \partial\Omega. \quad (3.16)$$

Proof. By the boundary conditions of problem (1.3), we have on $\partial\Omega$ that

$$Du = u_{\gamma}\gamma, \quad (3.17)$$

and

$$(u_{\gamma})_j = (u_i\gamma_i)_j = u_{ij}\gamma_i + u_iD_j\gamma_i = u_{ij}\gamma_i + u_{\gamma}\gamma_iD_j\gamma_i = u_{ij}\gamma_i. \quad (3.18)$$

Then

$$(u_{\gamma})_{\gamma} = u_{ij}\gamma_i\gamma_j := u_{\gamma\gamma}. \quad (3.19)$$

Since $u_{\gamma} = c_0$ on $\partial\Omega$, for any tangential direction τ , we have $(u_{\gamma})_{\tau} = 0$, then

$$Du_{\gamma} = (u_{\gamma})_{\gamma}\gamma = u_{ij}\gamma_i\gamma_j\gamma. \quad (3.20)$$

Therefore,

$$u_{ij}\gamma_i = (u_{\gamma})_j = u_{kl}\gamma_k\gamma_l\gamma_j = (u_{\gamma})_{\gamma}\gamma_j = u_{\gamma\gamma}\gamma_j. \quad (3.21)$$

It follows that

$$u_{ij}u_i = u_{\gamma\gamma}u_j. \quad (3.22)$$

Now we derive (3.16) by induction. When $k=1$, (3.16) holds obviously since $\sigma_1^{ij}(D^2u) = \delta_{ij}$. Suppose (3.16) holds for $k-1$, then

$$\begin{aligned} \sigma_k^{ij}(D^2u)x_i\gamma_j|Du|^2 &= (\sigma_{k-1}(D^2u)\delta_{ij} - \sigma_{k-1}^{js}(D^2u)u_{si})x_j\gamma_i|Du|^2 \\ &= \sigma_{k-1}(D^2u)|Du|^2x \cdot \gamma - \sigma_{k-1}^{js}(D^2u)u_{\gamma\gamma}\gamma_sx_j|Du|^2 \\ &= \sigma_{k-1}(D^2u)|Du|^2x \cdot \gamma - \sigma_{k-1}^{ij}(D^2u)u_iu_jx \cdot \gamma u_{\gamma\gamma} \\ &= (\sigma_{k-1}(D^2u)|Du|^2 - \sigma_{k-1}^{ij}(D^2u)u_{ii}u_j) x \cdot \gamma \\ &= \sigma_k^{ij}(D^2u)u_iu_jx \cdot \gamma, \end{aligned} \tag{3.23}$$

where we use (2.8) in the first and last equality, (3.21) in the second equality, and (3.22) in the fourth equality. \square

Moreover, by the following lemma, the boundary integral can be turned into an integral on Ω .

Lemma 3.5. *Let $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ be a solution to the problem (1.3), then*

$$\int_{\partial\Omega} \sigma_k^{ij}(D^2u)u_iu_jx \cdot \gamma d\sigma = (n-k+1)c_0^2 \int_{\Omega} \sigma_{k-1}(D^2u) dx. \tag{3.24}$$

Proof. From (2.21),

$$\sigma_k^{ij}(D^2u)u_iu_jx \cdot \gamma = H_{k-1}|Du|^{k+1}x \cdot \gamma \quad \text{on } \partial\Omega. \tag{3.25}$$

Integrating it on $\partial\Omega$, and using Minkowskian integral formula (2.19), we obtain,

$$\int_{\partial\Omega} \sigma_k^{ij}(D^2u)u_iu_jx \cdot \gamma d\sigma = \int_{\partial\Omega} H_{k-1}|Du|^{k+1}x \cdot \gamma d\sigma = \frac{n-k+1}{k-1} \int_{\partial\Omega} H_{k-2}|Du|^{k+1} d\sigma. \tag{3.26}$$

By (2.21) and (3.17), we have

$$H_{k-2}|Du|^{k+1} = \sigma_{k-1}^{ij}(D^2u)u_iu_j|Du| = c_0^2\sigma_{k-1}^{ij}(D^2u)u_i\gamma_j \quad \text{on } \partial\Omega. \tag{3.27}$$

Applying the divergence theorem, we get

$$\int_{\partial\Omega} H_{k-2}|Du|^{k+1} d\sigma = c_0^2 \int_{\Omega} (\sigma_{k-1}^{ij}(D^2u)u_i)_j dx. \tag{3.28}$$

Since $\sigma_k^{ij}(D^2u)$ is divergence free, it follows that

$$(\sigma_{k-1}^{ij}(D^2u)u_i)_j = \sigma_{k-1}^{ij}(D^2u)u_{ij} = (k-1)\sigma_{k-1}(D^2u). \tag{3.29}$$

Putting (3.26), (3.28) and (3.29) together, we deduce (3.24). \square

Proof of Theorem 1.1. By Lemmas 3.3–3.5, we obtain

$$\frac{n-k+1}{2} \int_{\Omega} \sigma_{k-1}(D^2u)(|Du|^2 - c_0^2) dx = kC_n^k \int_{\Omega} u dx. \quad (3.30)$$

By maximum principle and Lemma 3.2, we have

$$P = |Du|^2 - 2u \leq c_0^2 \quad \text{in } \Omega. \quad (3.31)$$

Substituting it into (3.30), we get

$$kC_n^k \int_{\Omega} u dx \leq (n-k+1) \int_{\Omega} \sigma_{k-1}(D^2u) u dx. \quad (3.32)$$

On the other hand, by MacLaurin inequalities (2.6),

$$\sigma_{k-1}(D^2u) \geq C_n^{k-1} \left(\frac{\sigma_k(D^2u)}{C_n^k} \right)^{\frac{k-1}{k}} = C_n^{k-1} \quad \text{in } \Omega.$$

Since $u < 0$ in Ω , we have

$$(n-k+1)\sigma_{k-1}(D^2u)u \leq kC_n^k u. \quad (3.33)$$

It follows from (3.32) and (3.33) that

$$\sigma_{k-1}(D^2u) = C_n^{k-1} \quad \text{in } \Omega. \quad (3.34)$$

By MacLaurin inequalities (2.6), eigenvalues of D^2u are all equal to 1. Then $u = \frac{|x-x_0|^2 - c_0^2}{2}$ and $\Omega = B_{c_0}(x_0)$ for some x_0 . Hence we complete the proof of Theorem 1.1. \square

4 Overdetermined problem in \mathbb{H}^n

In this section, we present a Rellich-Pohozaev type identity for $\sigma_k(D^2u - uI) = C_n^k$ with zero Dirichlet boundary condition, and establish a differential inequality for the P function $P = |Du|^2 - u^2 - 2u$. After a similar argument as in the third section, we give a proof of Theorem 1.2.

Paralleled with Lemma 3.1, we prove the following lemma.

Lemma 4.1. *Let $\Omega \subset \mathbb{H}^n$ be a bounded C^2 domain and $u \in C^2(\overline{\Omega})$ is a solution to the problem (1.5), then u is k -admissible in Ω .*

Proof. The proof is almost the same as the proof in [6], where they prove that u is a k -convex function if it is a solution to the problem (1.3). From the boundedness and the smoothness of Ω , there exists a point $x_0 \in \partial\Omega$, the principal curvatures $\kappa_1, \dots, \kappa_{n-1}$ of $\partial\Omega$ at x_0 are nonnegative. By choosing a suitable coordinate system centered at x_0 , the first $n-1$

axes lay in the principal directions of curvature and the last one points in the direction of outer normal γ_{x_0} of $\partial\Omega$ at x_0 . Then D^2u has the following form:

$$D^2u = \begin{pmatrix} c_0\kappa_1 & \cdots & 0 & u_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & c_0\kappa_{n-1} & u_{n-1n} \\ u_{n1} & \cdots & u_{nn-1} & u_{nn} \end{pmatrix}. \tag{4.1}$$

Using the boundary condition in Eq. (1.5), a straightforward calculation shows that $D^2u(x_0)$ is in fact in diagonal form. Hence,

$$0 < S_k[u](x_0) = u_{nn}(x_0)c_0^{k-1}H_{k-1}(x_0) + c_0^kH_k(x_0). \tag{4.2}$$

Then from Newton inequalities (2.4), we deduce that

$$\begin{aligned} S_{k-1}[u](x_0) &= u_{nn}(x_0)c_0^{k-2}H_{k-2}(x_0) + c_0^{k-1}H_{k-1}(x_0) \\ &> -c_0^{k-1} \frac{H_{k-2}(x_0)H_k(x_0)}{H_{k-1}(x_0)} + c_0^{k-1}H_{k-1}(x_0) \geq 0. \end{aligned} \tag{4.3}$$

Repeat the process until we get $S_j[u](x_0) > 0$ for all $1 \leq j \leq k$. Hence $D^2u(x_0) - u(x_0)I \in \Gamma_k$. Since $u \in C^2(\overline{\Omega})$, by the argument of lemma in [24], u is k -admissible in Ω . \square

Distinct from the Euclidean case, the extra term $-uI$ in Eq. (1.5) prompts us to consider the P function $P = |Du|^2 - u^2 - 2u$, as in [10, 13].

Lemma 4.2. *Let $u \in C^3(\Omega)$ be an admissible solution of $\sigma_k(D^2u - uI) = C_n^k$ in $\Omega \subset \mathbb{H}^n$. Then the following P function*

$$\tilde{P} := |Du|^2 - u^2 - 2u \tag{4.4}$$

satisfies

$$\sigma_k^{ij}(D^2u - uI)\tilde{P}_{ij} \geq 0. \tag{4.5}$$

Proof. We may choose a suitable coordinate such that $g_{ij} = \delta_{ij}$ and $g_{ij,k} = 0$ at $p \in \Omega$. The following computation is done at p . Since

$$\begin{aligned} \frac{1}{2}(|Du|^2)_{ij} &= u_{li}u_{lj} + u_lu_{lij} = u_{li}u_{lj} + u_lu_{ijl} - u_lu_p(\delta_{pl}\delta_{ij} - \delta_{pj}\delta_{il}) \\ &= (u_{li} - u\delta_{li})(u_{lj} - u\delta_{lj}) + 2u(u_{ij} - u\delta_{ij}) + u^2\delta_{ij} + u_l(u_{ij} - u\delta_{ij})_l + u_iu_j, \end{aligned} \tag{4.6}$$

and

$$\frac{1}{2}(u^2)_{ij} = uu_{ij} + u_iu_j = u(u_{ij} - u\delta_{ij}) + u^2\delta_{ij} + u_iu_j. \tag{4.7}$$

We have

$$\begin{aligned}
\frac{1}{2}\sigma_k^{ij}(D^2u-uI)\tilde{P}_{ij} &= \sigma_k^{ij}(D^2u-uI)((u_{li}-u\delta_{li})(u_{lj}-u\delta_{lj})+u(u_{ij}-u\delta_{ij})) \\
&\quad + u_l(u_{ij}-u\delta_{ij})_l - (u_{ij}-u\delta_{ij}) - u\delta_{ij}) \\
&= (S_1[u]S_k[u] - (k+1)S_{k+1}[u] - kS_k[u]) \\
&\quad - u(-kS_k[u] + (n-k+1)S_{k-1}[u]) \\
&\geq 0,
\end{aligned} \tag{4.8}$$

where the first term in the third line is dealt as in Lemma 3.2 and the second term can be handled with by MacLaurin inequalities (2.6), since u is negative in Ω . \square

Now we establish the following Rellich-Pohozaev type identity.

Lemma 4.3. *Let $\Omega \subset \mathbb{H}^n$ be a bounded C^2 domain. Let $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ be a solution to the problem*

$$\begin{cases} \sigma_k(D^2u-uI) = C_n^k & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.9}$$

Then

$$\begin{aligned}
&\frac{1}{2} \int_{\partial\Omega} \sigma_k^{ij}(D^2u-uI)V_i\gamma_j|Du|^2 d\sigma \\
&= -\frac{n-k+1}{2} \int_{\Omega} \sigma_{k-1}(D^2u-uI)u^2V dx - kC_n^k \int_{\Omega} uV dx \\
&\quad + \frac{n-k+1}{2} \int_{\Omega} \sigma_{k-1}(D^2u-uI)|Du|^2V dx.
\end{aligned} \tag{4.10}$$

Proof. Multiplying the Eq. (4.9) by uV , we obtain

$$\begin{aligned}
kC_n^k uV &= \sigma_k^{ij}(D^2u-uI)(u_{ij}-u\delta_{ij})uV \\
&= \sigma_k^{ij}(D^2u-uI)u_{ij}uV - (n-k+1)\sigma_{k-1}(D^2u-uI)u^2V \\
&=: I + II.
\end{aligned} \tag{4.11}$$

Since $D^2V = VI$, by (2.13), we have

$$\begin{aligned}
I &:= \sigma_k^{ij}(D^2u-uI)u_{ij}uV = \sigma_k^{ij}(D^2u-uI)u_{il}uV_{lj} \\
&= (\sigma_k^{ij}(D^2u-uI)u_{il}uV_{lj})_j - \sigma_k^{ij}(D^2u-uI)u_{ilj}uV_l - \sigma_k^{ij}(D^2u-uI)u_{il}u_jV_l.
\end{aligned} \tag{4.12}$$

Using $u_{ilj} = u_{ijl} - u_l\delta_{ij} + u_j\delta_{il}$, it follows that

$$\begin{aligned}
 I &= (\sigma_k^{ij}(D^2u - uI)u_{il}uV_l)_j - \sigma_k^{ij}(D^2u - uI)(u_{ij} - u\delta_{ij})_l uV_l \\
 &\quad - \sigma_k^{ij}(D^2u - uI)u_j uV_i - \sigma_k^{ij}(D^2u - uI)u_{il}u_j V_i.
 \end{aligned}
 \tag{4.13}$$

Differentiating the equation $\sigma_k(D^2u - uI) = C_n^k$, we get

$$\sigma_k^{ij}(D^2u - uI)(u_{ij} - u\delta_{ij})_l = 0.
 \tag{4.14}$$

By (2.13), we compute that

$$\begin{aligned}
 \sigma_k^{ij}(D^2u - uI)u_j uV_i &= \frac{1}{2}\sigma_k^{ij}(D^2u - uI)(u^2)_j V_i \\
 &= \frac{1}{2}(\sigma_k^{ij}(D^2u - uI)u^2 V_i)_j - \frac{1}{2}\sigma_k^{ij}(D^2u - uI)u^2 V_{ij} \\
 &= \frac{1}{2}(\sigma_k^{ij}(D^2u - uI)u^2 V_i)_j - \frac{n-k+1}{2}\sigma_{k-1}(D^2u - uI)u^2 V.
 \end{aligned}
 \tag{4.15}$$

By (2.13) and (2.15), we also have that

$$\begin{aligned}
 \sigma_k^{ij}(D^2u - uI)u_{il}u_j V_l &= \sigma_k^{il}(D^2u - uI)u_{ij}u_j V_l = \frac{1}{2}\sigma_k^{ij}(D^2u - uI)(|Du|^2)_i V_j \\
 &= \frac{1}{2}(\sigma_k^{ij}(D^2u - uI)|Du|^2 V_j)_i - \frac{n-k+1}{2}\sigma_{k-1}(D^2u - uI)|Du|^2 V.
 \end{aligned}
 \tag{4.16}$$

Substituting (4.13)–(4.16) into (4.11), we obtain

$$\begin{aligned}
 kC_n^k uV &= (\sigma_k^{ij}(D^2u - uI)u_{il}uV_l)_j - (n-k+1)\sigma_{k-1}(D^2u - uI)u^2 V \\
 &\quad - \frac{1}{2}(\sigma_k^{ij}(D^2u - uI)u^2 V_i)_j + \frac{n-k+1}{2}\sigma_{k-1}(D^2u - uI)u^2 V \\
 &\quad - \frac{1}{2}(\sigma_k^{ij}(D^2u - uI)|Du|^2 V_j)_i + \frac{n-k+1}{2}\sigma_{k-1}(D^2u - uI)|Du|^2 V.
 \end{aligned}
 \tag{4.17}$$

Applying divergence theorem and noting that $u = 0$ on $\partial\Omega$, we deduce

$$\begin{aligned}
 kC_n^k \int_{\Omega} uV dx &= \frac{n-k+1}{2} \int_{\Omega} \sigma_{k-1}(D^2u - uI)|Du|^2 V dx - \frac{n-k+1}{2} \int_{\Omega} \sigma_{k-1}(D^2u - uI)u^2 V dx \\
 &\quad - \frac{1}{2} \int_{\partial\Omega} \sigma_k^{ij}(D^2u - uI)|Du|^2 V_j \gamma_i d\sigma.
 \end{aligned}
 \tag{4.18}$$

This completes the proof. □

To handle with the boundary integral term, we need the following lemma. We omit the proof which is the same as Lemma 3.4.

Lemma 4.4. Let $\Omega \subset \mathbb{H}^n$ be a C^2 bounded domain, $u \in C^3(\overline{\Omega})$ be a solution to problem (1.5), then

$$\sigma_k^{ij}(D^2u - uI)V_j\gamma_i|Du|^2 = \sigma_k^{ij}(D^2u - uI)u_iu_jV_\gamma \quad \text{on } \partial\Omega. \quad (4.19)$$

Furthermore, we turn the boundary integral into an integral on Ω .

Lemma 4.5. Let $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ be a solution to problem (1.5), then

$$\int_{\partial\Omega} \sigma_k^{ij}(D^2u - uI)u_iu_jV_\gamma d\sigma = (n-k+1)c_0^2 \int_{\Omega} \sigma_{k-1}(D^2u - uI)V dx. \quad (4.20)$$

Proof. Using Minkowskian integral formula (2.19) and (2.21), we obtain

$$\begin{aligned} \int_{\partial\Omega} \sigma_k^{ij}(D^2u)u_iu_jV_\gamma d\sigma &= \int_{\partial\Omega} H_{k-1}|Du|^{k+1}V_\gamma d\sigma = \frac{n-k+1}{k-1} \int_{\partial\Omega} H_{k-2}|Du|^{k+1}V d\sigma \\ &= \frac{n-k+1}{k-1} \int_{\partial\Omega} \sigma_{k-1}^{ij}(D^2u)u_iu_j|Du|V d\sigma. \end{aligned} \quad (4.21)$$

Since $u=0$ and $|Du|=u_\gamma=c_0$ on $\partial\Omega$, we have

$$\begin{aligned} \int_{\partial\Omega} \sigma_{k-1}^{ij}(D^2u)u_iu_j|Du|V d\sigma &= c_0^2 \int_{\partial\Omega} \sigma_{k-1}^{ij}(D^2u - uI)u_iV_\gamma d\sigma \\ &= c_0^2 \int_{\Omega} (\sigma_{k-1}^{ij}(D^2u - uI)u_iV)_j dx. \end{aligned} \quad (4.22)$$

Applying Proposition 2.2, we get

$$(\sigma_{k-1}^{ij}(D^2u - uI)u_iV)_j = \sigma_{k-1}^{ij}(D^2u - uI)u_{ij}V + \sigma_{k-1}^{ij}(D^2u - uI)u_iV_j. \quad (4.23)$$

By computation,

$$\sigma_{k-1}^{ij}(D^2u - uI)u_iV_j = (\sigma_{k-1}^{ij}(D^2u - uI)uV_j)_i - \sigma_{k-1}^{ij}(D^2u - uI)uV\delta_{ij}. \quad (4.24)$$

Substituting (4.23) and (4.24) into (4.22), and noting that $u=0$ on $\partial\Omega$, it follows that

$$\begin{aligned} \int_{\partial\Omega} \sigma_{k-1}^{ij}(D^2u)u_iu_j|Du|V d\sigma &= c_0^2 \int_{\Omega} \sigma_{k-1}^{ij}((D^2u - uI)(u_{ij} - u\delta_{ij}))V dx \\ &= (k-1)c_0^2 \int_{\Omega} \sigma_{k-1}(D^2u - uI)V dx. \end{aligned} \quad (4.25)$$

Putting (4.21) and (4.25) together, we obtain (4.20). \square

Proof of Theorem 1.2. Combining Lemmas 4.3, 4.4 with Lemma 4.5, we obtain

$$kC_n^k \int_{\Omega} uV dx = \frac{n-k+1}{2} \int_{\Omega} \sigma_{k-1}(D^2u - uI)V(|Du|^2 - u^2 - c_0^2) dx. \quad (4.26)$$

By Lemma 4.2, maximum principle can be applied to \tilde{P} , thus

$$\tilde{P} = |Du|^2 - u^2 - 2u \leq c_0^2 \quad \text{in } \Omega. \quad (4.27)$$

Putting it into (4.26), we deduce

$$kC_n^k \int_{\Omega} Vudx \leq (n-k+1) \int_{\Omega} \sigma_{k-1}(D^2u - uI) Vudx. \quad (4.28)$$

On the other hand, using MacLaurin inequalities (2.6), we have

$$\sigma_{k-1}(D^2u - uI) \geq C_n^{k-1} \left(\frac{\sigma_k(D^2u - uI)}{C_n^k} \right)^{\frac{k-1}{k}} = C_n^{k-1}. \quad (4.29)$$

Since $u < 0$ in Ω , we get

$$(n-k+1)\sigma_{k-1}(D^2u - uI)u \leq kC_n^k u \quad \text{in } \Omega. \quad (4.30)$$

It follows from (4.28) and (4.30) that

$$\sigma_{k-1}(D^2u - uI) = C_n^{k-1} \quad \text{in } \Omega. \quad (4.31)$$

By MacLaurin inequalities (2.6), eigenvalues of $D^2u - uI$ are all equal to 1. Follows from an Obata type result ([25], See also [10, 26, 27]), Ω must be a ball B_R and u depends only on the distance from the center of B_R , where $R = \tanh^{-1} c_0$. It is easy to see that u is of the form

$$u = \frac{\cosh r}{\cosh R} - 1. \quad (4.32)$$

Hence we complete the proof of Theorem 1.2. \square

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