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Serrin-Type Overdetermined Problems for Hessian Quotient Equations and Hessian Quotient Curvature Equations

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Abstract

We consider overdetermined problems for Hessian quotient equations and Hessian quotient curvature equations, which are fully nonlinear elliptic equations. We establish Rellich–Pohozaev-type identities for Hessian quotient equations and Hessian quotient curvature equations. Based on these identities and the maximum principle for P functions, the symmetry of solutions can be proved in the Euclidean space. We also prove the related result for Hessian quotient equations in the hyperbolic space. Our results generalize the overdetermined problems for k-Hessian equations and k-curvature equations.

Keywords Overdetermined problem · Hessian quotient equation · Hessian quotient curvature equation · P functions · Rellich–Pohozaev identity

Mathematics Subject Classification 35J60 · 35N25

1 Introduction

Serrin [20] considered the following symmetry problem:

$$\begin{cases} \Delta u = n & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ \frac{\partial u}{\partial \gamma} = 1 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is C^2 bounded domain in $\mathbb{R}^n$ and γ is the unit outer normal to $\partial\Omega$. If $u \in C^2(\overline{\Omega})$ solves (1.1), Serrin [20] proved that up to a translation $u = \frac{|x|^2-1}{2}$ and Ω is the unit ball. Serrin's proof was based on the moving plane method and it was applied to more general uniformly elliptic equations. Based on the maximum principle for the P function and a Rellich–Pohozaev type identity, Weinberger [23] gave another proof.

There were lots of generalizations of Serrin and Weinberger's work to quasilinear elliptic equations (see e.g. [6, 7, 9] and references therein) and fully nonlinear equations such as the k-Hessian equation and Weingarten curvature equations (see e.g. [1, 12, 22]). In the Euclidean space, Brandolini–Nitsch–Salani–Trombetti [1] solved the overdetermined problem for the k-Hessian equation i.e. $S_k(D^2u) = C_n^k$ using a Rellich–Pohozaev type identity and Newton inequalities. The same problem was later proved by Bao–Wang [22] where they used the method of moving planes.

P functions are very useful in the study of elliptic partial differential equations. For example, Ma [13] gave the P function for 2-dimensional Monge–Ampère equation. For k-Hessian equations and k-curvature equations, P functions were given by Philippin and Safoui [15].

Let Ω be a bounded C^2 domain and γ be the unit outer normal to $\partial\Omega$. Let k, l be integers such that $0 \le l < k \le n$. In the first part of this paper, we consider the following overdetermined problem for Hessian quotient equations in the Euclidean space $\mathbb{R}^n$,

$$\begin{cases} S_k(D^2u) = \frac{C_n^k}{C_n^l} S_l(D^2u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \gamma} = 1 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $S_k(D^2u)$ is the k-th elementary symmetric function of D^2u (see Sect. 2).

Our first result is the following.

Theorem 1.1 *Let Ω be a C^2 bounded domain in $\mathbb{R}^n$, $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ be a solution to (1.2) with the integer k, l satisfying $0 \le l < k \le n$ and $S_l(D^2u) > 0$ in $\overline{\Omega}$. Then up to a translation $u = \frac{|x|^2-1}{2}$ and Ω is the unit ball with the center at 0.*

Since $S_l(D^2u) > 0$, similar as the argument in [1], we can prove that u is k-convex which means $S_i(D^2u) > 0$, $1 \le i \le k$ in $\overline{\Omega}$. Then by maximum principle, $u < 0$ in Ω, and the solution to Dirichlet problem of $S_k(D^2u) = \frac{C_n^k}{C_n^l} S_l(D^2u)$ is unique.

In the second part, we consider the following Hessian quotient type equations in the hyperbolic space $\mathbb{H}^n$,

$$\begin{cases} S_k(D^2u - uI) = \frac{C_n^k}{C_n^l} S_l(D^2u - uI) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \gamma} = 1 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Our result is as follows.

Theorem 1.2 *Let Ω be a C^2 bounded domain in $\mathbb{H}^n$, $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ be a solution to* (1.3) *with the integer k, l satisfying $0 \le l < k \le n$ and $S_l(D^2u - uI) > 0$ in $\overline{\Omega}$. Then, up to a translation $u = \frac{\cosh r}{\cosh R} - 1$ and Ω is the ball of radius* $\tanh^{-1} 1$ *with the center at* 0.

In the third part, we consider Hessian quotient curvature equations in the Euclidean space $\mathbb{R}^n$,

$$\begin{cases} S_k(D(\frac{Du}{w})) = \frac{C_n^k}{C_n^l} S_l(D(\frac{Du}{w})) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \gamma} = 1 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

where $w = \sqrt{1 + |Du|^2}$.

Our result is as follows.

Theorem 1.3 *Let Ω be a C^2 bounded domain in $\mathbb{R}^n$, $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ be a solution to* (1.4) *with the integer k, l satisfying $0 \le l < k \le n$ and $S_l(D(\frac{Du}{w})) > 0$ in $\overline{\Omega}$. Then up to a translation $u = -\sqrt{1 - |x|^2} + \frac{1}{\sqrt{2}}$ and Ω is the ball of radius $\frac{1}{\sqrt{2}}$.*

The organization of this paper is as follows. In Sect. 2, we recall some preliminaries for the Hessian operator in the Euclidean space and the hyperbolic space and some known facts about Weingarten hypersurfaces in the Euclidean space. In Sect. 3, we first prove a Rellich–Pohozae-type identity for Problem (1.2) in the Euclidean space. Then by the Rellich–Pohozaev-type identity and the P function, we prove the Theorem 1.1. In Sect. 4, we derive the Rellich–Pohozaev-type identity for Problem (1.3) in the hyperbolic space, then combing with a P function, we give the proof of Theorem 1.2. In the last section, we prove Theorem 1.3.

2 Preliminaries

2.1 Elementary Symmetric Functions

We denote by $A = (a_{ij})$ a matrix in $\mathbb{R}^{n \times n}$. Recall the definition of k-th elementary symmetric functions of A,

$$S_k(A) = \frac{1}{k!} \sum_{1 \le i_1, \ldots, i_k, j_1, \ldots, j_k \le n} \delta_{i_1 \ldots i_k}^{j_1 \ldots j_k} A_{i_1 j_1} \ldots A_{i_k j_k}.$$

where $\delta_{i_1 \ldots i_k}^{j_1 \cdots j_k}$ is the Kronecker symbol. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$,

$$S_k(\lambda) = S_k(\lambda_1, \ldots, \lambda_n) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \ldots \lambda_{i_k}.$$

It is clear that $\S_k(A) = \S_k(\lambda(A))$ provided that the eigenvalues $\lambda(A)$ of A are all real.

Denote by

$$S_k^{ij}(A) := \frac{\partial S_k(A)}{\partial a_{ij}},$$

then it is easy to see from the definition above that

$$\sum_{i,j=1}^{n} S_k^{ij}(A) a_{ij} = k S_k(A), \quad \text{and} \quad \sum_{i=1}^{n} S_k^{ii}(A) = (n-k+1) S_{k-1}(A). \tag{2.1}$$

For $1 \leq k \leq n-1$, we have the following Newton's inequality

$$(n-k+1)(k+1) S_{k-1}(A) S_{k+1}(A) \leq k(n-k) S_k^2(A). \tag{2.2}$$

For $1 \leq k \leq n$, recall that the Gårding cone is defined as

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : S_1(\lambda) > 0, \ldots, S_k(\lambda) > 0\}.$$

For $\lambda(A) \in \Gamma_k$ and $k > l \geq 0$, $r > s \geq 0$, $k \geq r$, $l \geq s$, we have the following Maclaurin's inequality.

$$\left(\frac{S_k(A)/C_n^k}{S_l(A)/C_n^l}\right)^{\frac{1}{k-l}} \leq \left(\frac{S_r(A)/C_n^r}{S_s(A)/C_n^s}\right)^{\frac{1}{r-s}}. \tag{2.3}$$

In particular,

$$\left(\frac{S_k(A)}{C_n^k}\right)^{\frac{1}{k}} \leq \left(\frac{S_l(A)}{C_n^l}\right)^{\frac{1}{l}}, \quad \forall\, k \geq l \geq 1, \tag{2.4}$$

and

$$\frac{S_k(A)/C_n^k}{S_{k-1}(A)/C_n^{k-1}} \leq \frac{S_l(A)/C_n^l}{S_{l-1}(A)/C_n^{l-1}}, \quad \forall\, k \geq l \geq 1. \tag{2.5}$$

By (2.3), we also have

$$\frac{S_{k+1}(A)/C_n^{k+1}}{S_k(A)/C_n^k} \leq \left(\frac{S_k(A)/C_n^k}{S_l(A)/C_n^l}\right)^{\frac{1}{k-l}} \leq \frac{S_{l+1}(A)/C_n^{l+1}}{S_l(A)/C_n^l}. \tag{2.6}$$

The equalities in (2.3), (2.4), (2.5) and (2.6) hold if the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A are equal to each other. The following proposition in [17] is very useful.

Proposition 2.1 *For any $n \times n$ symmetric matrix A, we have*

$$S_k^{ij}(A) = S_{k-1}(A)\delta_{ij} - \sum_{l=1}^{n} S_{k-1}^{il}(A) a_{jl}. \tag{2.7}$$

In the following, we write D, D^2 and Δ for the gradient, Hessian and Laplacian on $\mathbb{R}^n$. We also follow Einstein's summation convention.

2.2 Hessian Operators

2.2.1 Hessian Operators in Euclidean Space

Let Ω be an open subset of $\mathbb{R}^n$ and let $u \in C^2(\Omega)$. The k-Hessian operator $S_k(D^2u)$ is defined as the k-th elementary symmetric function of D^2u. Notice that

$$S_1(D^2u) = \Delta u \quad \text{and} \quad S_n(D^2u) = \det D^2u. \tag{2.8}$$

A function u is called k-convex in Ω, if $D^2u(x) \in \Gamma_k$ for any $x \in \Omega$. A direct computation yields that $(S_k^{1j}(D^2u), \ldots, S_k^{nj}(D^2u))$ is divergence free, that is

$$\frac{\partial}{\partial x_i} S_k^{ij}(D^2u) = 0. \tag{2.9}$$

By using (2.7), it is easy to see that

$$S_k^{ij}(D^2u)u_{il} = S_k^{il}(D^2u)u_{ij}. \tag{2.10}$$

2.2.2 Hessian Operators in Hyperbolic Space

Let Ω be an open subset of $\mathbb{H}^n$ and let $u \in C^2(\Omega)$. The k-Hessian operator $S_k[u]$ is defined as the k-th elementary symmetric function $S_k(D^2u - uI)$ of $D^2u - uI$. Notice that

$$S_1[u] = \Delta u - nu \quad \text{and} \quad S_n[u] = \det(D^2u - uI). \tag{2.11}$$

A function u is called *k-admissible* in Ω if $\lambda(D^2u(x) - u(x)I) \in \Gamma_k$ for any $x \in \Omega$. We list the following propositions proven in [8], which will be used in Sect. 4.

Proposition 2.2 *Suppose $u \in C^3(\Omega)$, then*

$$D_i(S_k^{ij}(D^2u - uI)) = 0.$$

Proposition 2.3 *Let $u \in C^2(\Omega)$, then*

$$S_k^{ij}(D^2u - uI)u_{il} = S_k^{il}(D^2u - uI)u_{ij}.$$

2.3 Weingarten Hypersurfaces

Let $\mathcal{M}$ be a hypersurface in $\mathbb{R}^{n+1}$, which is locally represented as a graph $x_{n+1} = u(x)$, $x = (x_1, \ldots, x_n) \in \Omega$, where $\Omega \subset \mathbb{R}^n$. Then $\nu = \frac{(-Du,1)}{\sqrt{1+|Du|^2}}$ is the unit outer normal of $\mathcal{M}$. Recall that $w = \sqrt{1+|Du|^2}$. The first and second fundamental forms can be respectively expressed as

$$g_{ij} = \delta_{ij} + u_i u_j, \quad \text{and} \quad b_{ij} = \frac{u_{ij}}{w}.$$

Then, the principal curvatures $\lambda_1, \ldots, \lambda_n$ of $\mathcal{M}$ are the eigenvalues of the second fundamental form with respect to the first fundamental form i.e. $a_{ij} := g^{ik} b_{kj}$, where $g^{ij} = \delta_{ij} - \frac{u_i u_j}{w^2}$ is the inverse matrix to g_{ij}. Hence

$$a_{ij} = g^{ik} b_{kj} = \frac{u_{ij}}{w} - \frac{u_i u_k u_{kj}}{w^3} = \left(\frac{u_i}{w}\right)_j.$$

We say u is k-admissible if $\lambda(a_{ij}) \in \Gamma_k$.

The following divergence free property is due to Reilly [17].

Proposition 2.4 *Suppose $u \in C^3(\Omega)$, $A_{ij} = (\frac{u_i}{w})_j$. Then,*

$$D_j S_k^{ij}(A) = 0.$$

For general (non-symmetric) matrices, we need the following formula proved by Pietra–Gavitone–Xia [5].

Proposition 2.5 *For any $n \times n$ matrix $A = (a_{ij})$, we have*

$$S_k^{ij}(A) = S_{k-1}(A)\delta_{ij} - \sum_{l=1}^{n} S_{k-1}^{il}(A) a_{jl}.$$

The following proposition can be inferred from above proposition, we omit the proof.

Proposition 2.6 *For any $n \times n$ matrix $A = (a_{ij})$, we have*

$$S_k^{il}(A) a_{jl} = S_k^{lj}(A) a_{li}.$$

2.4 Minkowskian Integral Formulas

Let Ω be a C^2 bounded domain, and $\partial\Omega$ is the boundary of Ω. Denote the principle curvatures of $\partial\Omega$ by $\kappa = (\kappa_1, \ldots, \kappa_{n-1})$. For $1 \le k \le n-1$, the k-th curvature of $\partial\Omega$ is defined as

$$H_k := S_k(\kappa).$$

Ω is called *k-convex*, if $H_i > 0$ for all $1 \le i \le k$. In particular, $(n-1)$-convex is strictly convex, 1-convex is also called mean convex.

We refer to [10, 11, 19] for Minkowskian integral formula in $\mathbb{R}^n$ and $\mathbb{H}^n$. Suppose Ω is a bounded C^2 domain of $\mathbb{R}^n$, then the Minkowskian integral formula says

$$\int_{\partial\Omega} \frac{H_k}{C_{n-1}^k} x \cdot \gamma \mathrm{d}\sigma = \int_{\partial\Omega} \frac{H_{k-1}}{C_{n-1}^{k-1}} \mathrm{d}\sigma. \tag{2.12}$$

Suppose Ω is a domain of $\mathbb{H}^n$. Let $p \in \Omega$, r be the distance from p. Let $V(x) = \cosh(r(x))$. Then the Minkowskian integral formula says

$$\int_{\partial\Omega} \frac{H_k}{C_{n-1}^k} V_\gamma \mathrm{d}\sigma = \int_{\partial\Omega} \frac{H_{k-1}}{C_{n-1}^{k-1}} V \mathrm{d}\sigma. \tag{2.13}$$

2.5 Curvatures of Level Sets

Let u be a smooth function in $\mathbb{R}^n$ or $\mathbb{H}^n$, for any regular $c \in \mathbb{R}$ of u (that is, $Du(x) \neq 0$ for any $x \in \mathbb{R}^n$ such that $u(x) = c$), the level set $\Sigma_c := u^{-1}(c)$ is a smooth hypersurface by the implicit function theorem. The k-th order curvature H_k of the level set Σ_c is given by

$$H_{k-1} = \frac{S_k^{ij} u_i u_j}{|Du|^{k+1}}, \tag{2.14}$$

which can be found in [14, 17].

At the last of this section, we introduce some notations. For convenience, we use S_m and S_m^{ij} instead of $S_m(D^2u)$ and $S_m^{ij}(D^2u)$ in Sect. 3, instead of instead of $S_m(D^2u - uI)$ and $S_m^{ij}(D^2u - uI)$ in Sect. 4, instead of $S_m(D(\frac{Du}{w}))$ and $S_m^{ij}(D(\frac{Du}{w}))$ in Sect. 5.

3 Overdetermined Problem for Hessian Quotient Equations in Euclidean Space

In this section, we present a Rellich–Pohozaev type identity for Hessian quotient equations in $\mathbb{R}^n$ with zero Dirichlet boundary condition, and use a P-function to give a proof of Theorem 1.1.

The following type lemma was proven in [1] for k-Hessian equation, which implies the solution to overdetermined problem for k-Hessian equation is k-convex. We also have the following lemma for Hessian quotient equation, which ensures MacLaurin inequalities (2.4) can be applied.

Lemma 3.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2 domain and $u \in C^2(\overline{\Omega})$ is a solution to (1.2) with $S_l > 0$ on $\overline{\Omega}$, then u is k-convex in Ω.*

Remark 3.2 The proof is almost the same as that in [1], except that a key inequality turns into

$$0 < \frac{1}{S_l}(u_{nn}H_{k-1} + H_k).$$

So we need $S_l > 0$ to ensure that $u_{nn} \geq -\frac{H_k}{H_{k-1}}$.

Based on Lemma 3.1, we are able to derive the lemma below, which was proved in [15]. So we can apply the maximum principle on the P function. For completeness, we present the proof.

Lemma 3.3 ([15]) *Let $u \in C^3(\Omega)$ be an admissible (i.e. u is k-convex) solution of*

$$S_k = \frac{C_n^k}{C_n^l} S_l \quad in \ \ \Omega \subset \mathbb{R}^n, \tag{3.1}$$

with $0 \leq l < k \leq n$. Then, the maximum of $P := |Du|^2 - 2u$ is attained only on the boundary $\partial\Omega$ unless P is constant.

Proof Denote by

$$F^{ij} := \frac{\partial}{\partial u_{ij}} \frac{S_k}{S_l} = \frac{1}{S_l^2}(S_k^{ij} S_l - S_k S_l^{ij}).$$

By (2.1), we have

$$F^{ij} u_{ij} = (k-l)\frac{S_k}{S_l},$$
$$F^{ij} u_{si} u_{sj} = \frac{1}{S_l^2}\big((l+1)S_{l+1}S_k - (k+1)S_{k+1}S_l\big).$$

Differentiating the Eq. (3.1), we have

$$F^{ij} u_{ijs} = 0.$$

Then, we have

$$\begin{aligned} F^{ij} P_{ij} &= 2F^{ij}\big(u_{si}u_{sj} + u_s u_{sij} - u_{ij}\big) \\ &= 2F^{ij} u_{si}u_{sj} - 2F^{ij}u_{ij} \\ &= \frac{2S_k}{S_l}\Big((l+1)\frac{S_{l+1}}{S_l} - (k+1)\frac{S_{k+1}}{S_k} - (k-l)\Big). \end{aligned} \tag{3.2}$$

By (2.6) and (3.1), we have

$$\frac{S_{l+1}/C_n^{l+1}}{S_l/C_n^l} \geq \left(\frac{S_k/C_n^k}{S_l/C_n^l}\right)^{\frac{1}{k-l}} = 1, \tag{3.3}$$

and

$$\frac{S_{k+1}/C_n^{k+1}}{S_k/C_n^k} \leq \left(\frac{S_k/C_n^k}{S_l/C_n^l}\right)^{\frac{1}{k-l}} = 1. \tag{3.4}$$

That is

$$(l+1)\frac{S_{l+1}}{S_l} \geq n-l, \tag{3.5}$$

and

$$(k+1)\frac{S_{k+1}}{S_k} \leq n-k. \tag{3.6}$$

Plugging (3.5) and (3.6) into (3.2), we obtain

$$F^{ij}P_{ij} \geq 0.$$

By maximum principle, the maximum of P is attained on $\partial\Omega$. □

The Rellich–Pohozaev type identity for k-Hessian equation has already been found [1, 21]. Brandolini–Nitsch–Salani–Trombetti [1] gave the Rellich–Pohozaev type identity for $S_k = f(u)$ in Ω, with $u = 0$ on $\partial\Omega$,

$$\frac{n-2k}{k(k+1)}\int\limits_\Omega S_k^{ij}u_iu_j\mathrm{d}x + \frac{1}{k+1}\int\limits_{\partial\Omega} x\cdot\gamma|Du|^{k+1}H_{k-1}\mathrm{d}\sigma = n\int\limits_\Omega F(u)\mathrm{d}x,$$

where $F(u) = \int\limits_u^0 f(s)\mathrm{d}s$.

For Hessian quotient equations, we first prove identities for $\int\limits_\Omega S_k u$ and $\int\limits_\Omega S_l u$ in which the bad terms $\int\limits_\Omega \partial_s(S_k)x_s u$ and $\int\limits_\Omega \partial_s(S_l)x_s u$ arise. By differentiating the Hessian quotient equation, these two terms can be cancelled. Then we can prove the following Rellich-Pohozaev type identity.

Lemma 3.4 *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2 domain, $0 \leq l < k \leq n$. If $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ is a solution to the problem*

$$\begin{cases} S_k = \frac{C_n^k}{C_n^l}S_l & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Suppose $S_l > 0$ on $\overline{\Omega}$, then

$$(n-k+1)C_n^l\int\limits_\Omega S_{k-1}|Du|^2\mathrm{d}x - (n-l+1)C_n^k\int\limits_\Omega S_{l-1}|Du|^2\mathrm{d}x$$

$$
-C_n^l\int_{\partial\Omega} S_k^{ij}|Du|^2 x_i\gamma_j \mathrm{d}\sigma + C_n^k\int_{\partial\Omega} S_l^{ij}|Du|^2 x_i\gamma_j \mathrm{d}\sigma - 2(k-l)C_n^l\int_{\Omega} S_k u = 0. \tag{3.8}
$$

Proof By direct computation,

$$
kS_k u = S_k^{ij}u_{ij} = S_k^{ij}u_{is}u(x_s)_j = \left(S_k^{ij}u_{is}ux_s\right)_j - x_s\partial_s S_k u - S_k^{ij}u_{is}u_jx_s. \tag{3.9}
$$

By (2.10), we have

$$
S_k^{ij}u_{is}u_jx_s = S_k^{is}u_{ij}u_jx_s = \frac{1}{2}S_k^{ij}(|Du|^2)_ix_j. \tag{3.10}
$$

Using (2.1) and (2.9), we get

$$
S_k^{ij}(|Du|^2)_ix_j = (S_k^{ij}|Du|^2x_j)_i - (n-k+1)S_{k-1}|Du|^2. \tag{3.11}
$$

Putting (3.9), (3.10) and (3.11) together, we find

$$
2x_s\partial_s S_k u = (S_k^{ij}u_{is}u(|x|^2)_s)_j - (S_k^{ij}|Du|^2x_j)_i + (n-k+1)S_{k-1}|Du|^2 - 2kS_ku. \tag{3.12}
$$

Similarly, we obtain

$$
2x_s\partial_s S_l u = (S_l^{ij}u_{is}u(|x|^2)_s)_j - (S_l^{ij}|Du|^2x_j)_i + (n-l+1)2S_{l-1}|Du|^2 - 2lS_lu. \tag{3.13}
$$

Differentiating the Eq. (3.7), we have

$$
C_n^l\partial_s S_k = C_n^k\partial_s S_l. \tag{3.14}
$$

By (3.7), (3.12), (3.13) and (3.14), we obtain

$$
\begin{aligned}
&(n-k+1)C_n^l\int_{\Omega} S_{k-1}|Du|^2\mathrm{d}x - (n-l+1)C_n^k\int_{\Omega} S_{l-1}|Du|^2\mathrm{d}x \\
&\quad - C_n^l\int_{\partial\Omega} S_k^{ij}|Du|^2x_i\gamma_j\mathrm{d}\sigma + C_n^k\int_{\partial\Omega} S_l^{ij}|Du|^2x_i\gamma_j\mathrm{d}\sigma - 2(k-l)C_n^l\int_{\Omega} S_ku = 0
\end{aligned}
$$

□

The following two lemmas in [8] help us to handle with boundary term arise in (3.8).

Lemma 3.5 *Let $\Omega \subset \mathbb{R}^n$ be a C^2 bounded domain, $u \in C^2(\overline{\Omega})$ satisfies $u = 0$ and $u_\gamma = 1$ and $\partial\Omega$, then*

$$S_k^{ij} x_j \gamma_i |Du|^2 = S_k^{ij} u_i u_j x \cdot \gamma \quad on\ \partial\Omega.$$

Lemma 3.6 *Let $\Omega \subset \mathbb{R}^n$ be a C^2 bounded domain, $u \in C^2(\overline{\Omega})$ satisfies $u = 0$ and $u_\gamma = 1$ and $\partial\Omega$, then*

$$\int_{\partial\Omega} S_k^{ij} u_i u_j x \cdot \gamma \mathrm{d}\sigma = (n-k+1) \int_\Omega S_{k-1} \mathrm{d}x.$$

Proof of Theorem 1.1. By Lemmas 3.4, 3.5 and 3.6, we obtain

$$\begin{aligned} 2(k-l)C_n^l \int_\Omega u S_k \mathrm{d}x =& (n-k+1)C_n^l \int_\Omega S_{k-1}\big(|Du|^2 - 1\big)\mathrm{d}x \\ &- (n-l+1)C_n^k \int_\Omega S_{l-1}\big(|Du|^2 - 1\big)\mathrm{d}x, \end{aligned}$$

or equivalently,

$$2(k-l)\int_\Omega (-u)S_k \mathrm{d}x = (n-k+1)\int_\Omega \big(1-|Du|^2\big)S_{k-1}\Big(1 - \frac{l}{k}\frac{S_{l-1}/C_n^{l-1}}{S_{k-1}/C_n^{k-1}}\Big)\mathrm{d}x. \tag{3.15}$$

By (1.2) and (2.3), we have

$$\left(\frac{S_{k-1}/C_n^{k-1}}{S_{l-1}/C_n^{l-1}}\right)^{\frac{1}{k-l}} \geq \left(\frac{S_k/C_n^k}{S_l/C_n^l}\right)^{\frac{1}{k-l}} = 1 \geq \frac{S_k/C_n^k}{S_{k-1}/C_n^{k-1}}.$$

So

$$\frac{S_{l-1}}{S_{k-1}}\frac{(n-l+1)C_n^k}{(n-k+1)C_n^l} \leq \frac{l}{k} \quad \text{and} \quad S_k \leq \frac{n-k+1}{k}S_{k-1}. \tag{3.16}$$

By Lemma 3.3,

$$1 - |Du|^2 \geq -2u > 0 \text{ in } \Omega.$$

Then, substituting (3.16) into (3.15), we get

$$\int_\Omega S_{k-1}(|Du|^2 - 2u - 1)\mathrm{d}x \geq 0.$$

By Lemma 3.3,

$$P \le \max_{\partial\Omega} P = 1.$$

It follows that

$$P = |Du|^2 - 2u \equiv 1 \quad \text{in } \Omega.$$

Since the derivatives vanish, by (3.2) and (3.3), we obtain

$$0 \ge n - k - (k+1)\frac{S_{k+1}}{S_k}. \tag{3.17}$$

Hence, $S_{k+1} > 0$ and the equality in (3.4) holds. By (2.3), the eigenvalues of D^2u are equal to 1. Using the boundary condition in (1.2), we derive that $u = \frac{|x|^2-1}{2}$. Hence, we complete the proof of Theorem 1.1 □

4 Overdetermined Problem for Hessian Quotient Equations on the Hyperbolic Space

In this section, we present a Rellich–Pohozaev type identity for Hessian quotient equations in $\mathbb{H}^n$ with zero Dirichlet boundary condition, and use a P-function to give a proof of Theorem 1.2.

The following lemma can be proved almost the same as [8], which implies the solution to (1.3) is k-admissible.

Lemma 4.1 *Let $\Omega \subset \mathbb{H}^n$ be a bounded C^2 domain and $u \in C^2(\overline{\Omega})$ is a solution to the problem (1.2) with $S_l > 0$ on $\partial\Omega$, then u is k-admissible in Ω.*

$P = |Du|^2 - u^2 - 2u$ was proven to be a P function for the k-Hessian equation on $\mathbb{H}^n$ in [8]. In the following lemma, we prove it is also a P function for the Hessian quotient equation on $\mathbb{H}^n$. By the appointment in Sect. 2, we use S_k and S_k^{ij} instead of $S_k(D^2u - uI)$ and $S_k^{ij}(D^2u - uI)$ in this section.

Lemma 4.2 *Let $u \in C^3(\Omega)$ be a solution to*

$$S_k = \frac{C_n^k}{C_n^l} S_l \quad \text{in } \Omega, \tag{4.1}$$

with $0 \le l < k \le n$. Then the maximum of $P = |Du|^2 - u^2 - 2u$ is attained on $\partial\Omega$.

Proof Let $F^{ij} = \frac{\partial}{\partial u_{ij}} \frac{S_k}{S_l}$, then

$$F^{ij} = \frac{1}{S_l^2}\left(S_k^{ij} S_l - S_k S_l^{ij}\right).$$

Let $A_{ij}=u_{ij}-u\delta_{ij}$ for short. Then by direct calculations, we have

$$\frac{1}{2}P_{ij}=A_{si}A_{sj}-A_{ij}+u_rA_{ij,r}+u(A_{ij}-\delta_{ij}).$$

Contracting with F^{ij}, we obtain

$$\begin{aligned}\frac{1}{2}F^{ij}P_{ij}&=F^{ij}(A_{si}A_{sj}-A_{ij}+u_rA_{ij,r}+u(A_{ij}-\delta_{ij}))\\&=F^{ij}(A_{si}A_{sj}-A_{ij})+(-u)F^{ij}(\delta_{ij}-A_{ij})\\&=\frac{1}{S_l^2}\Big((S_1S_k-(k+1)S_{k+1})S_l-(S_1S_l-(l+1)S_{l+1})S_k-(k-l)S_kS_l\Big)\\&\quad-\frac{u}{S_l^2}\Big((n-k+1)S_{k-1}S_l-(n-l+1)S_{l-1}S_k-(k-l)S_kS_l\Big).\end{aligned}\tag{4.2}$$

The same argument as in Sect. 3 yields

$$\frac{1}{S_l^2}\Big(\big(S_1S_k-(k+1)S_{k+1}\big)S_l-\big(S_1S_l-(l+1)S_{l+1}\big)S_k-(k-l)S_kS_l\Big)\geq 0.\tag{4.3}$$

By (2.3) and (4.1), we have

$$\left(\frac{S_{k-1}/C_n^{k-1}}{S_l/C_n^l}\right)^{\frac{1}{k-l-1}}\geq\left(\frac{S_k/C_n^k}{S_l/C_n^l}\right)^{\frac{1}{k-l}}=1,$$

and

$$\left(\frac{S_k/C_n^k}{S_{l-1}/C_n^{l-1}}\right)^{\frac{1}{k-l-1}}\geq\left(\frac{S_k/C_n^k}{S_l/C_n^l}\right)^{\frac{1}{k-l}}=1.$$

So

$$\frac{S_{k-1}}{S_l}\geq\frac{C_n^{k-1}}{C_n^l}\quad\text{and}\quad\frac{S_{l-1}}{S_k}\leq\frac{C_n^{l-1}}{C_n^k}.\tag{4.4}$$

Hence

$$\begin{aligned}&-\frac{u}{S_l^2}\Big((n-k+1)S_{k-1}S_l-(n-l+1)S_{l-1}S_k-(k-l)S_kS_l\Big)\\&=(-u)\Big((n-k+1)\frac{S_{k-1}}{S_l}-(n-l+1)\frac{S_{l-1}}{S_k}\big(\frac{C_n^k}{C_n^l}\big)^2-(k-l)\frac{C_n^k}{C_n^l}\Big)\\&\geq(-u)\frac{C_n^k}{C_n^l}\Big((n-k+1)\frac{C_n^{k-1}}{C_n^k}-(n-l+1)\frac{C_n^{l-1}}{C_n^k}-(k-l)\Big)\\&=0,\end{aligned}\tag{4.5}$$

where we use (4.1) in the second line and (4.4) in the third line. Substituting (4.3) and (4.5) into (4.2), we finally have

$$F^{ij}P_{ij} \geq 0. \tag{4.6}$$

By maximum principle, the maximum of P is attained on $\partial\Omega$. □

We will use the following Rellich–Pohozaev type identity.

Lemma 4.3 *Let $u \in C^1(\overline{\Omega}) \cap C^3(\Omega)$ be a solution of*

$$\begin{cases} S_k = \frac{C_n^k}{C_n^l} S_l & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.7}$$

Then, there holds

$$\begin{aligned} &\frac{n-k+1}{2} C_n^l \int\limits_{\Omega} S_{k-1}(|Du|^2 - u^2) V \mathrm{d}x - \frac{n-l+1}{2} C_n^k \int\limits_{\Omega} S_{l-1}\big(|Du|^2 - u^2\big) V \mathrm{d}x \\ &- \frac{1}{2} C_n^l \int\limits_{\partial\Omega} S_k^{is} |Du|^2 V_s \gamma_i \mathrm{d}\sigma \\ &+ \frac{1}{2} C_n^k \int\limits_{\partial\Omega} S_l^{is} |Du|^2 V_s \gamma_i \mathrm{d}\sigma - (k-l) C_n^l \int\limits_{\Omega} S_k u V \mathrm{d}x = 0. \end{aligned} \tag{4.8}$$

Proof Multiplying the equation by uV, we obtain

$$kS_k uV = S_k^{ij}(u_{ij} - u\delta_{ij})uV = S_k^{ij} u_{ij} uV - (n-k+1) S_{k-1} u^2 V. \tag{4.9}$$

Since $D^2 V = VI$, we have

$$S_k^{ij} u_{ij} uV = S_k^{ij} u_{ir} u V_{rj} = (S_k^{ij} u_{ir} u V_r)_j - S_k^{ij} u_{irj} u V_r - S_k^{ij} u_{ir} u_j V_r.$$

Using $u_{irj} = u_{ijr} - u_r\delta_{ij} + u_j\delta_{ir} = (u_{ij} - u\delta_{ij})_r + u_j\delta_{ir}$, we have

$$\begin{aligned} S_k^{ij} u_{ij} uV &= (S_k^{ij} u_{ir} u V_r)_j - S_k^{ij}(u_{ij} - u\delta_{ij})_r u V_r - S_k^{ij} u_j u V_i - S_k^{ij} u_{ir} u_j V_r \\ &= (S_k^{ij} u_{ir} u V_r)_j - u V_r D_r S_k - S_k^{ij} u_j u V_i - S_k^{ij} u_{ir} u_j V_r. \end{aligned} \tag{4.10}$$

Furthermore, we have

$$\begin{aligned} S_k^{ij} u_j u V_i &= \frac{1}{2}(S_k^{ij} u^2 V_i)_j - \frac{1}{2} S_k^{ij} u^2 V_{ij} = \frac{1}{2}(S_k^{ij} u^2 V_i)_j \\ &\quad - \frac{n-k+1}{2} S_{k-1} u^2 V. \end{aligned} \tag{4.11}$$

By Proposition 2.3, we obtain

$$S_k^{ij}u_{ir}u_jV_r = S_k^{ir}u_{ij}u_jV_r = \frac{1}{2}S_k^{ij}(|Du|^2)_iV_j = \frac{1}{2}(S_k^{ij}|Du|^2V_j)_i - \frac{n-k+1}{2}S_{k-1}|Du|^2V. \tag{4.12}$$

Putting (4.9)–(4.12) together, we obtain

$$\begin{aligned} uV_rD_rS_k =& (S_k^{ij}u_{ir}uV_r)_j - \frac{1}{2}(S_k^{ij}u^2V_i)_j - \frac{1}{2}(S_k^{ij}|Du|^2V_i)_j \\ &+ \frac{n-k+1}{2}S_{k-1}(|Du|^2-u^2)V - kS_kuV. \end{aligned} \tag{4.13}$$

Similarly, we have

$$\begin{aligned} uV_rD_rS_l =& (S_l^{ij}u_{ir}uV_r)_j - \frac{1}{2}(S_l^{ij}u^2V_i)_j - \frac{1}{2}(S_l^{ij}|Du|^2V_i)_j \\ &+ \frac{n-l+1}{2}S_{l-1}(|Du|^2-u^2)V - lS_luV. \end{aligned} \tag{4.14}$$

By (4.7), we have

$$C_n^lD_rS_k = C_n^kD_rS_l. \tag{4.15}$$

Substituting (4.13) and (4.14) into (4.15), we have

$$\begin{aligned} C_n^l\Big(&(S_k^{ij}u_{ir}uV_r)_j - \frac{1}{2}\big(S_k^{ij}u^2V_i\big)_j - \frac{1}{2}(S_k^{ij}|Du|^2V_i)_j \\ &+ \frac{n-k+1}{2}S_{k-1}(|Du|^2-u^2)V - kS_kuV\Big) \\ =& C_n^k\Big((S_l^{ij}u_{ir}uV_r)_j - \frac{1}{2}(S_l^{ij}u^2V_i)_j - \frac{1}{2}(S_l^{ij}|Du|^2V_i)_j \\ &+ \frac{n-l+1}{2}S_{l-1}(|Du|^2-u^2)V - lS_luV\Big). \end{aligned} \tag{4.16}$$

Integrating the above on Ω and use (4.7), we finally obtain (4.8), thus finish the proof. □

The following two lemmas are from [8], we use them to deal with the boundary terms appear in (4.8).

Lemma 4.4 *Let $u \in C^2(\overline{\Omega})$ satisfying $u = 0$ and $u_\gamma = 1$ on $\partial\Omega$, then*

$$S_k^{ij}V_j\gamma_i|Du|^2 = S_k^{ij}u_iu_jV_\gamma \quad \text{on } \partial\Omega.$$

Lemma 4.5 *Let $u \in C^2(\overline{\Omega})$ satisfying $u = 0$ and $u_\gamma = 1$ on $\partial\Omega$, then*

$$\int_{\partial\Omega} S_k^{ij} u_i u_j V_\gamma \mathrm{d}\sigma = (n-k+1) \int_{\Omega} S_{k-1} V \mathrm{d}x.$$

Proof of Theorem 1.3. By Lemmas 4.3, 4.4 and 4.5 we obtain

$$\begin{aligned}
&(n-k+1)C_n^l \int_{\Omega} S_{k-1}(|Du|^2 - u^2 - 1)V \mathrm{d}x \\
&\quad - (n-l+1)C_n^k \int_{\Omega} S_{l-1}\big(|Du|^2 - u^2 - 1\big)V \mathrm{d}x \\
&\quad = 2(k-l)C_n^l \int_{\Omega} S_k u V \mathrm{d}x.
\end{aligned}$$

That is

$$\begin{aligned}
&(n-k+1) \int_{\Omega} S_{k-1}(|Du|^2 - u^2 - 1)\Big(1 - \frac{(n-l+1)C_n^k S_{l-1}}{(n-k+1)C_n^l S_{k-1}}\Big)V \mathrm{d}x \\
&\quad = 2(k-l) \int_{\Omega} S_k u V \mathrm{d}x.
\end{aligned}$$

By (2.3) and (4.7), we have

$$\left(\frac{S_{k-1}/C_n^{k-1}}{S_{l-1}/C_n^{l-1}}\right)^{\frac{1}{k-l}} \geq \left(\frac{S_k/C_n^k}{S_l/C_n^l}\right)^{\frac{1}{k-l}} = 1 \geq \frac{S_k/C_n^k}{S_{k-1}/C_n^{k-1}}.$$

So

$$\frac{S_{l-1}}{S_{k-1}} \frac{(n-l+1)C_n^k}{(n-k+1)C_n^l} \leq \frac{l}{k} \quad \text{and} \quad S_k \leq \frac{n-k+1}{k} S_{k-1}.$$

Note that by Lemma 4.2, we have

$$|Du|^2 - u^2 - 1 \leq 2u \quad \text{in } \Omega.$$

So

$$\begin{aligned}
&(n-k+1) \int_{\Omega} S_{k-1}(|Du|^2 - u^2 - 1)\Big(1 - \frac{(n-l+1)C_n^k S_{l-1}}{(n-k+1)C_n^l S_{k-1}}\Big)V \mathrm{d}x \\
&\quad \leq \frac{2(n-k+1)(k-l)}{k} \int_{\Omega} S_{k-1} u V \mathrm{d}x.
\end{aligned}$$

On the other hand,

$$2(k-l)\int_{\Omega} S_k u V \mathrm{d}x \geq \frac{2(n-k+1)(k-l)}{k}\int_{\Omega} S_{k-1} u V \mathrm{d}x.$$

So we have

$$\frac{n-k+1}{k} S_{k-1} = S_k.$$

By (2.5), we infer the eigenvalues of $D^2u - uI$ are all equal to 1. Follows from an Obata type result ([18], see also [2–4, 16]), Ω must be a ball B_R and u depends only on the distance from the center of B_R, where $R = \tanh^{-1} 1$. It is easy to see that u is of the form

$$u = \frac{\cosh r}{\cosh R} - 1.$$

□

5 Overdetermined Problem for Quotient Curvature Equations

In this section, we present a Rellich–Pohozaev type identity for Hessian quotient curvature equations with zero Dirichlet boundary condition, and use a P-function to give a proof of Theorem 1.3.

The following lemma can be proved almost the same as [12], which implies the solution to (1.4) is k-admissible.

Lemma 5.1 *Let Ω be a C^2 bounded domain of $\mathbb{R}^n$ and $u \in C^2(\overline{\Omega})$ be a solution to problem (1.4) with $S_l > 0$ on $\partial\Omega$, then u is a k-admissible function in Ω and Ω is $(k-1)$-convex.*

The following lemma is from [12].

Lemma 5.2 *Let $\Omega \subset \mathbb{R}^n$ be a C^2 bounded domain, $u \in C^2(\overline{\Omega})$ satisfies $u = 0$ and $u_\gamma = 1$ on $\partial\Omega$, then*

$$\int_{\partial\Omega} \frac{S_k^{ij} x_i \gamma_j}{w} \mathrm{d}\sigma = \frac{n-k+1}{\sqrt{2}} \int_{\Omega} S_{k-1} \mathrm{d}x.$$

In [12], Jia prove that $P = \frac{1}{w} + u$ is a P function to constant curvature equation $S_k = C_n^k$. The following lemma implies it is also a P function for $S_k = \frac{C_n^k}{C_n^l} S_l$. As agreed in Sect. 2, we use S_k and S_k^{ij} instead of $S_k(D(\frac{Du}{w}))$ and $S_k^{ij}(D(\frac{Du}{w}))$ in this section.

Lemma 5.3 *Let Ω be a C^2 bounded domain of $\mathbb{R}^n$, $u \in C^2(\overline{\Omega}) \cap C^3(\Omega)$ be a solution to problem (1.4), the minimum of $\frac{1}{w}$, $-u$ and $P = \frac{1}{w} + u$ is attained on $\partial\Omega$.*

Proof Let $X=(x,u)\in\mathcal{M}$ and $e_{n+1}=(0,\ldots,0,1)\in\mathbb{R}^n$. Choose a local orthogonal frame $\{e_1,\ldots.e_n\}$ on $\mathcal{M}$. The unit normal is $N=\frac{(-Du,1)}{w}$. Then we have

$$\frac{1}{w}=\langle N,e_{n+1}\rangle,\quad u=\langle X,e_{n+1}\rangle,\quad X_i=e_i,$$

and

$$e_{ij}=h_{ij}N,\quad N_i=-h_{ij}e_j,$$

where h_{ij} is the coefficient of the second fundamental form. Thus

$$P=\frac{1}{w}+u=\langle N,e_{n+1}\rangle+\langle X,e_{n+1}\rangle.$$

By direct computation, we obtain

$$P_{ij}=-h_{ij,r}\langle e_r,e_{n+1}\rangle-\frac{1}{w}h_{im}h_{mj}+\frac{1}{w}h_{ij}.$$

Let $F^{ij}=\frac{\partial}{\partial h_{ij}}\frac{S_k(\{h_{st}\})}{S_l(\{h_{rq}\})}$, then

$$F^{ij}=\frac{1}{S_l^2(\{h_{st}\})}\big(S_k^{ij}(\{h_{st}\})S_l(\{h_{st}\})-S_l^{ij}(\{h_{st}\})S_k(\{h_{st}\})\big).$$

Note that $F^{ij}h_{ij,r}=0$, we have

$$\begin{aligned}F^{ij}P_{ij}=&\frac{1}{wS_l^2(\{h_{st}\})}\big((S_k^{ij}(\{h_{st}\})S_l(\{h_{st}\})-S_k(\{h_{st}\})S_l^{ij}(\{h_{st}\}))(h_{ij}-h_{im}h_{mj})\big)\\=&\frac{1}{wS_l^2(\{h_{st}\})}\Big((k-l)S_k(\{h_{st}\})S_l(\{h_{st}\})+(l+1)S_{l+1}(\{h_{st}\})S_k(\{h_{st}\})\\&-(k+1)S_{k+1}(\{h_{st}\})S_l(\{h_{st}\})\Big)\\\le&\,0,\end{aligned}$$

where the last inequality can be derived similar as in Sects. 3 and 4. By maximum principle, the minimum of P is attained on $\partial\Omega$. The same argument leads the conclusion for $\frac{1}{w}$ and $-u$. □

We prove the following Rellich–Pohozaev identity.

Lemma 5.4 *Let $\Omega\subset\mathbb{R}^n$ be a C^2 domain, $u\in C^2(\overline{\Omega})\cap C^3(\Omega)$ be a solution to the following problem*

$$\begin{cases}S_k=\frac{C_n^k}{C_n^l}S_l & \text{in }\Omega,\\u=0 & \text{on }\partial\Omega,\end{cases}\tag{5.1}$$

with $0 \le l < k \le n$*. Then*

$$C_n^l \int\limits_{\partial\Omega} S_k^{si} \frac{x_s \gamma_i}{w} \mathrm{d}\sigma - C_n^k \int\limits_{\partial\Omega} S_l^{si} \frac{x_s \gamma_i}{w} \mathrm{d}\sigma - (k-l) C_n^l \int\limits_{\Omega} u S_k \mathrm{d}x$$
$$- (n-k+1) C_n^l \int\limits_{\Omega} \frac{S_{k-1}}{w} + (n-l+1) C_n^k \int\limits_{\Omega} \frac{S_{l-1}}{w} = 0. \quad (5.2)$$

Proof By direct computation, we obtain

$$\left(\frac{u_i}{w}\right)_j u_i = \frac{1}{2} w \left(\frac{|Du|^2}{w^2}\right)_j = \frac{1}{2} w \left(1 - \frac{1}{w^2}\right)_j = -\left(w^{-1}\right)_i. \quad (5.3)$$

Multiplying the equation with ku, we obtain

$$k S_k u = S_k^{ij} \left(\frac{u_i}{w}\right)_s (x_s)_j u = \left(S_k^{ij} \left(\frac{u_i}{w}\right)_s x_s u\right)_j - u x_s D_s S_k - S_k^{ij} \left(\frac{u_i}{w}\right)_s x_s u_j. \quad (5.4)$$

By using (5.3), we obtain

$$S_k^{ij} \left(\frac{u_i}{w}\right)_s x_s u_j = S_k^{si} \left(\frac{u_j}{w}\right)_i x_s u_j = -S_k^{si} (w^{-1})_i x_s = -\left(S_k^{si} (w^{-1}) x_s\right)_i + \frac{(n-k+1) S_{k-1}}{w}. \quad (5.5)$$

Putting (5.4) and (5.5) together, we obtain

$$u x_s D_s S_k = \left(S_k^{ij} \left(\frac{u_i}{w}\right)_s x_s u\right)_j + \left(S_k^{si} \frac{x_s}{w}\right)_i - (n-k+1) \frac{S_{k-1}}{w} - k u S_k. \quad (5.6)$$

Note that from (5.2)

$$C_n^l D_s S_k = C_n^k D_s S_l. \quad (5.7)$$

By (5.6) and (5.2) we have

$$C_n^l \left(\left(S_k^{ij} \left(\frac{u_i}{w}\right)_s x_s u\right)_j + (S_k^{si} \frac{x_s}{w})_i - (n-k+1) \frac{S_{k-1}}{w} - k u S_k \right)$$
$$= C_n^k \left(\left(S_l^{ij} \left(\frac{u_i}{w}\right)_s x_s u\right)_j + (S_l^{si} \frac{x_s}{w})_i - (n-l+1) \frac{S_{l-1}}{w} - l u S_l \right).$$

Integrate it on Ω and use (5.2), we obtain the identity we want. $\square$

Proof of Theorem 1.3. By Lemmas 5.2 and 5.4, we obtain

$$(n-k+1)C_n^l\int\limits_\Omega S_{k-1}\left(\frac{1}{\sqrt{2}}-\frac{1}{w}\right)\mathrm{d}x-(n-l+1)C_n^k\int\limits_\Omega S_{l-1}\left(\frac{1}{\sqrt{2}}-\frac{1}{w}\right)\mathrm{d}x$$
$$=(k-l)C_n^l\int\limits_\Omega uS_k\mathrm{d}x.$$

That is

$$(n-k+1)\int\limits_\Omega S_{k-1}\left(\frac{1}{\sqrt{2}}-\frac{1}{w}\right)\left(1-\frac{(n-l+1)C_n^kS_{l-1}}{(n-k+1)C_n^lS_{k-1}}\right)\mathrm{d}x=(k-l)\int\limits_\Omega uS_k. \tag{5.8}$$

By (2.3) and (5.1), we have

$$\left(\frac{S_{k-1}/C_n^{k-1}}{S_{l-1}/C_n^{l-1}}\right)^{\frac{1}{k-l}}\geq\left(\frac{S_k/C_n^k}{S_l/C_n^l}\right)^{\frac{1}{k-l}}=1\geq\frac{S_k/C_n^k}{S_{k-1}/C_n^{k-1}}.$$

So it follows

$$\frac{S_{l-1}}{S_{k-1}}\frac{(n-l+1)C_n^k}{(n-k+1)C_n^l}\leq\frac{l}{k}, \tag{5.9}$$

and

$$S_k\leq\frac{n-k+1}{k}S_{k-1}. \tag{5.10}$$

Note that by Lemma 5.3, we have

$$\frac{1}{\sqrt{2}}-\frac{1}{w}\leq u\quad\text{in }\Omega. \tag{5.11}$$

So by (5.9) and (5.11), we obtain

$$(n-k+1)\int\limits_\Omega S_{k-1}\left(\frac{1}{\sqrt{2}}-\frac{1}{w}\right)\left(1-\frac{(n-l+1)C_n^kS_{l-1}}{(n-k+1)C_n^lS_{k-1}}\right)\mathrm{d}x$$
$$\leq\frac{(n-k+1)(k-l)}{k}\int\limits_\Omega S_{k-1}u\mathrm{d}x. \tag{5.12}$$

On the other hand, by maximum principle, we have $u<0$ in Ω, it follows by (5.10)

$$(k-l)\int\limits_\Omega S_kuV\mathrm{d}x\geq\frac{2(n-k+1)(k-l)}{k}\int\limits_\Omega S_{k-1}uV\mathrm{d}x. \tag{5.13}$$

So we have

$$\frac{n-k+1}{k}S_{k-1} = S_k.$$

By (2.5), we infer the eigenvalues of $D\left(\frac{Du}{w}\right)$ are all equal to 1. Combining with the boundary conditions in (1.4), we obtain

$$u = -\sqrt{1-|x|^2} + \frac{1}{\sqrt{2}},$$

and Ω is a ball with radius $\frac{1}{\sqrt{2}}$. □

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Declarations

Conflict of interest The authors have no conflict of interest.

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