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Nonparametric Mean Curvature Flow with Nearly Vertical Contact Angle Condition

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Dedicated to Professors Sun-Yung Alice Chang and Paul C. Yang on their 70th birthdays

Abstract. For any bounded strictly convex domain Ω in $\mathbb{R}^n$ with smooth boundary, we find the prescribed contact angle which is nearly perpendicular such that nonparametric mean curvature flow with contact angle boundary condition converge to ones which move by translation. Subsequently, the existence and uniqueness of smooth solutions to the capillary problem without gravity on strictly convex domain are also discussed.

AMS subject classifications: 35K59, 35J93.

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1 Introduction

In this paper, we are interested in the study of the evolution of graphs defined over bounded strict convex domains $\Omega\subset\mathbb{R}^n$ by the nonparametric mean curvature flow, whose speed in the direction of their normal is equal to their mean curvature and with a prescribed contact angle to $\partial\Omega$.

Various results have been obtained for mean curvature flow of hypersurfaces with Dirichlet boundary conditions [26], zero-Neumann boundary condition [15], [18] and

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general Neumann boundary condition [25]. We study the evolution of graphs for $u=u(x,t)$ with the speed depending on the mean curvature of the surface $\{(x,u(x,t)):x\in\Omega\}$ and with the prescribed contact angle boundary condition, that is,

$$\begin{cases} u_t=\sqrt{1+|Du|^2}\mathcal{H}(u) & \text{in } \Omega\times(0,\infty),\\ \langle\gamma,\nu\rangle=\cos\theta & \text{on } \partial\Omega\times(0,\infty),\\ u(x,0)=u_0(x) & \text{on } \overline{\Omega}, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$, $n\geq 2$, is a compact domain with smooth boundary $\partial\Omega$, $\theta:\partial\Omega\to\mathbb{R}$ is the angle (*contact angle*) between the graph and the boundary, given by $\langle\gamma,\nu\rangle=\cos\theta$, which is equivalent to $u_\nu=-\cos\theta\sqrt{1+|Du|^2}$, where ν is the unit inner normal of $\partial\Omega$. Remark that one may extend θ to $\overline{\Omega}$ with $\theta\in C^\infty(\overline{\Omega})$. And $u_0(x)$ is also a smooth function satisfying the compatible condition

$$u_{0,\nu}=-\cos\theta\sqrt{1+|Du_0|^2} \quad \text{on} \quad \partial\Omega.$$

While $\mathcal{H}$ is the mean curvature operator

$$\mathcal{H}(u):=\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right),$$

and γ is the upward normal of the graph $\{(x,u(x,t)):x\in\Omega\}$, which is given by

$$\gamma:=\frac{(-Du,1)}{\sqrt{1+|Du|^2}},$$

and we denote by $Q_T:=\Omega\times[0,T)$ for convenience.

For the prescribed contact angle boundary condition, a more general type of problem is to study the following equation, which has an extra term $F(x,u,Du)$ (some called the transport term) compared to (1.1), i.e.

$$u_t=\sqrt{1+|Du|^2}\mathcal{H}(u)-F(x,u,Du) \quad \text{in} \quad Q_T. \tag{1.2}$$

Guan [14] proved the global existence of solutions to (1.2) with prescribed contact angle condition for general bounded domain Ω. Recently, Zhou generalized Guan's results to the domain Ω on Riemannian manifold in [30].

As for studying the asymptotic behavior of $u(x,t)$ in (1.2), Guan [14] or Zhou [30] only obtained the convergence results for $F(x,u,Du)$ with specific form, say $F:=\phi(x,u)\cdot\sqrt{1+|Du|^2}$ with $\phi_u\geq c_0>0$, which excluded $F\equiv 0$. In [15], Huisken studied the fixed vertical contact angle case of (1.1), i.e. $\theta(x)\equiv\frac{\pi}{2}$, so $u_\nu=0$ on $\partial\Omega$. By using the Sobolev-type inequalities and an iteration method, Huisken proved that the solution $u(\cdot,t)$ of (1.1) converges to a constant function as $t\to+\infty$. For the non-perpendicular case, Altschuler and Wu in [1] firstly considered the problem (1.1) with fixed contact angle boundary condition in one dimension, and showed that $u(x,t)$ converges to translating solitons. Subsequently, they studied in [2] for two dimension case and proved that the solutions of (1.1) converge to one which moves only by translation, under the condition that $\Omega\subset\mathbb{R}^2$ is strictly convex and $\|D\theta\|_{C^0}<\min_{\partial\Omega}\kappa$, where κ is the curvature of the curve $\partial\Omega$. The convergence results in [1] and [2] are only known now for one and two dimension cases respectively. In particular, the uniform gradient estimate is still unknown for higher space dimension. It is an open question whether the results in [1] and [2] also hold for higher space dimension? In the first part of this paper, we give a partial positive answer to this question, when the contact angle is close to $\frac{\pi}{2}$ and the domain is strictly convex.

From above discussions, we rewrite (1.1) into the following equivalent form,

$$\begin{cases} u_t=\sum_{i,j=1}^n a_{ij}u_{ij} & \text{in } \Omega\times[0,T), \\ u_\nu=-\cos\theta(x)\sqrt{1+|Du|^2} & \text{on } \partial\Omega\times[0,T), \\ u(x,0)=u_0(x) & \text{on } \overline{\Omega}, \end{cases} \tag{1.3}$$

where $a_{ij}:=\delta_{ij}-\frac{u_iu_j}{1+|Du|^2}$ and the other quantities are just the same as the ones in (1.1). Our first main result is the following convergence theorem for (1.3).

Theorem 1.1. *Let $\Omega\subset\mathbb{R}^n$ ($n\geq 2$) be a strictly convex, bounded domain and $\partial\Omega\in C^3$. There exists $\varepsilon_0>0$ depending only on the convexity of Ω such that if $\theta\in C^3(\overline{\Omega})$ satisfies*

$$|\cos\theta|\leq\varepsilon_0<1, \quad \text{and} \quad \|D\theta\|_{C^1(\overline{\Omega})}\leq\varepsilon_0, \tag{1.4}$$

then the flow $u(x,t)$ in (1.3) exists for all time and converges to a translating solution to the following mean curvature equation

$$\begin{cases} \sum_{i,j=1}^n a_{ij}u_{ij}=\tau & \text{in } \Omega, \\ u_\nu=-\cos\theta(x)\sqrt{1+|Du|^2} & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

That is, the solution of (1.3) converges to $w(x)+\tau t$ as $t\to\infty$, which means that

$$\lim_{t\to+\infty}\|u(\cdot,t)-(w(\cdot)+\tau t)\|_{C^0(\overline{\Omega})}=0,$$

where (τ,w) is a suitable solution solving (1.5).

The crucial part of the proof is to derive an a priori estimate for the spatial gradient of $u(x,t)$, which is time-independent. This will be achieved by choosing an appropriate auxiliary function and combining with the maximum principle. Our auxiliary function and approach are motivated by methods used in [10, 24, 25, 27].

Remark 1.1. When $\theta(x)\equiv\frac{\pi}{2}$, Theorem 1.1 was firstly proved by Huisken in [15]. If one denotes $\theta(x):=\frac{\pi}{2}+\gamma(x)$, then condition (1.4) is equivalently reduced to: there exists $\varepsilon_0>0$ satisfying

$$\|\gamma\|_{C^2(\overline{\Omega})}\leq\varepsilon_0.$$

It is worthwhile to notice that, when the contact angle $\theta(x)\equiv\theta_0$ is a fixed constant, the evolution equation (1.3) is related to the so-called *mean curvature flow of surface clusters*, also called *space partitions* (networks, in the plane), see [3], [8] and references therein for more interesting physical background. As a direct corollary of Theorem 1.1, we have the following result for the fixed contact angle.

Corollary 1.1. Let $\Omega\subset\mathbb{R}^n$ $(n\geq 2)$ be a strictly convex, bounded domain and $\partial\Omega\in C^3$. Assume that $\theta(x)\equiv\theta_0\in(0,\frac{\pi}{2}]$ in (1.3), then there exists $\varepsilon_0>0$ depending only on the convexity of Ω such that if

$$0<\frac{\pi}{2}-\varepsilon_0<\theta_0\leq\frac{\pi}{2},$$

the unique smooth solution $u(x,t)$ of (1.3) converges to $w(x)+\lambda t$ as $t\to\infty$, which means that

$$\lim_{t\to+\infty}\|u(\cdot,t)-(w(\cdot)+\lambda t)\|_{C^0(\overline{\Omega})}=0,$$

where (λ,w) is a suitable solution to (1.5) with $\theta(x)\equiv\theta_0$ and $\tau:=\lambda$.

In fact, the existence of solutions to (1.5) is closely related to the capillary problem. Precisely, the capillary problem is referred to study the following equations.

$$\begin{cases}\mathcal{H}(u)=\tau+ku & \text{in } \Omega,\\ \langle\gamma,\nu\rangle=\cos\theta & \text{on } \partial\Omega,\end{cases}\tag{1.6}$$

where k is usually referred to as the *capillarity constant* (see [8], Chapter 1). Results about the positive gravity $k>0$ case are extensively studied and quite well-known. Ural'tseva [29], Simon-Spruck [28] and Gerhardt [10] had obtained the existence results of (1.6) for any dimensions. More results related to positive gravity capillary problem also could be seen in [12], the wonderful exposition book by Finn in [8] and references therein. We only discuss and focus on $k=0$ (gravity free) in (1.6) in the rest part of this paper, i.e.

$$\begin{cases}\operatorname{div}\Big(\dfrac{Du}{\sqrt{1+|Du|^2}}\Big)=\tau & \text{in } \Omega,\\ u_\nu=-\cos\theta\sqrt{1+|Du|^2} & \text{on } \partial\Omega.\end{cases}\tag{1.7}$$

If there exists a solution to (1.7) with constant angle $\theta\equiv\theta_0$, integrating by parts on Ω yields that

$$\tau=\frac{\int_{\partial\Omega}\cos\theta d\sigma}{|\Omega|}=\frac{|\partial\Omega|}{|\Omega|}\cos\theta_0.\tag{1.8}$$

As pointed out by Concus and Finn in [6], Eq. (1.7) may not have any solution, even for constant angle $\theta\equiv\theta_0\in[0,\frac{\pi}{2}]$. In [12], Giusti proved the following results.

Theorem 1.2 ([12]). *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a Lipschitz bounded domain. If $\theta_0 \in (0, \frac{\pi}{2}]$ and there exists $\varepsilon_0 > 0$ such that*

$$\Big| \frac{|\partial\Omega|}{|\Omega|} |\Omega'| - |\partial\Omega \cap \overline{\Omega'}| \Big| \cdot \cos\theta_0 \leq (1-\varepsilon_0) |\partial\Omega' \cap \Omega|, \tag{1.9}$$

holds for all proper subdomains $\Omega' \subset \Omega$. Then there exists a solution $u \in BV(\Omega)$ solving (1.7) *in weak sense, with $\theta \equiv \theta_0$ be a fixed constant.*

Nevertheless, one should note that the criterion (1.9) in Theorem 1.2 is often quite complicated and difficult to be verified, since it involves infinitely many subdomains $\Omega' \subset \Omega$. Subsequently, in 2 dimension, Giusti ([13], Appendix) showed that (1.9) in Theorem 1.2 holds under the curvature condition $0 < \kappa \leq \frac{|\partial\Omega|}{|\Omega|}$ (this implies that Ω is strictly convex), where κ denotes the curvature of curve $\partial\Omega$. Also we mention the papers [7] and [23], where Finn and afterwards Lieberman provided another interesting viewpiont to replace criterion (1.9) with the existence of a vector field criterion. Based on those consideration and motivation, we provide below with another sufficient condition for any dimension $(n \geq 2)$, which can ensure the existence of smooth solution to (1.7). Precisely, we obtain the following result.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a strictly convex, bounded domain and $\partial\Omega \in C^\infty$. Assume $\theta \in C^\infty(\overline{\Omega})$, then there exists a small constant $\varepsilon_0 > 0$ only depending on the convexity of Ω such that if θ satisfies*

$$|\cos\theta| \leq \varepsilon_0 < 1, \quad \text{and} \quad \|D\theta\|_{C^1(\overline{\Omega})} \leq \varepsilon_0, \tag{1.10}$$

then there exists a unique $\tau \in \mathbb{R}$ and a function $u \in C^\infty(\overline{\Omega})$ satisfying (1.7)*. In particular, the solution is unique up to an additive constant.*

Remark 1.2. The convexity condition of the domain is necessary in the sense that Finn-Giusti [9] gave an example of nonexistence for the equation (1.7) when the domain is non-convex. And if the domain is nonsmooth, there are already many works related to the generalized solution using the variational methods, see e.g. [8] (particularly Chapters 6, 7), [9], [12], and references therein.

The main difference and difficulty between the positive gravity ($k > 0$ in (1.6)) and free gravity (1.7) is that there is no C^0 estimate for the solutions to (1.7), since a solution plus any constant is still a solution to (1.7). Thus one can not use the continuity method to get the existence. In order to overcome this difficulty, we use an approximation argument and obtain the uniform gradient estimate of the approximation equation, which is independent of $\|u\|_{C^0}$. This approach has been used previously in several different settings, see, e.g., [16, 17, 25, 27]. Those results also motivate our work here. Additionally, we want to point out that this approach is different with many former methods about the capillary problem, say e.g. [8, 12, 13], where they usually proved the existence of generalized solutions firstly, hereafter to show that the generalized solutions possess some regularity. Here our methods are able to get the existence of smooth solution directly.

This article is structured as follows. In Section 2, the uniform gradient estimate is established for (1.3). In Section 3, the asymptotic behavior of solution to (1.3), i.e. Theorem 1.1 is demonstrated and followed as the same as the approach used in [2], once we get the uniform gradient estimate. The last section is devoted to prove Theorem 1.3, after obtaining the uniform gradient estimate for the solutions to approximation equations.

2 Uniform gradient estimate for mean curvature flow

In this section, in order to study the asymptotic behavior of the nonparametric mean curvature flow with prescribed contact angle boundary condition, we establish the uniform gradient estimate for the solution to (1.3) under the condition (1.4).

We have the following facts when Ω is a strictly convex smooth domain. By the classical result (see for example Caffarelli- Nirenberg-Spruck [4] Section 2, and we can take $g'(0)=1$ in their definition of u in page 275), there exists a smooth defining function h for Ω such that $h<0$ in Ω and $h=0$ on $\partial\Omega$, $\{h_{ij}\}\geq k_1\{\delta_{ij}\}$ for some constant $k_1>0$ and $\sup_\Omega|Dh|\leq 1$, $h_\nu=-1$ and $|Dh|=1$ on $\partial\Omega$. Because of the strict convexity of the domain, we may assume that the curvature matrix of $\partial\Omega$ satisfies

$$\{\kappa_{ij}\}_{1\leq i,j\leq n-1}\geq\kappa_0\{\delta_{ij}\}_{1\leq i,j\leq n-1}$$

for some constant $\kappa_0>0$. For convenience, we denote by

$$M_1:=\sup_{\overline{\Omega}}|D^2h|,\quad M_2:=\sup_{\overline{\Omega}}|D^3h|,$$

and define the big O notation $O(s)$, which means that there exists a constant $C>0$, such that $|O(s)|\leq Cs$ for s large enough. In particular, we have the positive constant C only depending on M_1,M_2 and n in the rest setting of this paper.

Using the maximum principle, the same as in [2], we have a priori bound on $|u_t|^2$.

Lemma 2.1 ([2], Lemma 2.2)**.** *If $u(x,t)$ is a smooth solution to* (1.3)*, then*

$$\sup_{Q_T}|u_t|^2=\sup_{\Omega}|u_t|^2|_{t=0}$$

holds. So there exists a constant $C=C(u_0)>0$ such that $\sup_{\overline{Q}_T}|u_t|\leq C$.

Next we obtain the uniform gradient estimate for (1.3), which turns the quasilinear evolution equation (1.3) into a uniformly parabolic equation and the infinite time existence of smooth solutions follows by standard regularity theory.

Theorem 2.1. *Let $\Omega\subset\mathbb{R}^n$ $(n\geq 2)$ be a smooth strictly convex bounded domain. There is a small constant $\varepsilon_0>0$ depending only on the convexity of Ω such that if $\theta\in C^3(\overline{\Omega})$ satisfying condition* (1.4)*, and if $u(x,t)\in C^{3,2}(\overline{Q}_T)$ is a solution to* (1.1)*, then there exists a constant $C\sim n,\Omega,u_0,\theta$ such that*

$$\sup_{\overline{Q}_T}|Du|\leq C. \tag{2.1}$$

Proof. To get the uniform bound of $|Du|$ in Q_T, we only need to prove that: for $0<T'<T$, the bound of $|Du|$ on $\overline{Q}_{T'}$ is independent of T' and then one takes a limit argument.

Let

$$\Phi(x,t):=\log w(x,t)+\alpha h(x),$$

where $w(x,t):=v-\sum_{l=1}^n u_l h_l \cos\theta$ and α is a positive constant to be determined later, $v:=\sqrt{1+|Du|^2}$.

Assume that $\Phi(x,t)$ attains the maximum value at $(x_0,t_0)\in\overline{Q}_{T'}$. We divide it into the following three cases to complete the proof.

Case 1: $x_0\in\partial\Omega\times[0,T']$. At x_0, we choose the coordinate such that $\frac{\partial}{\partial x_n}$ be the inner normal direction of $\partial\Omega$, which is exactly equal to ν, and let $\{x_i\}_{i=1}^{n-1}$ be the geodesic coordinate of $x_0\in\partial\Omega$. Along the geodesic $x_n=t$ $(0<t\le\varepsilon)$, one takes the parallel transport of tangential direction $\frac{\partial}{\partial x_i}$ to establish the geodesic coordinate in the neighborhood around the point x_0 in $\overline{\Omega}$. Denote ∇' as the induced connection on $\partial\Omega$ by D. We denote $D_{ij}=D_iD_j$.

Firstly, we notice from boundary condition in (1.3) that

$$w=v+u_n\cos\theta=v\sin^2\theta \quad \text{on } \partial\Omega.$$

We denote $\nabla' u$ and u_n as the tangential and normal part of Du on the boundary by our choice of coordinate above. We also denote $\nabla'_i(u_n):=u_{ni}$. From the boundary condition $u_n=-v\cos\theta$, we deduce that

$$u_n^2=v^2\cos^2\theta=\cos^2\theta(1+|\nabla' u|^2+u_n^2),$$

so it directly follows that

$$u_n^2=\cot^2\theta(1+|\nabla' u|^2), \tag{2.2}$$

and in particular, we have

$$w=v\sin^2\theta=\sqrt{1+|\nabla' u|^2+u_n^2}\sin^2\theta=\sqrt{1+|\nabla' u|^2}\sin\theta,$$

and

$$vw=1+|\nabla' u|^2.$$

From Gauss-Weingarten equation and directly computation, we have

$$\begin{aligned} D_n v&=\frac{1}{v}\sum_{k=1}^n u_k D_{kn}u=\frac{1}{v}\sum_{i=1}^{n-1}u_i D_{in}u-\cos\theta D_{nn}u\\ &=\frac{1}{v}\sum_{i=1}^{n-1}\left(u_i u_{ni}+\sum_{j=1}^{n-1}u_i b_{ij}u_j\right)-\cos\theta D_{nn}u, \end{aligned}$$

where b_{ij} is the second fundamental form of $\partial\Omega$. Then at $x_0\in\partial\Omega$, it follows that

$$\begin{aligned}
0\geq D_n\Phi(x_0)&=\frac{D_n w}{w}+\alpha h_n\\
&=\frac{1}{w}\left[D_n v-D_n(\sum_{k=1}^n u_k h_k)\cos\theta-D_n(\cos\theta)\cdot\sum_{k=1}^n u_k h_k\right]-\alpha\\
&=\frac{1}{w}\left[D_n v-\sum_{k=1}^n D_{nk}u h_k\cos\theta-\sum_{k=1}^n u_k D_{nk}h\cos\theta+\sum_{k=1}^n u_k h_k\sin\theta\theta_n\right]-\alpha\\
&=\frac{1}{w}\left[D_n v+D_{nn}u\cos\theta-\sum_{k=1}^n u_k D_{nk}h\cos\theta-\sin\theta\theta_n u_n\right]-\alpha\\
&=\frac{1}{vw}\sum_{i=1}^{n-1}\left(u_i u_{ni}+\sum_{j=1}^{n-1}b_{ij}u_i u_j\right)-\frac{1}{w}\sum_{k=1}^n u_k D_{nk}h\cos\theta+\cot\theta\theta_n-\alpha. \qquad (2.3)
\end{aligned}$$

And for all $1\leq i\leq n-1$, since $\{x_i\}$ is the tangential direction, we obtain

$$0=\nabla'_i\Phi(x_0)=\frac{\nabla'_i w}{w}=\frac{1}{w}\left[\nabla'_i v+u_{ni}\cos\theta-u_n\sin\theta\nabla'_i\theta\right],$$

this implies that

$$\nabla'_i v=-u_{ni}\cos\theta+u_n\sin\theta\nabla'_i\theta. \qquad (2.4)$$

On the other hand, by taking the tangential derivative to the boundary condition of (1.3) and combining with (2.4), it yields that

$$u_{ni}=\nabla'_i(-\cos\theta v)=-\cos\theta\nabla'_i v+\sin\theta\nabla'_i\theta v=\cos^2\theta u_{ni}-\cos\theta\sin\theta u_n\nabla'_i\theta+\sin\theta\nabla'_i\theta v,$$

then it follows that

$$u_{ni}=(\cos\theta\cot\theta+\csc\theta)\nabla'_i\theta v,\quad \text{for } i=1,\cdots,n.$$

Hence, we get

$$\frac{1}{vw}\sum_{i=1}^{n-1}\left(u_i u_{ni}+\sum_{j=1}^{n-1}b_{ij}u_i u_j\right)=\frac{1}{vw}\sum_{i,j=1}^{n-1}b_{ij}u_i u_j+\frac{1}{w}\left(\cos\theta\cot\theta+\csc\theta\right)\sum_{i=1}^{n-1}u_i\nabla'_i\theta, \qquad (2.5)$$

$$-\frac{1}{w}\sum_{k=1}^n u_k D_{nk}h\cos\theta=-\frac{1}{w}\left(\sum_{i=1}^{n-1}u_i D_{ni}h+u_n D_{nn}h\right)\cos\theta. \qquad (2.6)$$

Substituting Eqs. (2.5) and (2.6) into (2.3), and note that we have condition (1.4) with

$\varepsilon_0 < 1$, thus we have

$$\begin{aligned}
0 \geq D_n\Phi(x_0) =& \frac{1}{vw}\sum_{i,j=1}^{n-1} b_{ij}u_iu_j + \frac{1}{w}\big(\cos\theta\cot\theta + \csc\theta\big)\sum_{i=1}^{n-1} u_i\theta_i \\
&- \frac{1}{w}\left(\sum_{i=1}^{n-1} u_iD_{ni}h + u_nD_{nn}h\right)\cos\theta + \cot\theta\theta_n - \alpha \\
\geq& \frac{\kappa_0}{vw}|\nabla' u|^2 - (\cot^2\theta + \csc^2\theta)\frac{|\nabla' u|}{\sqrt{1+|\nabla' u|^2}}|\nabla'\theta| - \frac{|\nabla' u|}{\sqrt{1+|\nabla' u|^2}}|\cot\theta|M_1 \\
&+ \cot^2\theta k_1 + \cot\theta\theta_n - \alpha \\
\geq& \kappa_0 \cdot \frac{|\nabla' u|^2}{1+|\nabla' u|^2} - \frac{2}{\sin^2\theta}|\nabla'\theta| - \big(M_1 + |D\theta|\big)|\cot\theta| - \alpha \\
\geq& \kappa_0 \cdot \frac{|\nabla' u|^2}{1+|\nabla' u|^2} - \frac{\varepsilon_0}{1-\varepsilon_0^2}(3+M_1) - \alpha,
\end{aligned}$$

where $M_1 := \sup_{\overline{\Omega}} |D^2h|$. Then it yields that

$$\left[\kappa_0 - \frac{\varepsilon_0}{1-\varepsilon_0^2}(M_1+3) - \alpha\right]|\nabla' u|^2 \leq \frac{\varepsilon_0}{1-\varepsilon_0^2}(M_1+3) + \alpha. \tag{2.7}$$

By choosing $\alpha, \varepsilon_0 > 0$ such that

$$0 < \alpha \leq \frac{\kappa_0}{3}, \quad \text{and any} \quad \varepsilon_0 \in \left(0, \frac{\kappa_0}{9(M_1+3)}\right], \tag{2.8}$$

it follows from (2.7) that we have $|\nabla' u|^2 \leq 2$, so the estimate of $|Du|$ follows by combining this with equation (2.2).

Case 2: $x_0 \in \Omega$ and $t_0 = 0$, then we have

$$\Phi(x,t) \leq \Phi(x_0,0) = \log(\sqrt{1+|Du_0|^2} - \langle Du_0, Dh\rangle\cos\theta) + \alpha h \leq C(u_0, \sup_{\Omega}|h|, \alpha).$$

Since $|\cos\theta| \leq \varepsilon_0 < 1$, it yields from above that

$$\sup_{\Omega_T} v \leq C(u_0, \sup_{\Omega}|h|). \tag{2.9}$$

Case 3: $x_0 \in \Omega$ and $T' \geq t_0 > 0$, so at (x_0, t_0), we have

$$0 = \Phi_i(x_0,t_0) = \frac{w_i}{w} + \alpha h_i, \tag{2.10}$$

and

$$\begin{aligned}
0 \geq& \sum_{i,j=1}^{n} a_{ij}\Phi_{ij}(x_0,t_0) - \Phi_t(x_0,t_0) \\
=& \left(\sum_{i,j=1}^{n} \frac{a_{ij}w_{ij}}{w} - \frac{w_t}{w}\right) - \alpha^2\sum_{i,j=1}^{n} a_{ij}h_ih_j + \alpha\sum_{i,j=1}^{n} a_{ij}h_{ij} =: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned} \tag{2.11}$$

We choose a proper coordinates at (x_0,t_0) such that $|Du|(x_0,t_0)=u_1(x_0,t_0)$ and $\{u_{ij}(x_0,t_0)\}_{2\le i,j\le n}$ is diagonal. Then at (x_0,t_0),

$$a_{11}=\frac{1}{v^2},\ a_{ij}=0 \quad \text{for } i\neq j \text{ and } a_{ii}=1 \text{ for } i\ge 2.$$

We always assume that $u_1(x_0,t_0)$ is large enough in the below computation, such that u_1, $v=\sqrt{1+u_1^2}$, and $w=v-u_1h_1\cos\theta$ (since we assume $|\cos\theta|\le\varepsilon_0<1$) are comparable at (x_0,t_0), that is, if we let $u_1\ge 1$ (otherwise $|u_1|$ is bounded by 1), then

$$u_1\le v\le\sqrt{2}u_1,\quad (1-\varepsilon_0)u_1\le w\le(\sqrt{2}+\varepsilon_0)u_1,\quad (1-\varepsilon_0)v\le w\le(1+\varepsilon_0)v.$$

All the computation below will be done at the point (x_0,t_0). We have

$$\mathrm{II}=-\alpha^2\sum_{i,j=1}^n a_{ij}h_ih_j=-\alpha^2\left(\frac{h_1^2}{v^2}+\sum_{i=2}^n h_i^2\right)\ge-\alpha^2\left[\frac{1}{v^2}+1\right], \tag{2.12}$$

and

$$\mathrm{III}=\alpha\sum_{i,j=1}^n a_{ij}h_{ij}=\alpha\left(\frac{h_{11}}{v^2}+\sum_{i=2}^n h_{ii}\right)\ge\alpha k_1\left[\frac{1}{v^2}+(n-1)\right]. \tag{2.13}$$

We denote by $\mathrm{J}:=\sum_{i,j=1}^n a_{ij}w_{ij}-w_t$. From (2.10), we have

$$\sum_{l=1}^n\left(\frac{u_lu_{li}}{v}-u_{li}h_l\cos\theta-u_lh_{li}\cos\theta+u_lh_l\sin\theta\theta_i\right)=-\alpha h_iw.$$

If we denote by

$$\mathrm{S}_l:=\frac{u_l}{v}-h_l\cos\theta,\quad \text{for } l=1,\cdots,n,$$

then we have the bound as $2\ge\mathrm{S}_1\ge\frac14$ if we assume $u_1^2\ge 1$ and $\varepsilon_0\le\frac14$. Hence, then we obtain

$$\sum_{l=1}^n \mathrm{S}_lu_{li}=h_{1i}\cos\theta u_1-h_1\sin\theta\theta_iu_1-\alpha h_iw. \tag{2.14}$$

It follows that for $i=2,\cdots,n$,

$$\begin{aligned}u_{1i}&=-\frac{\mathrm{S}_i}{\mathrm{S}_1}u_{ii}+\frac{h_{1i}\cos\theta}{\mathrm{S}_1}u_1-\frac{h_1\sin\theta\theta_i}{\mathrm{S}_1}u_1-\frac{\alpha h_i}{\mathrm{S}_1}w\\&=-\frac{\mathrm{S}_i}{\mathrm{S}_1}u_{ii}+O(|\cos\theta|+|D\theta|)u_1+O(\alpha)w,\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}u_{11}=&-\sum_{k=2}^{n}\frac{\mathrm{S}_k}{\mathrm{S}_1}u_{1k}+\frac{h_{11}\cos\theta}{\mathrm{S}_1}u_1-\frac{h_1\sin\theta\theta_1}{\mathrm{S}_1}u_1-\frac{\alpha h_1}{\mathrm{S}_1}w\\ =&\sum_{k=2}^{n}\frac{\mathrm{S}_k^2}{\mathrm{S}_1^2}u_{kk}+O(|\cos\theta|+|D\theta|)u_1+O(\alpha)w. \end{aligned}\tag{2.16}$$

To handle the term I, we take the first derivative with respect to x_k to the equation in (1.3),

$$u_{kt}=\Big(\sum_{i,j=1}^{n}a_{ij}u_{ij}\Big)_k=\sum_{i,j=1}^{n}a_{ij,k}u_{ij}+\sum_{i,j=1}^{n}a_{ij}u_{ijk}.$$

By direct computation, we have

$$\begin{aligned}w_t=&v_t-\Big(\sum_{l=1}^{n}u_lh_l\cos\theta\Big)_t=\sum_{l=1}^{n}\Big(\frac{u_lu_{lt}}{v}-u_{lt}h_l\cos\theta\Big)=\sum_{l=1}^{n}\mathrm{S}_lu_{lt}\\ =&\sum_{k=1}^{n}\mathrm{S}_k\Big[\sum_{i,j=1}^{n}a_{ij,k}u_{ij}+\sum_{i,j=1}^{n}a_{ij}u_{ijk}\Big]. \end{aligned}\tag{2.17}$$

and

$$\begin{aligned}w_{ij}=&v_{ij}-\sum_{k=1}^{n}(u_kh_k\cos\theta)_{ij}\\ =&\sum_{k=1}^{n}\frac{u_{ki}u_{kj}}{v}+\sum_{k=1}^{n}\frac{u_ku_{kij}}{v}-\sum_{k,l=1}^{n}\frac{u_ku_{ki}u_lu_{lj}}{v^3}-\sum_{l=1}^{n}u_{li}h_{lj}\cos\theta-\sum_{l=1}^{n}u_{lj}h_{li}\cos\theta-u_1h_{1ij}\cos\theta\\ &-\sum_{l=1}^{n}u_{lij}h_l\cos\theta-\sum_{k=1}^{n}(u_kh_k)_i(\cos\theta)_j-\sum_{k=1}^{n}(u_kh_k)_j(\cos\theta)_i-\sum_{k=1}^{n}u_kh_k(\cos\theta)_{ij}. \end{aligned}\tag{2.18}$$

By (2.17) and (2.18), thus it follows that

$$\begin{aligned}\mathrm{J}=&w\mathrm{I}=\sum_{i,j=1}^{n}a_{ij}w_{ij}-w_t\\ =&\sum_{i,j=1}^{n}a_{ij}\Big(\sum_{k=1}^{n}\frac{u_{ki}u_{kj}}{v}-\sum_{k,l=1}^{n}\frac{u_ku_{ki}u_lu_{lj}}{v^3}\Big)\\ &+\sum_{i,j,k=1}^{n}\Big[a_{ij}\big(\frac{u_ku_{kij}}{v}-u_{kij}h_k\cos\theta\big)-\mathrm{S}_ka_{ij,k}u_{ij}-\mathrm{S}_ka_{ij}u_{ijk}\Big]\\ &-\sum_{i,j,k=1}^{n}a_{ij}u_k\Big(h_{kij}\cos\theta-2h_{ki}\sin\theta\theta_j-h_k\cos\theta\theta_i\theta_j-h_k\sin\theta\theta_{ij}\Big)\\ &-2\sum_{i,j,k=1}^{n}a_{ij}u_{ki}\big(h_{kj}\cos\theta-h_k\sin\theta\theta_j\big)\\ =:&\mathrm{J}_1+\mathrm{J}_2+\mathrm{J}_3+\mathrm{J}_4. \end{aligned}\tag{2.19}$$

We can derive that

$$\begin{aligned}
\mathrm{J}_3 =& -\sum_{i,j,k=1}^{n} a_{ij}u_k\Big(h_{kij}\cos\theta - 2h_{ki}\sin\theta\theta_j - h_k\cos\theta\theta_i\theta_j - h_k\sin\theta\theta_{ij}\Big)\\
=& -h_{111}\cos\theta\frac{u_1}{v^2} - \sum_{i=2}^{n}h_{1ii}\cos\theta u_1 + 2h_{11}\sin\theta\theta_1\frac{u_1}{v^2} + 2\sum_{i=2}^{n}h_{1i}\sin\theta\theta_i u_1\\
&+h_1\cos\theta\theta_1^2\frac{u_1}{v^2} + \sum_{i=2}^{n}h_1\cos\theta\theta_i^2 u_1 + h_1\sin\theta\theta_{11}\frac{u_1}{v^2} + \sum_{i=2}^{n}h_1\sin\theta\theta_{ii}u_1\\
\geq& -C(|\cos\theta| + |D\theta| + |D^2\theta|)u_1.
\end{aligned} \tag{2.20}$$

Using (2.15) and (2.16), we get

$$\begin{aligned}
\mathrm{J}_4 =& -2\sum_{i,j,k=1}^{n} a_{ij}u_{ki}\big(h_{kj}\cos\theta - h_k\sin\theta\theta_j\big)\\
=& -2\sum_{k=1}^{n}\big(h_{k1}\cos\theta - h_k\sin\theta\theta_1\big)\frac{u_{1k}}{v^2} - 2\sum_{k=1}^{n}\sum_{i=2}^{n}\big(h_{ki}\cos\theta - h_k\sin\theta\theta_i\big)u_{ki}\\
=& -2\big(h_{11}\cos\theta - h_1\sin\theta\theta_1\big)\frac{u_{11}}{v^2} - 2\sum_{k=2}^{n}\big(h_{k1}\cos\theta - h_k\sin\theta\theta_1\big)\frac{u_{1k}}{v^2}\\
&-2\sum_{i=2}^{n}\big(h_{1i}\cos\theta - h_1\sin\theta\theta_i\big)u_{1i} - 2\sum_{i=2}^{n}\big(h_{ii}\cos\theta - h_i\sin\theta\theta_i\big)u_{ii}\\
=& -2\big(h_{11}\cos\theta - h_1\sin\theta\theta_1\big)\left[\sum_{k=2}^{n}\frac{\mathrm{S}_k^2}{\mathrm{S}_1^2}u_{kk} + O(|\cos\theta| + |D\theta|)u_1 + O(\alpha)w\right]\frac{1}{v^2}\\
&-2\sum_{k=2}^{n}\big(h_{k1}\cos\theta - h_k\sin\theta\theta_1\big)\left[-\frac{\mathrm{S}_k}{\mathrm{S}_1}u_{kk} + O(|\cos\theta| + |D\theta|)u_1 + O(\alpha)w\right]\frac{1}{v^2}\\
&-2\sum_{i=2}^{n}\big(h_{1i}\cos\theta - h_1\sin\theta\theta_i\big)\left[-\frac{\mathrm{S}_i}{\mathrm{S}_1}u_{ii} + O(|\cos\theta| + |D\theta|)u_1 + O(\alpha)w\right]\\
&-2\sum_{i=2}^{n}\big(h_{ii}\cos\theta - h_i\sin\theta\theta_i\big)u_{ii}\\
\geq& -C(|\cos\theta| + |D\theta|)\left(\sum_{i=2}^{n}|u_{ii}| + u_1\right).
\end{aligned} \tag{2.21}$$

It follows that

$$\mathrm{J}_3 + \mathrm{J}_4 \geq -C(|\cos\theta| + |D\theta| + |D^2\theta|)u_1 - C(|\cos\theta| + |D\theta|)\sum_{i=2}^{n}|u_{ii}|. \tag{2.22}$$

A direct computation gives

$$\begin{aligned}
\mathrm{J}_2=&\sum_{i,j,k=1}^{n}\left[a_{ij}\big(\frac{u_k u_{kij}}{v}-u_{kij}h_k\cos\theta\big)-\mathrm{S}_k a_{ij,k}u_{ij}-\mathrm{S}_k a_{ij}u_{ijk}\right]\\
=&-\sum_{i,j,k=1}^{n}\mathrm{S}_k a_{ij,k}u_{ij}=-\sum_{i,j,k=1}^{n}\mathrm{S}_k u_{ij}\left(-2\frac{u_{ik}u_j}{v^2}+2\sum_{l=1}^{n}\frac{u_i u_j u_{lk}u_l}{v^4}\right)\\
=&2\frac{1}{v^2}\sum_{i=1}^{n}u_1u_{1i}\left(\sum_{l=1}^{n}\mathrm{S}_l u_{il}\right)-\frac{2}{v^4}u_1^3u_{11}\sum_{l=1}^{n}\mathrm{S}_l u_{1l}\\
=&2\frac{u_1u_{11}}{v^4}\sum_{l=1}^{n}\mathrm{S}_l u_{1l}+2\sum_{i=2}^{n}u_{1i}\frac{u_1}{v^2}\left(\sum_{l=1}^{n}\mathrm{S}_l u_{il}\right)\\
=&\frac{2u_1u_{11}}{v^4}\left[\mathrm{S}_1u_{11}+\sum_{l=2}^{n}\mathrm{S}_l u_{1l}\right]+\frac{2u_1}{v^2}\left[\mathrm{S}_1\sum_{i=2}^{n}u_{1i}^2+\sum_{i=2}^{n}\mathrm{S}_i u_{ii}u_{1i}\right]\\
=&\frac{2u_1\mathrm{S}_1}{v^4}u_{11}^2+\frac{2u_1u_{11}}{v^4}\sum_{l=2}^{n}\mathrm{S}_l u_{1l}+\frac{2u_1\mathrm{S}_1}{v^2}\sum_{i=2}^{n}u_{1i}^2+\frac{2u_1}{v^2}\sum_{i=2}^{n}\mathrm{S}_i u_{ii}u_{1i}.
\end{aligned}\tag{2.23}$$

By expanding the sum in J_1, it is easy to obtain that

$$\mathrm{J}_1=\frac{1}{v^5}u_{11}^2+\frac{2}{v^3}\sum_{i=2}^{n}u_{1i}^2+\frac{1}{v}\sum_{i=2}^{n}u_{ii}^2.\tag{2.24}$$

So we have

$$\begin{aligned}
\mathrm{J}_1+\mathrm{J}_2=&\left[\frac{1}{v^5}+\frac{2u_1\mathrm{S}_1}{v^4}\right]u_{11}^2+\left[\frac{2}{v^3}+\frac{2u_1\mathrm{S}_1}{v^2}\right]\sum_{i=2}^{n}u_{1i}^2\\
&+\frac{1}{v}\sum_{i=2}^{n}u_{ii}^2+\frac{2u_1u_{11}}{v^4}\sum_{l=2}^{n}\mathrm{S}_l u_{1l}+\frac{2u_1}{v^2}\sum_{i=2}^{n}\mathrm{S}_i u_{ii}u_{1i}.
\end{aligned}\tag{2.25}$$

From $2\geq \mathrm{S}_1\geq \frac{1}{4}$ and for $2\leq i\leq n, |\mathrm{S}_i|\leq|\cos\theta|$, we can use the Cauchy inequality

$$\left|\frac{2u_1u_{11}}{v^4}\sum_{l=2}^{n}\mathrm{S}_l u_{1l}\right|\leq\frac{2u_1\mathrm{S}_1}{v^4}u_{11}^2+\frac{(n-1)|\cos\theta|^2u_1}{2\mathrm{S}_1v^4}\sum_{i=2}^{n}u_{1i}^2,\tag{2.26}$$

$$\left|\frac{2u_1}{v^2}\sum_{i=2}^{n}\mathrm{S}_i u_{ii}u_{1i}\right|\leq\frac{u_1\mathrm{S}_1}{v^2}\sum_{i=2}^{n}u_{1i}^2+\frac{u_1|\cos\theta|^2}{\mathrm{S}_1v^2}\sum_{i=2}^{n}u_{ii}^2.\tag{2.27}$$

Substituting (2.26) and (2.27) into (2.25), if we assume $|\cos\theta|\leq\frac{1}{4}$, we obtain

$$\mathrm{J}_1+\mathrm{J}_2\geq\frac{1}{2v}\sum_{i=2}^{n}u_{ii}^2.\tag{2.28}$$

Substituting (2.19), (2.22) and (2.28), we get

$$\begin{aligned}\mathrm{J}\geq&\frac{1}{2v}\sum_{i=2}^{n}u_{ii}^2-C(|\cos\theta|+|D\theta|)\sum_{i=2}^{n}|u_{ii}|-C(|\cos\theta|+|D\theta|+|D^2\theta|)u_1\\ \geq&\frac{1}{4v}\sum_{i=2}^{n}u_{ii}^2-C(|\cos\theta|+|D\theta|+|D^2\theta|)v.\end{aligned}$$

Hence, finally we derive that

$$\mathrm{I}=\frac{\mathrm{J}}{w}\geq -C(|\cos\theta|+|D\theta|+|D^2\theta|).$$

Combining this with (2.11)–(2.13) together, we obtain

$$0\geq \mathrm{I}+\mathrm{II}+\mathrm{III}\geq -C(|\cos\theta|+|D\theta|+|D^2\theta|)-\alpha^2\left(\frac{1}{v^2}+1\right)+\alpha k_1\left(\frac{1}{v^2}+(n-1)\right).$$

By taking $\alpha:=\min\{\frac{k_1}{2},\frac{\kappa_0}{3},1\}:=\alpha_0$ and $\varepsilon_0=\min\{\frac{\kappa_0}{9(M_1+3)},\frac{\alpha_0 k_1}{2C_1},\frac{1}{4}\}$ in (1.4), we obtain

$$v(x_0,t_0)\leq \tilde{C},$$

where $\tilde{C}$ is independent of T'.

Combining three cases above together, we get the uniform estimate for $|Du|$ which is independent of T' and then Theorem 2.1 is proved. □

3 Elliptic interlude and asymptotic behavior

As the approach in the two dimension case in the paper [2], our gradient estimate also can be used to solve the elliptic version of the problem. The elliptic version of equation (1.3) is

$$\begin{cases}\displaystyle\sum_{i,j=1}^{n}\left(\delta_{ij}-\frac{u_iu_j}{1+|Du|^2}\right)u_{ij}=\tau & \text{in } \Omega,\\ u_\nu=-\cos\theta\sqrt{1+|Du|^2} & \text{on } \partial\Omega,\end{cases}\tag{3.1}$$

where $\tau\in\mathbb{R}$ is a uniquely determined constant. In fact, by using the integration by parts, one can see that

$$\tau=\frac{\int_{\partial\Omega}\cos\theta d\sigma}{\int_{\Omega}(1+|Du|^2)^{-\frac{1}{2}}dx}.\tag{3.2}$$

We can obtain the following existence result for (3.1) in high space dimension case under the condition (1.4). For 2 dimension, this result was proved by Altschuler-Wu (see Theorem 2.6 in [2]) under more generally condition on Ω and θ.

Theorem 3.1. *Let $\Omega\subset\mathbb{R}^n$ ($n\geq 3$) be a smooth strictly convex bounded domain. There is a small constant $\varepsilon_0>0$ such that if $\theta\in C^3(\overline{\Omega})$ satisfying condition* (1.4)*, then there exists a unique τ and a smooth solution u for* (3.1)*. In particular, the solution u is unique up to an additive constant.*

Proof. Firstly one wants to consider the following problem,

$$\begin{cases} \sum_{i,j=1}^{n}(\delta_{ij}-\frac{u_iu_j}{1+|Du|^2})u_{ij}=\varepsilon u & \text{in} \quad \Omega, \\ u_\nu=-\cos\theta\sqrt{1+|Du|^2} & \text{on} \quad \partial\Omega. \end{cases} \tag{3.3}$$

Using the barrier argument in [2] (see the proof of Theorem 2.6), one obtains $\sup_{\overline{\Omega}}|\varepsilon u|\leq C_0$, where C_0 is independent of ε. Under the condition (1.4), we want to show the solution to (3.3) has the uniform gradient estimate, which is independent of ε, i.e. $|Du|\leq C$. This can be proved by the same procedure as the proof of Theorem 2.1 by now replacing u_t in (1.3) with εu.

We choose the same $\Phi(x):=\log w(x)+\alpha h(x)$ and the boundary case is treated by the Case 1 in Section 2. Now we assume the $\Phi(x)$ attains its maximum at $x_0\in\Omega$.

So at x_0, we have $0=\Phi_i(x_0)=\frac{w_i}{w}+\alpha h_i$ and

$$0\geq\sum_{i,j=1}^{n}a_{ij}\Phi_{ij}(x_0)=\sum_{i,j=1}^{n}\frac{a_{ij}w_{ij}}{w}-\alpha^2\sum_{i,j=1}^{n}a_{ij}h_ih_j+\alpha\sum_{i,j=1}^{n}a_{ij}h_{ij}=:\mathrm{I}+\mathrm{II}+\mathrm{III}. \tag{3.4}$$

We choose a proper coordinates at x_0 such that $|Du|(x_0)=u_1(x_0)$ and $\{u_{ij}(x_0)\}_{2\leq i,j\leq n}$ is diagonal. Then at x_0,

$$\mathrm{II}+\mathrm{III}\geq-\alpha^2\left(\frac{1}{v^2}+1\right)+\alpha k_1\left(\frac{1}{v^2}+(n-1)\right). \tag{3.5}$$

We denote by $\mathrm{J}:=\sum_{i,j=1}^{n}a_{ij}w_{ij}$ and as in (2.19), it follows that

$$\mathrm{J}=w\mathrm{I}:=\mathrm{J}_1+\widetilde{J}_2+\mathrm{J}_3+\mathrm{J}_4, \tag{3.6}$$

where $\mathrm{J}_1,\mathrm{J}_3,\mathrm{J}_4$ are defined as in (2.19). Using Eq. (3.3) we have

$$\begin{aligned}\widetilde{J}_2:=&\sum_{i,j,k=1}^{n}a_{ij}(\frac{u_ku_{kij}}{v}-u_{kij}h_k\cos\theta)=\sum_{i,j,k=1}^{n}\mathrm{S}_k a_{ij}u_{ijk}\\ =&-\sum_{i,j,k=1}^{n}\mathrm{S}_k a_{ij,k}u_{ij}+\sum_{k=1}^{n}\mathrm{S}_k(\varepsilon u)_k=\mathrm{J}_2+\varepsilon u_1\mathrm{S}_1\geq\mathrm{J}_2.\end{aligned} \tag{3.7}$$

By the same procedure as the proof of Theorem 2.1 we can get the the uniform gradient estimates which is independent of ε.

From the C^0 and C^1 estimates, we get the existence of the solution to (3.3) by standard theory in [11] for each $\varepsilon>0$.

The uniform estimate of $|Du|$ also implies that $|D(\varepsilon u)|\to 0$ as $\varepsilon\to 0$. Thus one can conclude that $\varepsilon u\to\tau$ as $\varepsilon\to 0$ for some $\tau\in\mathbb{R}$ (One can see the detail proof on the existence part for the similar limited equation in Theorem 1.3).

To show the uniqueness, as in [2] (see the proof of Theorem 2.6 there) one assumes that if (u_1,τ_1) and (u_2,τ_2) are two solutions to (3.1). Without loss of generality, we assume $\tau_1\le\tau_2$, denote $u:=u_1-u_2$. By direct computation, we obtain that u is the super-solution of the following elliptic operator

$$Lu:=\sum_{i,j=1}^{n}\hat{a}_{ij}u_{ij}+\sum_{l=1}^{n}b_l u_l,$$

where

$$\hat{a}_{ij}:=a_{ij}(Du_1),\quad b_l:=\sum_{i,j=1}^{n}\int_0^1 a_{ij,p_l}(tDu_1+(1-t)Du_2)dt\cdot(u_2)_{ij}.$$

From the maximum principle, u attains the maximum value on the boundary, say at $x_0\in\partial\Omega$. Thus combining with Hopf lemma, we have $\nabla' u(x_0)=0$ and $D_\nu u(x_0)<0$, that is, $|\nabla' u_1|=|\nabla' u_2|=q$ and $D_\nu u_1<D_\nu u_2$, where we denote ∇' and D_ν as the tangential and normal part of D on boundary respectively. On the other hand, from the boundary condition in (3.1), it follows that

$$\frac{D_\nu u_1}{\sqrt{1+q^2+|D_\nu u_1|^2}}=\frac{D_\nu u_2}{\sqrt{1+q^2+|D_\nu u_2|^2}},$$

but this is a contradiction with the fact that function $\frac{x}{\sqrt{1+q^2+x^2}}$ is strictly increasing with respect to $x\in\mathbb{R}$ and $D_\nu u_1(x_0)<D_\nu u_2(x_0)$. So u must be a constant, and $\tau_1=\tau_2$. Therefore, we have completed the proof. □

Remark 3.1. For any $\varepsilon>0$, even for any bounded smooth domain, the existence of solution to (3.3) could be got from the standard method on the capillary problem with positive gravity, see for example Theorem 9.12 in Lieberman [22] or [19].

Remark 3.2. If one considers the following problem,

$$\begin{cases}\sum_{i,j=1}^{n}\left(\delta_{ij}-\frac{u_iu_j}{1+|Du|^2}\right)u_{ij}=\varepsilon u(1+|Du|^2)^{\beta} & \text{in }\Omega,\\ u_\nu=-\cos\theta\sqrt{1+|Du|^2} & \text{on }\partial\Omega,\end{cases}\tag{3.8}$$

then for $0\le\beta<\frac{1}{2}$, using the same calculation we can get the uniform gradient estimates of u for any $\varepsilon>0$. When $\beta=\frac{1}{2}$, we need make a more careful computation in Section 4 so as to get the corresponding existence theorem.

For a solution $w=w(x)$ to (3.1), it is obvious that $\tilde{w}=w(x)+\lambda t$ solves the following parabolic problem,

$$\begin{cases} u_t=(\delta_{ij}-\dfrac{u_iu_j}{1+|Du|^2})u_{ij} & \text{in } \Omega\times(0,\infty),\\ u_\nu=-\cos\theta\sqrt{1+|Du|^2} & \text{on } \partial\Omega\times(0,\infty),\\ u(x,0)=w(x) & \text{on } \overline{\Omega}, \end{cases}$$

Corollary 3.1. For a solution $u=u(x,t)$ to (1.1), there exists a time-indenpendent positive constant C, such that

$$|u(x,t)-\lambda t|\le C.$$

Proof. Let $z(x,t)=u(x,t)-\tilde{w}(x,t)$, then it satisfies the following equation

$$z_t=\tilde{a}_{ij}z_{ij}+b_iz_i,$$

where $\tilde{a}_{ij}=a_{ij}(Du)$ and $b_i=(\tilde{w})_{kl}\int_0^1 a_{kl,p_i}(\eta Du+(1-\eta)D\tilde{w})d\eta$. If z achieves its maximum and minimum on $\overline{\Omega}\times\{0\}$, then

$$\sup_{\Omega\times(0,\infty)}|u-w-\lambda t|=\sup_{\Omega\times(0,\infty)}|z|\le\sup_{\Omega\times\{0\}}|z|=\sup_{\Omega}|u_0-w|.$$

Therefore,

$$\sup_{\Omega\times(0,\infty)}|u-\lambda t|\le\sup_{\Omega}|w|+\sup_{\Omega}|u_0-w|.$$

If z attains its maximum or minimum on $\partial\Omega\times(0,\infty)$, then as in the uniqueness part proof of Theorem 3.1, Hopf's lemma tells us u_0-w must be a constant. Therefore $u-\tilde{w}$ also must be a constant on $\Omega\times(0,\infty)$ for the uniqueness of the solution to (1.1), so we have

$$\sup_{\Omega\times(0,\infty)}|u-\lambda t|\le\sup_{\Omega\times(0,\infty)}|u-\lambda t-w|+\sup_{\Omega}|w|=\sup_{\Omega}|u_0-w|+\sup_{\Omega}|w|. \qquad \square$$

Using the technique in [2], the uniform estimates in Lemma 2.1, Theorem 2.1 and Schauder estimates, we get the following result.

Lemma 3.1. *Let u_1 and u_2 be any two solutions to equation* (1.1)*, with initial data $u_{0,1}$ and $u_{0,2}$ respectively. Let $u=u_1-u_2$, then u converges to a constant function as $t\to\infty$. In particular, the limit of any solution to Eq.* (1.1) *is $\tilde{w}$ up to a constant.*

Proof. We now see that u satisfies a linear parabolic equation

$$\begin{cases} z_t=\tilde{a}_{ij}z_{ij}+b_iz_i & \text{in } \Omega\times(0,\infty),\\ \dfrac{D_\nu u_1}{\sqrt{1+|Du_1|^2}}=\dfrac{D_\nu u_2}{\sqrt{1+|Du_2|^2}} & \text{on } \partial\Omega\times(0,\infty), \end{cases}$$

where $\widetilde{a}_{ij} = a_{ij}(Du_1)$ and $b_i = (u_2)_{kl}\int_0^1 a_{kl,p_i}(\eta Du_1 + (1-\eta)Du_2)\mathrm{d}\eta$. The strong maximum principle implies that $osc(u)(t) = \max_{\overline{\Omega}} u(x,t) - \min_{\overline{\Omega}} u(x,t) \geq 0$ is a strictly decreasing function unless u is a constant.

We claim that

$$\lim_{t\to\infty} osc(u)(t) = 0.$$

Otherwise, if $\lim_{t\to\infty} osc(u)(t) = \delta$ for some $\delta > 0$, we will reach a contradiction. In fact, given a sequence $t_n \to +\infty$, we define

$$u_{1,n}(\cdot,t) = u_1(\cdot,t+t_n) - \lambda t_n$$

and

$$u_{2,n}(\cdot,t) = u_2(\cdot,t+t_n) - \lambda t_n.$$

By Corollary 3, for $i = 1,2$, we know $|u_{i,n} - \lambda t| \leq C$, remark that the uniform(independent of n) estimates on $\frac{\partial u_{i,n}}{\partial t}, |Du_{i,n}|$ have already been obtained in Lemma 2.1 and Theorem 2.1. According to Schauder theory [21], $u_{1,n}(\cdot,t)$ and $u_{2,n}(\cdot,t)$ are locally (in time) C^k uniformly bounded with respect to n for any k.

So, there exists a subsequence (still denoted by t_n) such that $u_{1,n}(\cdot,t)$ and $u_{2,n}(\cdot,t)$ converge locally uniformly in any C^k to $u_1^*(\cdot,t)$ and $u_2^*(\cdot,t)$ respectively. That is

$$u_1^*(\cdot,t) = \lim_{n\to\infty} u_{1,n}(\cdot,t), \quad u_2^*(\cdot,t) = \lim_{n\to\infty} u_{2,n}(\cdot,t).$$

Let $u^* = u_1^* - u_2^*$, then we deduce that

$$\begin{aligned} osc(u^*)(t) &= osc(u_1^* - u_2^*) \\ &= \lim_{n\to\infty} osc(u_1(x,t+t_n) - \lambda t_n - u_2(x,t+t_n) + \lambda t_n) \\ &= \lim_{n\to\infty} osc(u_1(x,t+t_n) - u_2(x,t+t_n)) \\ &= \lim_{n\to\infty} osc(u)(t+t_n) = \delta, \end{aligned} \tag{3.9}$$

where the second equality holds because of the uniform convergence of $u_{1,n}(\cdot,t)$ and $u_{2,n}(\cdot,t)$.

But u^* satisfies the uniformly parabolic equation

$$\begin{cases} z_t = \widetilde{a}_{ij} z_{ij} + b_i z_i & \text{in } \Omega\times(-\infty,\infty), \\ \dfrac{D_\nu u_1^*}{\sqrt{1+|Du_1^*|^2}} = \dfrac{D_\nu u_2^*}{\sqrt{1+|Du_2^*|^2}} & \text{on } \partial\Omega\times(-\infty,\infty), \end{cases}$$

where $\widetilde{a}_{ij} = a_{ij}(Du_1^*)$ and $b_i = (u_2^*)_{kl}\int_0^1 a_{kl,p_i}(\eta Du_1^* + (1-\eta)Du_2^*)\mathrm{d}\eta$.

By the strong maximum principle and Hopf's lemma, we know u^* is a constant. This makes a contradiction to $osc(u^*)(t) \equiv \delta$ and the claim now is proved.

According to the claim, we have

$$\lim_{t\to\infty}\max_{\overline{\Omega}} u=\lim_{t\to\infty}\min_{\overline{\Omega}} u=c_0 \quad \text{for some constant } c_0.$$

It then follows that $\lim\limits_{t\to\infty}|u-c_0|=0$ and we finish the proof of this lemma. □

Proof of Theorem 1.1. From Lemma 2.1 and Theorem 2.1 and Schauder estimate, we obtain uniform estimates in any C^k-norm for the derivatives of u, and locally (in time) uniform bounds for the C_0 norm. So we get longtime existence with uniform bounds on all higher derivatives of u. From Corollary 3 and Lemma 3.1, the limit of any solution to Eq. (1.1) is $\tilde{w}=w+\lambda t$ up to a constant, where (λ,w) is the solution to Eq. (3.1) by Theorem 3.1. □

4 Constant mean curvature equation with prescribed contact angle boundary value condition

In this section, we consider the capillary problem with prescribed contact angle boundary condition. We first obtain the following uniform gradient estimate of u for equations (4.1). Note that, for any fixed $\varepsilon>0$, the existence of solutions to (4.1) is well-known, see [8] for example. Now we can prove the following lemma.

Lemma 4.1. *Let $\Omega\subset\mathbb{R}^n$ $(n\geq 2)$ be a strictly convex, bounded domain and $\partial\Omega\in C^3$. There is a small constant ε_0 such that if $\theta\in C^3(\overline{\Omega})$ satisfying condition* (1.4)*, and if u is the solution to the following mean curvature type equation with prescribed contact angle boundary condition,*

$$\begin{cases}\operatorname{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right)=\varepsilon u & \text{in } \Omega,\\ u_\nu=-\cos\theta\sqrt{1+|Du|^2} & \text{on } \partial\Omega,\end{cases} \tag{4.1}$$

then there exists a constant $C\sim n,\Omega$ such that

$$\sup_{\overline{\Omega}}|Du|\leq C. \tag{4.2}$$

Remark 4.1. The idea and approach for proving Lemma 4.1 are similar with those showed in Theorem 3.1. However, as we remark in Remark 3.2 that there are new difficulties arising here. Moreover, it follows from the remark below Theorem 1 in the work [5] of Concus and Finn that $|u|\leq\frac{C_1}{\varepsilon}+C_2$. Denoting by $f=\varepsilon u$, we have

$$\sup_{\overline{\Omega}}|f|=\sup_{\overline{\Omega}}|\varepsilon u_\varepsilon|\leq C_0, \tag{4.3}$$

where C_0 is independent of ε. One can also derive the same result following the barrier argument in [2] (Just note that there is one more factor $\frac{1}{\sqrt{1+|Du|^2}}$ in (4.1) compared to the proof of Theorem 2.6 in [2], which does not affect the barrier argument).

Proof. Let

$$\Phi(x) = \log w(x) + \alpha h(x),$$

where $w(x) = v - \sum_{l=1}^{n} u_l h_l \cos\theta$ and α is a positive constant to be determined later, $v = \sqrt{1+|Du|^2}$. Denoting here by $a_{ij} := v^2\delta_{ij} - u_i u_j$, Eq. (4.1) now can be expressed to be

$$\sum_{i,j=1}^{n} a_{ij} u_{ij} = f v^3.$$

Assume that $\Phi(x)$ attains the maximum value at $x_0 \in \overline{\Omega}$. We divide it into the following two cases to complete the proof.

Case 1: If $x_0 \in \partial\Omega$, this is the same as in Case 1 in Theorem 2.1, since we retain the same boundary condition. By choosing the same α, ε_0 as in (2.8), we obtain the same conclusion.

Case 2: If $x_0 \in \Omega$, we have

$$0 = \Phi_i = \frac{w_i}{w} + \alpha h_i, \tag{4.4}$$

$$0 \geq \sum_{i,j=1}^{n} a_{ij}\Phi_{ij} = \sum_{i,j=1}^{n} \frac{a_{ij} w_{ij}}{w} - \sum_{i,j=1}^{n} \alpha^2 a_{ij} h_i h_j + \sum_{i,j=1}^{n} \alpha a_{ij} h_{ij} =: \mathrm{I} + \mathrm{II} + \mathrm{III}. \tag{4.5}$$

We choose a proper coordinates at (x_0, t_0) such that $|Du|(x_0) = u_1(x_0) > 0$ and $\{u_{ij}(x_0)\}_{2\leq i,j\leq n}$ is diagonal. Then at x_0, $a_{11} = 1$ and $a_{ii} = v^2$, for $i = 2, \cdots, n$.

We always assume that $u_1(x_0)$ is large enough in the below computation, such that u_1, v, and w (since we assume $|\cos\theta| \leq \varepsilon_0 < 1$) are equivalent to each other at x_0. Otherwise, we have completed the proof. All the computation below are at the point x_0.

We start to deal with the terms in (4.5).

$$\mathrm{II} + \mathrm{III} = \sum_{i,j=1}^{n} \alpha a_{ij} h_{ij} - \sum_{i,j=1}^{n} \alpha^2 a_{ij} h_i h_j \geq \alpha k_1 [1 + v^2(n-1)] - \alpha^2 [1 + v^2]. \tag{4.6}$$

We denote by $\mathrm{J} := \sum_{i,j=1}^{n} a_{ij} w_{ij}$. From (4.4), we have

$$\sum_{l=1}^{n} \left(\frac{u_l u_{li}}{v} - u_{li} h_l \cos\theta - u_l h_{li} \cos\theta + u_l h_l \sin\theta\theta_i \right) = -\alpha h_i w.$$

If we denote by

$$\mathrm{S}_l := \frac{u_l}{v} - h_l \cos\theta, \qquad \text{for } l = 1, \cdots, n,$$

then we obtain

$$\sum_{l=1}^{n} \mathrm{S}_l u_{li} = h_{1i} \cos\theta u_1 - h_1 \sin\theta\theta_i u_1 - \alpha h_i w. \tag{4.7}$$

We also have

$$u_{1i}=-\frac{S_i}{S_1}u_{ii}+\frac{1}{S_1}\sum_{l=1}^{n}S_l u_{li},\qquad \text{for } i=2,\cdots,n, \tag{4.8}$$

$$u_{11}=-\sum_{k=2}^{n}\frac{S_k}{S_1}u_{1k}+\frac{1}{S_1}\sum_{l=1}^{n}S_l u_{l1}=\sum_{i=2}^{n}\frac{S_i^2}{S_1^2}u_{ii}-\sum_{i=2}^{n}\frac{S_i}{S_1^2}\left(\sum_{l=1}^{n}S_l u_{li}\right)+\frac{1}{S_1}\sum_{l=1}^{n}S_l u_{l1}. \tag{4.9}$$

From the equation we have,

$$fv=\frac{u_{11}}{v^2}+\sum_{i=2}^{n}u_{ii}. \tag{4.10}$$

On the other hand, since

$$fv^3=\sum_{i,j=1}^{n}(v^2\delta_{ij}-u_iu_j)u_{ij}=v^2\Delta u-u_1^2u_{11},$$

we have

$$3fv-2\Delta u=fv-2\frac{u_1^2}{v^2}u_{11}=\left(\frac{1}{v^2}-2\frac{u_1^2}{v^2}\right)u_{11}+\sum_{i=2}^{n}u_{ii}. \tag{4.11}$$

Substituting (4.9) and (4.10) into (4.11), we obtain

$$\begin{aligned}&3fv-2\Delta u \\ =&\sum_{i=2}^{n}\left(\frac{1}{v^2}-\frac{2u_1^2}{v^2}\right)\frac{S_i^2}{S_1^2}u_{ii}+\sum_{i=2}^{n}u_{ii}-\sum_{i=2}^{n}\left(\frac{1}{v^2}-\frac{2u_1^2}{v^2}\right)\frac{S_i}{S_1^2}\Big(\sum_{l=1}^{n}S_l u_{li}\Big)+\Big(\frac{1}{v^2}-\frac{2u_1^2}{v^2}\Big)\frac{1}{S_1}\sum_{l=1}^{n}S_l u_{l1}.\end{aligned} \tag{4.12}$$

As in Section 2, by (2.18) and $J:=\sum_{i,j=1}^{n}a_{ij}w_{ij}$ we have

$$\begin{aligned}J=&\sum_{i,j,k=1}^{n}a_{ij}\left(\frac{u_{ki}u_{kj}}{v}-\sum_{l=1}^{n}\frac{u_ku_{ki}u_lu_{lj}}{v^3}\right)+\sum_{i,j,k=1}^{n}a_{ij}\left(\frac{u_ku_{kij}}{v}-u_{kij}h_k\cos\theta\right)\\ &-\sum_{i,j,k=1}^{n}a_{ij}u_k\big(h_{kij}\cos\theta-2h_{ki}\sin\theta\theta_j-h_k\cos\theta\theta_i\theta_j-h_k\sin\theta\theta_{ij}\big)\\ &-2\sum_{i,j,k=1}^{n}a_{ij}u_{ki}(h_{kj}\cos\theta-h_k\sin\theta\theta_j)\\ =:&J_1+J_2+J_3+J_4,\end{aligned} \tag{4.13}$$

where J_1, J_3 and J_4 are almost the same as in the proof of Theorem 2.1. We have

$$J_1=\sum_{i,j,k=1}^{n} a_{ij}\left(\frac{u_{ki}u_{kj}}{v}-\sum_{l=1}^{n}\frac{u_k u_{ki} u_l u_{lj}}{v^3}\right)=\frac{1}{v^3}u_{11}^2+\frac{2}{v}\sum_{i=2}^{n}u_{1i}^2+v\sum_{i=2}^{n}u_{ii}^2, \tag{4.14}$$

$$\begin{aligned} J_3=&-\sum_{i,j,k=1}^{n} a_{ij}u_k\left(h_{kij}\cos\theta-2h_{ki}\sin\theta\theta_j-h_k\cos\theta\theta_i\theta_j-h_k\sin\theta\theta_{ij}\right)\\ \geq&-C(|\cos\theta|+|D\theta|+|D^2\theta|)v^3, \end{aligned} \tag{4.15}$$

$$\begin{aligned} J_4=&-2\sum_{i,j,k=1}^{n} a_{ij}u_{ki}(h_{kj}\cos\theta-h_k\sin\theta\theta_j)\\ =&(|\cos\theta|+|D\theta|)[O(v^2)+O(v)]\sum_{i=2}^{n}|u_{ii}|+O(v^2u_1)(|\cos\theta|+|D\theta|)\\ \geq&-\frac{v}{4}\sum_{i=2}^{n}u_{ii}^2-C(|\cos\theta|+|D\theta|)v^3. \end{aligned} \tag{4.16}$$

A direct computation gives

$$\begin{aligned} J_2:=&\sum_{i,j,k=1}^{n} a_{ij}\Big(\frac{u_k u_{kij}}{v}-u_{kij}h_k\cos\theta\Big)=\sum_{i,j,l=1}^{n} a_{ij}S_l u_{lij}\\ =&\sum_{i,j,l=1}^{n} S_l(a_{ij}u_{ij})_l-\sum_{i,j,l=1}^{n} S_l a_{ij,l}u_{ij}=\sum_{l=1}^{n}S_l(fv^3)_l-\sum_{i,j,l=1}^{n} S_l a_{ij,l}u_{ij}\\ =&\sum_{l=1}^{n}S_l f_l v^3+3\sum_{k,l=1}^{n} S_l f v u_k u_{kl}-2\sum_{k,l=1}^{n} S_l u_k u_{kl}\Delta u+\sum_{i,j,l=1}^{n} 2S_l u_{il}u_j u_{ij}\\ =&u_1(3fv-2\Delta u)\sum_{l=1}^{n}S_l u_{1l}+2u_{11}u_1\sum_{l=1}^{n}S_l u_{1l}+2\sum_{i=2}^{n}u_{1i}u_1\Big(\sum_{l=1}^{n}S_l u_{il}\Big)+\sum_{l=1}^{n}S_l f_l v^3\\ =:&J_{21}+J_{22}+J_{23}+J_{24}. \end{aligned} \tag{4.17}$$

Note that $f_l=\varepsilon u_l$, and $S_1=\frac{u_1}{v}-h_1\cos\theta$, suppose that $|\cos\theta|<\frac{1}{2}$, we have

$$J_{24}=\sum_{l=1}^{n}S_l f_l v^3=S_1\varepsilon u_1 v^3>0. \tag{4.18}$$

It follows from (4.12) that

$$\begin{aligned} J_{21}=&u_1\sum_{l=1}^{n}S_l u_{1l}(3fv-2\Delta u) \qquad (4.19)\\ =&u_1\left(\sum_{l=1}^{n}S_l u_{1l}\right)\left[\sum_{i=2}^{n}\left(\frac{1}{v^2}-\frac{2u_1^2}{v^2}\right)\frac{S_i^2}{S_1^2}u_{ii}+\sum_{i=2}^{n}u_{ii}\right]\\ &-u_1\left(\sum_{m=1}^{n}S_m u_{1m}\right)\left[\sum_{i=2}^{n}\Big(\frac{1}{v^2}-\frac{2u_1^2}{v^2}\Big)\frac{S_i}{S_1^2}\left(\sum_{l=1}^{n}S_l u_{li}\right)-\left(\frac{1}{v^2}-\frac{2u_1^2}{v^2}\right)\frac{1}{S_1}\left(\sum_{l=1}^{n}S_l u_{l1}\right)\right]. \end{aligned}$$

Using (4.9), we have

$$\begin{aligned}\mathrm{J}_{22}=&2\sum_{l=1}^{n}\mathrm{S}_l u_{1l}u_1u_{11} \qquad (4.20)\\=&2\Big(\sum_{l=1}^{n}\mathrm{S}_l u_{1l}\Big)u_1\Big(\sum_{i=2}^{n}\frac{\mathrm{S}_i^2}{\mathrm{S}_1^2}u_{ii}\Big)-2\Big(\sum_{m=1}^{n}\mathrm{S}_m u_{1m}\Big)u_1\Big[\sum_{i=2}^{n}\frac{\mathrm{S}_i}{\mathrm{S}_1^2}\Big(\sum_{l=1}^{n}\mathrm{S}_l u_{li}\Big)-\frac{1}{\mathrm{S}_1}\Big(\sum_{l=1}^{n}\mathrm{S}_l u_{l1}\Big)\Big].\end{aligned}$$

Using (4.8), we have

$$\mathrm{J}_{23}=2\sum_{i=2}^{n}u_{1i}u_1\Big(\sum_{l=1}^{n}\mathrm{S}_l u_{il}\Big)=-2\sum_{i=2}^{n}u_1\frac{\mathrm{S}_i}{\mathrm{S}_1}u_{ii}\Big(\sum_{l=1}^{n}\mathrm{S}_l u_{il}\Big)+2\sum_{i=2}^{n}\frac{u_1}{\mathrm{S}_1}\Big(\sum_{l=1}^{n}\mathrm{S}_l u_{il}\Big)^2. \qquad (4.21)$$

Substituting (4.18)–(4.21) into (4.17), we obtain

$$\begin{aligned}\mathrm{J}_2\geq& u_1\Big(\sum_{l=1}^{n}\mathrm{S}_l u_{1l}\Big)\Big[\sum_{i=2}^{n}\Big(\frac{1}{v^2}-\frac{2u_1^2}{v^2}\Big)\frac{\mathrm{S}_i^2}{\mathrm{S}_1^2}u_{ii}+\sum_{i=2}^{n}u_{ii}\Big]\\&-u_1\Big(\sum_{m=1}^{n}\mathrm{S}_m u_{1m}\Big)\Big[\sum_{i=2}^{n}\Big(\frac{1}{v^2}-\frac{2u_1^2}{v^2}\Big)\frac{\mathrm{S}_i}{\mathrm{S}_1^2}\big(\sum_{l=1}^{n}\mathrm{S}_l u_{li}\big)-\Big(\frac{1}{v^2}-\frac{2u_1^2}{v^2}\Big)\frac{1}{\mathrm{S}_1}\Big(\sum_{l=1}^{n}\mathrm{S}_l u_{l1}\Big)\Big]\\&+2\Big(\sum_{l=1}^{n}\mathrm{S}_l u_{1l}\Big)u_1\Big(\sum_{i=2}^{n}\frac{\mathrm{S}_i^2}{\mathrm{S}_1^2}u_{ii}\Big)-2\Big(\sum_{m=1}^{n}\mathrm{S}_m u_{1m}\Big)u_1\Big[\sum_{i=2}^{n}\frac{\mathrm{S}_i}{\mathrm{S}_1^2}\big(\sum_{l=1}^{n}\mathrm{S}_l u_{li}\big)-\frac{1}{\mathrm{S}_1}\Big(\sum_{l=1}^{n}\mathrm{S}_l u_{l1}\Big)\Big]\\&-2\sum_{i=2}^{n}u_1\frac{\mathrm{S}_i}{\mathrm{S}_1}u_{ii}\Big(\sum_{l=1}^{n}\mathrm{S}_l u_{il}\Big)+2\sum_{i=2}^{n}\frac{u_1}{\mathrm{S}_1}\Big(\sum_{l=1}^{n}\mathrm{S}_l u_{il}\Big)^2\\=&\frac{3u_1}{v^2}\Big(\sum_{l=1}^{n}\mathrm{S}_l u_{1l}\Big)\Big(\sum_{i=2}^{n}\frac{\mathrm{S}_i^2}{\mathrm{S}_1^2}u_{ii}\Big)+\frac{3u_1}{v^2}\frac{1}{\mathrm{S}_1}\Big(\sum_{l=1}^{n}\mathrm{S}_l u_{l1}\Big)^2-\frac{3u_1}{v^2}\Big(\sum_{m=1}^{n}\mathrm{S}_m u_{1m}\Big)\sum_{i=2}^{n}\frac{\mathrm{S}_i}{\mathrm{S}_1^2}\Big(\sum_{l=1}^{n}\mathrm{S}_l u_{li}\Big)\\&+u_1\Big(\sum_{l=1}^{n}\mathrm{S}_l u_{1l}\Big)\Big(\sum_{i=2}^{n}u_{ii}\Big)-2u_1\sum_{i=2}^{n}\frac{\mathrm{S}_i}{\mathrm{S}_1}u_{ii}\Big(\sum_{l=1}^{n}\mathrm{S}_l u_{il}\Big)+2\sum_{i=2}^{n}\frac{u_1}{\mathrm{S}_1}\Big(\sum_{l=1}^{n}\mathrm{S}_l u_{il}\Big)^2\\=:&\mathrm{K}_1+\mathrm{K}_2+\mathrm{K}_3+\mathrm{K}_4+\mathrm{K}_5+\mathrm{K}_6. \qquad (4.22)\end{aligned}$$

Using the fact $ax^2+bx\geq-\frac{b^2}{4a}$ for any $a>0$, we obtain

$$\begin{aligned}&\mathrm{K}_1+\mathrm{K}_2+\mathrm{K}_5+\mathrm{K}_6\\=&\frac{3u_1}{v^2}\Big(\sum_{l=1}^{n}\mathrm{S}_l u_{1l}\Big)\Big(\sum_{i=2}^{n}\frac{\mathrm{S}_i^2}{\mathrm{S}_1^2}u_{ii}\Big)+\frac{3u_1}{v^2}\Big(\frac{1}{\mathrm{S}_1}\sum_{l=1}^{n}\mathrm{S}_l u_{l1}\Big)^2\\&-2u_1\sum_{i=2}^{n}\frac{\mathrm{S}_i}{\mathrm{S}_1}u_{ii}\Big(\sum_{l=1}^{n}\mathrm{S}_l u_{il}\Big)+2\sum_{i=2}^{n}\frac{u_1}{\mathrm{S}_1}\Big(\sum_{l=1}^{n}\mathrm{S}_l u_{il}\Big)^2\\\geq&-\frac{3(n-1)u_1}{4v^2}\sum_{i=2}^{n}\frac{\mathrm{S}_i^4}{\mathrm{S}_1^2}u_{ii}^2-\frac{u_1}{2}\sum_{i=2}^{n}\frac{\mathrm{S}_i^2}{\mathrm{S}_1}u_{ii}^2.\end{aligned}$$

Noting that $S_i = O(\cos\theta)$ for $i \geq 2$, u_1, v are comparable provided $u_1 \geq 1$, and by (4.7) we see that.

$$\mathrm{K}_3 = -\frac{3u_1}{v^2}\left(\sum_{l=1}^{n} \mathrm{S}_l u_{1l}\right)\sum_{i=2}^{n}\frac{\mathrm{S}_i}{\mathrm{S}_1^2}\left(\sum_{l=1}^{n}\mathrm{S}_l u_{li}\right) \geq -C(|\cos\theta|+|D\theta|)+|\alpha|)v.$$

Now we deal with term K_4, by (4.7), we have

$$\begin{aligned}
|\mathrm{K}_4| =& |u_1\left(\sum_{l=1}^{n}\mathrm{S}_l u_{1l}\right)\left(\sum_{i=2}^{n} u_{ii}\right)| \\
\leq & |(h_{11}\cos\theta - h_1\sin\theta\theta_1)|u_1^2\sum_{i=2}^{n}|u_{ii}| + |\alpha h_1 u_1 w\sum_{i=2}^{n} u_{ii}| \\
\leq & C(|\cos\theta|+|D\theta|)v^3 + C(|\cos\theta|+|D\theta|)v\sum_{i=2}^{n}u_{ii}^2 + \frac{u_1}{4}\sum_{i=2}^{n}u_{ii}^2 + (n-1)\alpha^2 u_1 w^2.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\mathrm{J}_2 \geq & -\frac{3(n-1)u_1}{4v^2}\sum_{i=2}^{n}\frac{\mathrm{S}_i^4}{\mathrm{S}_1^2}u_{ii}^2 - \frac{u_1}{2}\sum_{i=2}^{n}\frac{\mathrm{S}_i^2}{\mathrm{S}_1}u_{ii}^2 - \frac{u_1}{3}\sum_{i=2}^{n}u_{ii}^2 - \alpha^2(n-1)u_1 w^2 \\
& -C(|\cos\theta|+|D\theta|)v^3 - C(|\cos\theta|+|D\theta|+|\alpha|)v. \qquad (4.23)
\end{aligned}$$

Substituting (4.14), (4.23), (4.15) and (4.16) into (4.13), we have

$$\begin{aligned}
\mathrm{J} \geq & \frac{1}{v^3}u_{11}^2 + \frac{2}{v}\sum_{i=2}^{n}u_{1i}^2 + v\sum_{i=2}^{n}u_{ii}^2 - \frac{v}{4}\sum_{i=2}^{n}u_{ii}^2 \\
& -\frac{3(n-1)u_1}{4v^2}\sum_{i=2}^{n}\frac{\mathrm{S}_i^4}{\mathrm{S}_1^2}u_{ii}^2 - \frac{u_1}{2}\sum_{i=2}^{n}\frac{\mathrm{S}_i^2}{\mathrm{S}_1}u_{ii}^2 - \frac{u_1}{3}\sum_{i=2}^{n}u_{ii}^2 - \alpha^2(n-1)u_1 w^2 \\
& -C(|\cos\theta|+|D\theta|+|D^2\theta|)v^3 - C(|\cos\theta|+|D\theta|+|\alpha|)v \\
\geq & -C(|\cos\theta|+|D\theta|+|D^2\theta|)v^3 - \alpha^2(n-1)u_1 w^2, \qquad (4.24)
\end{aligned}$$

where we take $|\cos\theta| \leq \frac{1}{100}$ such that $\frac{1}{2}\frac{\mathrm{S}_i^2}{\mathrm{S}_1} \leq \frac{1}{4}$.

Note that $\frac{v}{2} < w < \frac{3v}{2}$, we have

$$\mathrm{I} := \frac{\mathrm{J}}{w} \geq -2C(|\cos\theta|+|D\theta|+|D^2\theta|)v^2 - 2n\alpha^2 v^2. \qquad (4.25)$$

Substituting (4.25) and (4.6) into (4.5), we obtain

$$0 \geq \mathrm{I}+\mathrm{II}+\mathrm{III} \geq \alpha k_1[1+v^2(n-1)] - \alpha^2[1+v^2] - 2C(|\cos\theta|+|D\theta|+|D^2\theta|)v^2 - 2n\alpha^2 v^2.$$

By taking $\alpha=\min\{\frac{k_1}{8},\frac{\kappa_0}{3},1\}:=\alpha_0$ and $\varepsilon_0:=\min\{\frac{\kappa_0}{9(M_1+3)},\frac{\alpha_0 k_1}{16C_1},\frac{1}{100}\}$ in (1.4), we obtain

$$v(x_0)\le\tilde{C}.$$

Finally, combining all above two cases together, thus we have $v(x_0)\le\tilde{C}$, where $\tilde{C}$ is independent of ε and $\|u\|_{C^0}$. □

With the preparations above, now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. To solve Eq. (1.7), we will estimate the solutions to the following family of equations $(\star_{\varepsilon,L})$.

$$\begin{cases}\operatorname{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right)=\varepsilon u-L & \text{in } \Omega,\\ u_\nu=-\cos\theta\sqrt{1+|Du|^2} & \text{on } \partial\Omega,\end{cases}\tag{$\star_{\varepsilon,L}$}$$

for any given $\varepsilon\in(0,1)$ and $L\in\mathbb{R}$.

Firstly, from [19] or [20], we know that there exists a unique solution $u_{\varepsilon,0}$ to problem $(\star_{\varepsilon,0})$ for $L=0$ and any $\varepsilon>0$. Hence for general $L\in\mathbb{R}$, one denotes $u_{\varepsilon,L}(x):=u_{\varepsilon,0}(x)+\frac{L}{\varepsilon}$, thus it solves $(\star_{\varepsilon,L})$ and also be the unique solution of problem $(\star_{\varepsilon,L})$.

Secondly, we show that, for any $\varepsilon\in(0,1)$, there exists a unique constant L_ε satisfying $\|u_{\varepsilon,L_\varepsilon}\|_{C^1(\overline{\Omega})}\le C$, where C is independent of ε. In fact, one can achieve this by just constructing the supersolution and subsolution of $(\star_{\varepsilon,0})$. Let ψ be the smooth fixed function satisfying

$$\psi_\nu=-\cos\theta\sqrt{1+|D\psi|^2} \text{ on } \partial\Omega \quad \text{and } \psi\in C^\infty(\overline{\Omega}).$$

Denotes $M:=\sup\limits_{\overline{\Omega}}|\psi|+\sup\limits_{\overline{\Omega}}|\operatorname{div}(\frac{D\psi}{\sqrt{1+|D\psi|^2}})|+1$, we consider the functions $\psi_\pm(x):=\psi(x)\pm\frac{M}{\varepsilon}$. It follows that, for any $\varepsilon\in(0,1)$, we have

$$\begin{aligned}\mathcal{L}(u_{\varepsilon,0}-\psi_+):=&\operatorname{div}\left(\frac{Du_{\varepsilon,0}}{\sqrt{1+|Du_{\varepsilon,0}|^2}}\right)-\operatorname{div}\left(\frac{D\psi}{\sqrt{1+|D\psi|^2}}\right)\\ \ge&\varepsilon u_{\varepsilon,0}-\varepsilon\psi-M=\varepsilon(u_{\varepsilon,0}-\psi_+),\end{aligned}$$

where the elliptic operator $\mathcal{L}u:=\sum\limits_{i,j=1}^n\frac{\partial}{\partial x_i}\big(\hat{a}^i_j(x)\partial_{x_j}u\big)$ with

$$\hat{a}^i_j(x):=\int_0^1\partial_{p_j}A^i\big(tDu_{\varepsilon,0}(x)+(1-t)D\psi(x)\big)\mathrm{d}t,\quad A^i(p):=\frac{p_i}{\sqrt{1+|p|^2}} \text{ for } 1\le i\le n.$$

Therefore, the maximum principle implies that $u_{\varepsilon,0}-\psi_+$ attains the nonnegative maximum value at the boundary, say $x_0\in\partial\Omega$ and $u_{\varepsilon,0}(x_0)-\psi_+(x_0)\ge0$. Since

$$\frac{D_\nu\psi}{\sqrt{1+|D\psi|^2}}=\frac{D_\nu u_{\varepsilon,0}}{\sqrt{1+|Du_{\varepsilon,0}|^2}},$$

by using the similar argument in Altschuler-Wu (see Theorem 2.6 in [2]) or the proof of Theorem 3.1, it yields a contradiction. Hence $u_{\varepsilon,0}\leq\psi_+$ in $\overline{\Omega}$. Similarly, from

$$\begin{aligned}\mathcal{L}(u_{\varepsilon,0}-\psi_-):=&\mathrm{div}\left(\frac{Du_{\varepsilon,0}}{\sqrt{1+|Du_{\varepsilon,0}|^2}}\right)-\mathrm{div}\left(\frac{D\psi}{\sqrt{1+|D\psi|^2}}\right)\\ \leq&\varepsilon u_{\varepsilon,0}-\varepsilon\psi+M=\varepsilon(u_{\varepsilon,0}-\psi_-),\end{aligned}$$

using the maximum principle and the boundary condition again, it follows that $u_{\varepsilon,0}\geq\psi_-$ in $\overline{\Omega}$. Consequently, we have

$$u_{\varepsilon,-M}(x)=u_{\varepsilon,0}(x)-\frac{M}{\varepsilon}\leq\psi(x)\leq u_{\varepsilon,0}(x)+\frac{M}{\varepsilon}=u_{\varepsilon,M}(x)\quad\text{in}\quad\overline{\Omega}.$$

Since $u_{\varepsilon,L}$ is strictly increasing with respect to $L\in\mathbb{R}$ for any fixed $\varepsilon\in(0,1)$, it follows that, there exists a unique constant $L_\varepsilon\in[-M,M]$ such that $u_{\varepsilon,L_\varepsilon}(x_0)=\psi(x_0)$ for some $x_0\in\Omega$. Combining this with Lemma 4.1, it yields that we also have the uniform C^0 estimate for $u_{\varepsilon,L_\varepsilon}$, then we obtain $\|u_{\varepsilon,L_\varepsilon}\|_{C^1(\overline{\Omega})}\leq C$, where C is independent of ε. Following the standard estimates [11] (Theorem 13.2), it implies that $\|u_{\varepsilon,L_\varepsilon}\|_{C^{k,\alpha}(\overline{\Omega})}\leq C$ for any $k\in\mathbb{N}$, where $\alpha=\alpha(n,\overline{\Omega})$. Hence, by the Arzela-Ascoli theorem and (4.3), there exists $\varepsilon_i\to0$, a subsequence $u_{\varepsilon_i,L_{\varepsilon_i}}$, L_{ε_i}, and a smooth function u_∞ such that

$$\varepsilon_i u_{\varepsilon_i,L_{\varepsilon_i}}-L_{\varepsilon_i}\to\tau,\quad\text{and}\quad u_{\varepsilon_i,L_{\varepsilon_i}}\to u_\infty.$$

And one can easily see that (u_∞,τ) solves (1.7).

As in the proof of Theorem 3.1 (or see [2], Theorem 2.6 there) we can get the uniqueness. Thus we have completed the proof. □

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