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The exterior Dirichlet problem for the homogeneous complex k-Hessian equation

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Abstract: In this article, we consider the homogeneous complex k-Hessian equation in an exterior domain $\mathbb{C}^n\backslash\Omega$. We prove the existence and uniqueness of the $C^{1,1}$ solution by constructing approximating solutions. The key point for us is to establish the uniform gradient estimate and the second-order estimate.

Keywords: exterior Dirichlet problem, complex k-Hessian equation, k-subharmonic function, gradient estimate

MSC code: 35B65, 35J25

1 Introduction

Let u be a real C^2 function in $\mathbb{C}^n$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the eigenvalues of the complex Hessian $\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right)$, the complex k-Hessian operator is defined by

$$H_k(u) := \sum_{1 \le i_1 < \cdots i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}, \tag{1.1}$$

where $1 \le k \le n$. Using the operators $d = \partial + \bar{\partial}$ and $d^c = \sqrt{-1}(\bar{\partial} - \partial)$, such that $dd^c = 2\sqrt{-1}\partial\bar{\partial}$, one obtains

$$(dd^c u)^k \wedge \omega^{n-k} = 4^n k!(n-k)! H_k(u) d\lambda,$$

where $\omega = dd^c|z|^2$ is the fundamental Kähler form and $d\lambda$ is the volume form. When $k = 1$, $H_1(u) = \frac{1}{4}\Delta u$. When $k = n$, $H_n(u) = \det u_{i\bar{j}}$ is the complex Monge-Ampère operator.

Let Ω be a bounded smooth domain in $\mathbb{C}^n$, the Dirichlet problem for the complex k-Hessian equation is as follows:

$$\begin{cases} H_k(u) = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where f and φ are given smooth functions. When $k = 1$, the k-Hessian equation is the Poisson equation. When $k = n$, it is the well-known complex Monge-Ampère equation.

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1.1 Some previous results

We briefly give some studies on the Dirichlet problem for the k-Hessian equation and the complex k-Hessian equation in the nondegenerate case, i.e., $f > 0$, and in the degenerate cases, i.e., $f \geq 0$. In general, the k-Hessian equation (the complex k-Hessian equation) is a fully nonlinear equation.

1.1.1 Results on bounded domains

For the k-Hessian equation in $\mathbb{R}^n$, if $f > 0$, Caffarelli et al. [7] solved the Dirichlet problem in a bounded $(k-1)$-convex domain. Guan [13] solved the Dirichlet problem by only assuming the existence of a subsolution. For the complex k-Hessian equation in $\mathbb{C}^n$, Li [30] solved (1.2) in a bounded $(k-1)$-pseudoconvex domain.

There are lots of studies on the Dirichlet problem in bounded domains in $\mathbb{R}^n$ of degenerate fully nonlinear equations. Caffarelli et al. [8] showed the $C^{1,1}$ regularity of the homogeneous Monge-Ampère equation, i.e., $f \equiv 0$. If $f^{\frac{1}{n-1}} \in C^{1,1}$, Guan et al. [20] proved the optimal $C^{1,1}$ regularity result due to the counterexample by Wang [36]. The k-Hessian equation case was proved by Krylov [23,24] and Ivochina et al. [22] by assuming $f^{\frac{1}{k}} \in C^{1,1}$. Dong [11] proved the $C^{1,1}$ regularity for some degenerate mixed-type Hessian equations.

For the Dirichlet problem of the degenerate complex Monge-Ampère equation, Lempert [25] showed that $(dd^c u)^n = 0$ in a punched strictly convex domain $\Omega\backslash\{z\}$ with logarithm growth near z admits a unique real analytic solution. Zeriahi [39] studied the viscosity solution to the Dirichlet problem of degenerate complex Monge-Ampère equation.

1.1.2 Results on unbounded domains

There are lots of results on the exterior Dirichlet problem for viscosity solutions of nondegenerate fully nonlinear equations. The C^0 viscosity solution for the Monge-Ampère equation: $\det D^2 u = 1$ with prescribed asymptotic behavior at infinity was obtained by Caffarelli and Li [6]. The k-Hessian equation case was showed by Bao et al. [4]. For the related results on other types of nondegenerate fully nonlinear equations, one can see [3,27,28,31].

The global $C^{k+2,\alpha}$ regularity of the homogeneous Monge-Ampère equation in a strip region was proved by Li and Wang [29] by assuming that the boundary functions are locally uniformly convex and $C^{k,\alpha}$. They showed that the uniform convexity of the boundary functions is necessary.

For $1 \leq k < \frac{n}{2}$, the $C^{1,1}$ regularity of the Dirichlet problem for the homogeneous k-Hessian equation in $\mathbb{R}^n\backslash\overline{\Omega}$ was proved by Xiao [38] by assuming that the domain Ω is $(k-1)$-convex and star-shaped. For $1 \leq k \leq n$, Ma and Zhang [33] proved the $C^{1,1}$ regularity of the k-Hessian equation when Ω is convex and strictly $(k-1)$ convex. The prescribed asymptotic behavior is $\log|x| + O(1)$ if $k = \frac{n}{2}$ and $|x|^{2-\frac{n}{k}} + O(1)$ if $k > \frac{n}{2}$.

1.2 Motivation

Our research is motivated by the study of regularity of extremal function. For the smoothly strictly convex domain Ω, Lempert [26] proved the pluricomplex Green function in $\mathbb{C}^n\backslash\Omega$ is smooth (analytic). In [17,18], Guan proved the $C^{1,1}$ regularity of the solution to the homogeneous complex Monge-Ampère equation in a ring domain. Then, he solved a conjecture of Chern-Levine-Nirenberg on the extended intrinsic norms. For the smoothly strongly pseudoconvex domain Ω, Guan [15] proved the $C^{1,1}$ regularity and decay estimates of pluricomplex Green function in $\mathbb{C}^n\backslash\Omega$ by considering the exterior Dirichlet problem for the homogeneous complex Monge-Ampère equation.

Another motivation is on the proof of geometric inequalities by considering the exterior problems of certain elliptic partial differential equations. When Ω is $(k-1)$-convex and star-shaped, Guan and Li [19] proved Alexandrov-Fenchel inequalities by the inverse curvature flows. If Ω is k-convex, Chang and Wang [9] and Qiu [34] proved Alexandrov-Fenchel inequalities by the optimal transport method. Whether Alexandrov-Fenchel inequalities hold for $(k-1)$-convex domain is still open. Recently, Agostiniani and Mazzieri [2] proved some geometric inequalities, such as Willmore inequality by considering the exterior Dirichlet problem of the Laplace equation. Fogagnolo et al. [12] showed the volumetric Minkowski inequality by considering the exterior Dirichlet problem of the p-Laplacian equation. Agostiniani et al. [1] removed the convexity assumption in [12] for the domain.

1.3 Our main result

In this article, we consider the following exterior Dirichlet problem for the complex k-Hessian equation.

For $1 \le k < n$, since the Green function in this case is $-|z|^{2-\frac{2n}{k}}$, we consider the k-Hessian equation as follows:

$$\begin{cases} (dd^c u)^k \wedge \omega^{n-k} = 0 & \text{in } \Omega^c := \mathbb{C}^n \backslash \overline{\Omega}, \\ u = -1 & \text{on } \partial\Omega, \\ u(z) \to 0 & \text{as } |z| \to \infty. \end{cases} \tag{1.3}$$

Theorem 1.1. *Assume* $1 \le k < n$. *Let* Ω *be a smoothly strongly pseudoconvex domain in* $\mathbb{C}^n$ *such that* $0 \in \Omega$ *and* $\overline{\Omega}$ *is holomorphically convex in ball centered at* 0. *There exists a unique k-subharmonic solution* $u \in C^{1,1}(\overline{\Omega^c})$ *of equation* (1.3). *Moreover, there exists uniform constant C such that, for any* $z \in \overline{\Omega^c}$, *the following holds*

$$\begin{cases} C^{-1}|z|^{2-\frac{2n}{k}} \le -u(z) \le C|z|^{2-\frac{2n}{k}}, \\ |Du|(z) \le C|z|^{1-\frac{2n}{k}}, \\ \Delta u(z) \le C|z|^{-\frac{2n}{k}}, \\ |Du|_{C^{0,1}(\Omega^c)} \le C. \end{cases} \tag{1.4}$$

Here, the k-subharmonic function is defined in Section 2 and we use the notation $\overline{\Omega^c} := \mathbb{C}^n \backslash \Omega$. Let r_0 be the constant such that $B_{r_0} \subset\subset \Omega$ and R_0 and S_0 be constants such that $\overline{\Omega}$ is holomorphically convex in B_{S_0} and $\Omega \subset\subset B_{R_0} \subset\subset B_{S_0}$, where B_{r_0}, B_{R_0}, and B_{S_0} are balls centered at 0 with radius r_0, R_0, and S_0, respectively.

To prove Theorem 1.1, we consider the following approximating equation:

$$\begin{cases} H_k(u^\varepsilon) = f^\varepsilon & \text{in } \Omega^c, \\ u^\varepsilon = -1 & \text{on } \partial\Omega, \\ u^\varepsilon(z) \to 0 & \text{as } |z| \to \infty, \end{cases}$$

where $f^\varepsilon = c_{n,k}\varepsilon^2(1+\varepsilon^2)^{n-k}(|x|^2+\varepsilon^2)^{-n-1}$ (see Section 4).

u^ε will be obtained by approximating solutions $u^{\varepsilon,R}$ defined on bounded domains: $\Sigma_R := B_R \backslash \overline{\Omega}$ (see Section 4). The existence and uniqueness of the smooth k-subharmonic solution of $u^{\varepsilon,R}$ follows from Li [30] if we can construct a subsolution. The key point is to establish the uniform C^2 estimates for $u^{\varepsilon,R}$.

In Section 2, we give some preliminaries. In Section 3, we solve the Dirichlet problem of the degenerate complex k-Hessian equation in a ring domain. Section 4 is the main part of this article. We show a uniform $C^{1,1}$ estimate of the solution which is the limit of the solutions of the nondegenerate complex k-Hessian equation. The key ingredient is to establish uniform gradient estimates and uniform second-order estimates. We use the idea of Hou et al. [21] (see also the real case by Chou and Wang [10]) to establish uniform second-order estimates. Theorem 1.1 will be proved in Section 5.

2 Preliminaries

2.1 k-Subharmonic solutions

In this section, we provide the definition of k-subharmonic functions and definition of k-subharmonic solutions.

The Γ_k-cone is defined by

$$\Gamma_k := \{\lambda \in \mathbb{R}^n | S_i(\lambda) > 0,\ \ 1 \le i \le k\}. \tag{2.1}$$

Recall $S_k(\lambda) := \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}$, and $S_k(A) := \delta^{j_1 \cdots j_k}_{i_1 \cdots i_k} A_{i_1 j_1} \cdots A_{i_k j_k}$, where $\delta^{j_1 \cdots j_k}_{i_1 \cdots i_k}$ is the Kronecker symbol, which has the value +1 (respectively, −1) if $i_1, i_2 \cdots i_k$ are distinct and $(j_1 j_2 \cdots j_k)$ is an even permutation (respectively, an odd permutation) of $(i_1 i_2 \cdots i_k)$ and has the value 0 in any other cases. We use the convention that $S_0(A) = 1$. It is clear that $S_k(A) = S_k(\lambda(A))$, where $\lambda(A)$ are the eigenvalues of A.

One can find the concavity property of $S_k^{\frac{1}{k}}$ in [7].

Lemma 2.1. *$S_k^{\frac{1}{k}}$ is a concave function in Γ_k. In particular, $\log S_k$ is concave in Γ_k.*

The following facts about elementary symmetric polynomials are useful in proving gradient estimates.

Proposition 2.2. *We have the following two inequalities:*
(a) If $\lambda \in \Gamma_k$, then

$$\frac{S_k^2(\lambda|i)}{S_{k-1}(\lambda|i)} \ge \frac{k+1}{k} \frac{n-k}{n-k-1} S_{k+1}(\lambda|i);$$

(b) If $\lambda \in \Gamma_k$, then

$$\frac{S_k(\lambda|i)}{S_{k-1}(\lambda|i)} \le \frac{1}{k} \frac{n-k}{n-1} S_1(\lambda|i).$$

Proof. Since $\lambda \in \Gamma_k$, we have $S_{k-1}(\lambda|i) > 0$. The first inequality follows from Newton inequality. Now, we prove (b). Since $\lambda \in \Gamma_k$, we have $S_h(\lambda|i) > 0$, $\forall h = 0, 1, \ldots, k-1$. If $S_k(\lambda|i) \le 0$, (b) holds naturally. If $S_k(\lambda|i) > 0$, the second inequality follows from the generalized Newton-MacLaurin inequality. □

The following two propositions enable us to adopt a case-wise argument to deal with the third-order terms as in [10] and [21].

Proposition 2.3. *Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \overline{\Gamma}_k$, and $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$. Then, there exists $\theta = \theta(n,k) > 0$ such that*

$$S_{k-1}(\lambda|k) \ge \theta \lambda_1 S_{k-2}(\lambda|1k),$$

from which it follows

$$S_{k-1}(\lambda|i) \ge \theta \lambda_1 \lambda_2 \cdots \lambda_{k-1}, \quad \forall i \ge k. \tag{2.2}$$

The following proposition was proven in [10]. In [21], Hou et al. provided a sharp constant $\theta = \frac{k}{n}$ in (2.3).

Proposition 2.4. *Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \overline{\Gamma}_k$, and $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$. Then, there exists $\theta = \theta(n,k) > 0$ such that*

$$\lambda_1 S_{k-1}(\lambda|i) \ge \theta S_k(\lambda). \tag{2.3}$$

Moreover, for any $\delta \in (0,1)$, there exists $K > 0$ such that if

$$S_k(\lambda) \le K \lambda_1^k \quad \text{or} \quad |\lambda_i| \le K \lambda_1 \quad \text{for any } i = k+1, k+2, \ldots, n,$$

we have

$$\lambda_1 S_{k-1}(\lambda|1) \geq (1-\delta) S_k(\lambda). \tag{2.4}$$

One can see the Lecture notes by Wang [37] for more properties of the k-Hessian operator and the study of Blocki [5] for those of the complex k-Hessian operator. We follow the definition by Blocki [5] to give the definition of k-subharmonic functions.

Definition 2.5. Let α be a real $(1,1)$-form in U, a domain of $\mathbb{C}^n$. We say that α is k-positive in U if the following inequalities hold:

$$\alpha^j \wedge \omega^{n-j} \geq 0, \quad \forall j = 1, \dots, k.$$

Definition 2.6. Let U be a domain in $\mathbb{C}^n$.
(1) A function $u : U \to \mathbb{R} \cup \{-\infty\}$ is called k-subharmonic if it is subharmonic, and for all k-positive real $(1,1)$-form $\alpha_1, \dots, \alpha_{k-1}$ in U,

$$dd^c u \wedge \alpha_1 \wedge \cdots \wedge \alpha_{k-1} \wedge \omega^{n-k} \geq 0.$$

The class of all k-subharmonic functions in U will be denoted by $\mathcal{SH}_k(U)$.
(2) A function $u \in C^2(U)$ is called *k-subharmonic* (*strictly k-subharmonic*) if $\lambda(\partial\bar{\partial}u) \in \bar{\Gamma}_k$ (λ $(\partial\bar{\partial}u) \in \Gamma_k$).

If $u \in \mathcal{SH}_k(U) \cap C(U)$, $(dd^c u)^k \wedge \omega^{n-k}$ is well defined in pluripotential theory by Blocki [5]. We need the following comparison principle by Błocki [5] to prove the uniqueness of the continuous solution of the problem (1.3).

Lemma 2.7. *Let U be a bounded domain in $\mathbb{C}^n$, $u, v \in \mathcal{SH}_k(U) \cap C(\bar{U})$ satisfy*

$$\begin{cases} (dd^c u)^k \wedge \omega^{n-k} \geq (dd^c v)^k \wedge \omega^{n-k} & \text{in } U, \\ u \leq v & \text{on } \partial U. \end{cases} \tag{2.5}$$

Then, $u \leq v$ in U.

2.2 The existence of the subsolution

Definition 2.8. ρ is called a defining function of C^1 domain U, if $U = \{z : \rho(z) < 0\}$ and $|D\rho| \neq 0$ on ∂U.

Definition 2.9. A C^2 domain U is called pseudoconvex (strictly pseudoconvex) if it is Levi pseudoconvex (strictly Levi pseudoconvex). That is, for a C^2 defining function of U defined in a neighborhood of U, the Levi form at every point $z \in \partial U$ defined by

$$L_{\partial U, z}(\xi) = \frac{1}{|D\rho(z)|} \sum_{j,k} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} \xi_j \bar{\xi}_k, \quad \xi \in {}^h T_{\partial U, z}$$

is nonnegative (positive). ${}^h T_{\partial U, z} := \{\xi \in \mathbb{C}^n | \sum_j \frac{\partial \rho}{\partial z_j} \xi_j = 0\}$ is the holomorphic tangent space to ∂U at z.

Definition 2.10. A C^2 domain U is called k-pseudoconvex (strictly k-pseudoconvex) if for a C^2 defining function of U defined in a neighborhood of U,

$$\lambda \left\{ \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j} \right\}_{1 \leq i,j \leq n-1} \in \bar{\Gamma}_k \ (\in \Gamma_k), \quad \forall z \in \partial U,$$

where $(z_1, \dots, z_{n-1})$ is a holomorphic coordinate system of ${}^h T_{\partial U, z}$ near z.

We need the following lemmas by Guan [17] to construct the subsolution of the k-Hessian equation in a ring.

Lemma 2.11. *Suppose that U is a bounded smooth domain in* $\mathbb{C}^n$. *For* $h, g \in C^m(U)$, $m \geq 2$, *for all* $\delta > 0$, *there exists* $H \in C^m(U)$ *such that*

(1) $H \geq \max\{h, g\}$ *and*

$$H(z) = \begin{cases} h(z), & \text{if } h(z) - g(z) > \delta, \\ g(z), & \text{if } g(z) - h(z) > \delta; \end{cases}$$

(2) *There exists* $|t(z)| \leq 1$ *such that*

$$\{H_{i\bar{j}}(z)\} \geq \left\{\frac{1 + t(z)}{2} g_{i\bar{j}} + \frac{1 - t(z)}{2} h_{i\bar{j}}\right\}, \quad \text{for all } z \in \{|g - h| < \delta\}.$$

By Lemma 2.1, we see H is k-subharmonic if h and g are both k-subharmonic.

The following lemma was proved by Guan [17].

Lemma 2.12. *Let* Ω_0 *and* Ω_1 *be smooth, strongly pseudoconvex domain in* $\mathbb{R}^n$ *with* $\Omega_1 \subset\subset \Omega_0$. *Assume that* Ω_1 *is holomorphically convex in* Ω_0. *Then, there exists a strictly plurisubharmonic function* $\underline{u} \in C^\infty(\overline{\Omega})$ *with* $\Omega \coloneqq \Omega_0 \backslash \overline{\Omega}_1$ *satisfying*

$$\begin{cases} H_k(\underline{u}) \geq \varepsilon_0, & \text{in } \Omega, \\ \underline{u} = \tau\rho_1, & \text{near } \partial\Omega_1, \\ \underline{u} = 1 + K\rho_0, & \text{near } \partial\Omega_0, \end{cases} \tag{2.6}$$

where ρ_0 *and* ρ_1 *are defining functions of* Ω_0 *and* Ω_1 and τ *and* K *are uniform constants.*

In [17], Guan considered the Dirichlet problem of homogeneous complex Monge-Ampère equation in a smooth ring as follows:

$$\begin{cases} (dd^c u)^n = 0 & \text{in } \Omega \coloneqq \Omega_0 \backslash \overline{\Omega}_1, \\ u = 0 & \text{on } \partial\Omega_1, \\ u = 1 & \text{on } \partial\Omega_0. \end{cases} \tag{2.7}$$

Guan [17] proved the following.

Theorem 2.13. *Let* Ω_0 *and* Ω_1 *be smooth, strongly pseudoconvex domains and assume that* Ω_1 *is holomorphically convex in* Ω_0. *There exists a unique solution* $u \in C^{1,1}(\overline{\Omega})$ *of equation* (2.7).

3 The Dirichlet problem for the homogeneous complex k-Hessian equations in the ring in $\mathbb{C}^n$

In this section, we consider the Dirichlet problem of the homogeneous complex k-Hessian equation in a smooth ring as follows:

$$\begin{cases} (dd^c u)^k \wedge \omega^{n-k} = 0, & \text{in } \Omega \coloneqq \Omega_0 \backslash \overline{\Omega}_1, \\ u = 0, & \text{on } \partial\Omega_1, \\ u = 1, & \text{on } \partial\Omega_0. \end{cases} \tag{3.1}$$

We assume that $\Omega_1 \subset\subset \Omega_0$ are smooth, strongly pseudoconvex domains and Ω_1 is holomorphically convex in Ω_0. Using Lemma 2.12, there exists a smooth, strictly plurisubharmonic subsolution $\underline{u}$ satisfying

$$\begin{cases} H_k(\underline{u}) \geq \varepsilon_0, & \text{in } \Omega, \\ \underline{u} = \tau\rho_1, & \text{near } \partial\Omega_1, \\ \underline{u} = 1 + \mathrm{K}\rho_0, & \text{near } \partial\Omega_0, \end{cases} \tag{3.2}$$

where τ and K are positive constants and ρ_i are defining functions of Ω_i.

Theorem 3.1. *Let* Ω_0 and Ω_1 *be smooth, strongly pseudoconvex domains and assume that* Ω_1 *is holomorphically convex in* Ω_0. *There exists a unique solution* $u \in C^{1,1}(\overline{\Omega})$ *of equation* (3.1).

The uniqueness follows from Lemma 2.7, the comparison principle for k-subharmonic solutions to complex k-Hessian equations. Next, we prove the existence and regularity of k-subharmonic solution by approximation. Indeed, for every $0 < \varepsilon < \varepsilon_0$, we consider the following problem:

$$\begin{cases} H_k(u^\varepsilon) = \varepsilon & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \partial\Omega_1, \\ u^\varepsilon = 1 & \text{on } \partial\Omega_0. \end{cases} \tag{3.3}$$

Since $\underline{u}$ in (3.2) is a subsolution to (3.3), by Li [30], the above problem has a unique smooth solution u^ε.

Next, we want to show the $C^{1,1}$ estimates of u^ε are independent of ε. First, by a maximum principal, $u^{\varepsilon_1} \geq u^{\varepsilon_2}$ for any $\varepsilon_1 \leq \varepsilon_2$. Thus, $u^0 := \lim_{\varepsilon\to\infty} u^\varepsilon$ exists. If we could prove uniform $C^{1,1}$ estimates, then u^0 is the $C^{1,1}$ solution of equation (3.1).

Theorem 3.2. *Let* u^ε *be the smooth k-subharmonic solution of* (3.3). *Then, there exists a uniform constant C independent of* ε *such that*

$$|u^\varepsilon|_{C^{1,1}(\overline{\Omega})} \leq C.$$

In the following subsections, for simplicity, we use u instead of u^ε.

3.1 C^1-estimates

Lemma 3.3. *There exists a uniform constant C such that*

$$|u|_{C^1(\overline{U})} \leq C. \tag{3.4}$$

Proof. Let h be the unique solution of the problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ h = 0 & \text{on } \partial\Omega_1, \\ h = 1 & \text{on } \partial\Omega_0. \end{cases} \tag{3.5}$$

By the maximal principle, we have $\underline{u} \leq u \leq h$. This gives uniform C^0 estimates.

Let $F^{ij} := \frac{\partial}{\partial u_{ij}} \log H_k(u) = \frac{\partial}{\partial u_{ij}} S_k(\partial\bar{\partial}u)$.

$$D_\xi = \sum_{i=1}^{n} \left(a_i \frac{\partial}{\partial x_i} + b_i \frac{\partial}{\partial y_i} \right) \quad \text{with} \quad \sum_{i=1}^{n} a_i^2 + b_i^2 = 1.$$

Then,

$$F^{ij}(D_\xi u)_{ij} = 0.$$

Thus, we have

$$\max_{\overline{U}}|Du| = \max_{\partial U}|Du|.$$

Since $\underline{u} \le u^{\varepsilon} \le h$ in Ω and $\underline{u} = u^{\varepsilon} = h$ on $\partial\Omega$, we have

$$h_{\nu} \le u^{\varepsilon}_{\nu} \le \underline{u}_{\nu},$$

where ν is the unit outer normal to $\partial\Omega$ (unit inner normal to $\partial\Omega_1$ and unit outer normal to $\partial\Omega_0$). Thus, we have

$$\max_{\overline{\Omega}}|Du| = \max_{\partial\Omega}|Du| \le C. \tag{3.6}$$

□

3.2 Second-order estimates

Lemma 3.4. *There exists a uniform constant C such that*

$$\max_{\overline{U}}|D^2u| \le C. \tag{3.7}$$

Proof. Denote by $D_{\xi}u = u_{\xi}$. Then,

$$L(u_{\xi\xi}) = -\frac{\partial^2 S_k^{\frac{1}{k}}(\partial\bar{\partial}u)}{\partial u_{j\bar{k}}\partial u_{l\bar{m}}} u_{j\bar{k}\xi}u_{l\bar{m}\xi} \ge 0.$$

Hence,

$$u_{\xi\xi}(z) \le \sup_{\partial\Omega}|D^2u|.$$

This implies $\forall i, j = 1, \ldots, n$,

$$u_{x_ix_i}, u_{y_iy_i} \le \sup_{\partial\Omega}|D^2u|, \quad u_{x_i\pm x_j}, u_{x_i\pm y_j}, u_{y_i\pm y_j} \le 2\sup_{\partial\Omega}|D^2u|.$$

On the other hand, $\Delta u(z) > 0$ implies

$$u_{x_ix_i}, u_{y_iy_i} \ge -(2n-1)\left(\sup_{\partial\Omega}|D^2u|\right).$$

Then,

$$\begin{aligned}
\pm\, u_{x_ix_j} &= u_{x_i\pm x_j} - u_{x_ix_i} - u_{x_jx_j} \le (4n-1)\left(\sup_{\partial\Omega}|D^2u|\right),\\
\pm\, u_{x_iy_j} &= u_{x_i\pm y_j} - u_{x_ix_i} - u_{y_jy_j} \le (4n-1)\left(\sup_{\partial\Omega}|D^2u|\right),\\
\pm u_{y_iy_j} &= u_{y_i\pm y_j} - u_{y_iy_i} - u_{y_jy_j} \le (4n-1)\left(\sup_{\partial\Omega}|D^2u|\right).
\end{aligned}$$

Thus, we have

$$\max_{\overline{\Omega}}|D^2u| \le C_n \max_{\partial\Omega}|D^2u|.$$

So we need to prove the second-order estimate on the boundary $\partial\Omega$. Here, we use the method by Guan [13,16,17] and Li [30].

Tangential derivative estimates on $\partial\Omega$

Consider a point $p \in \partial\Omega$. Without loss of generality, let p be the origin. Choose the coordinate $z_1, \dots, z_n$ such that the x_n axis is the inner normal direction to $\partial\Omega$ at 0. Suppose

$$t_1 = y_1, \quad t_2 = y_2, \dots, \quad t_n = y_n, \quad t_{n+1} = x_1, \quad t_{n+2} = x_2, \dots, \quad t_{2n} = x_n.$$

Denote by $t' = (t_1, \dots, t_{2n-1})$. Then, around the origin, $\partial\Omega$ can be represented as a graph

$$t_n = x_n = \rho(t') = B_{\alpha\beta} t_\alpha t_\beta + O(|t'|^3).$$

Since

$$(u - \underline{u})(t', \rho(t')) = 0,$$

we have

$$(u - \underline{u})_{t_\alpha t_\beta}(0) = -(u - \underline{u})_{t_n}(0) B_{\alpha\beta}, \quad \alpha, \beta = 1, \dots, 2n-1.$$

It follows by gradient estimate that

$$|u_{t_\alpha t_\beta}(0)| \le C, \quad \alpha, \beta = 1, \dots, 2n-1. \tag{3.8}$$

Tangential-normal derivative estimates on $\partial\Omega$

We use Guan's method [13,14,16]. Our barrier function here is simpler than before since u is constant on the boundary and the right-hand side of the approximating equation is a sufficiently small constant ε.

To estimate $u_{t_\alpha t_n}(0)$ for $\alpha = 1, \dots, 2n-1$ and $u_{t_n t_n}(0)$, we consider the auxiliary function

$$v = u - \underline{u} + td - \frac{N}{2} d^2$$

on $\Omega_\delta = \Omega \cap B_\delta(0)$ with constants $N, t,$ *and* δ to be determined later. The following lemma proven in [14] is needed.

Lemma 3.5. *For N sufficiently large and t* and δ *sufficiently small, there holds*

$$\begin{cases} Lv \le -\dfrac{\varepsilon}{4}(1 + \mathcal{F}) & \text{in } \Omega_\delta, \\ v \ge 0 & \text{on } \partial\Omega, \end{cases}$$

where $\varepsilon > 0$ *is a uniform constant depending only on subsolution* $\underline{u}$ *restricted in a small neighborhood of* $\partial\Omega$.

The following three lemmas were proven by Guan in [16].

Lemma 3.6. *Let* $F^{i\bar{j}} = \frac{\partial}{\partial u_{i\bar{j}}} S_k^{\frac{1}{k}}(\partial\bar{\partial}u)$. *Then, there is an index r such that*

$$\sum_{l=1}^{n-1} F^{i\bar{j}} u_{i\bar{l}} u_{l\bar{j}} \ge \frac{1}{2} \sum_{i \ne r} S_k^{\frac{1}{k}-1}(\lambda) S_{k-1}(\lambda|i) \lambda_i^2, \tag{3.9}$$

where $\lambda = (\lambda_1, \dots, \lambda_n)$ *are the eigenvalues of* $u_{i\bar{j}}$.

Lemma 3.7. *Suppose* $\lambda \in \overline{\Gamma}_k$. *If* $\lambda_r < 0$, *then*

$$\sum_{i \ne r} S_k^{\frac{1}{k}-1}(\lambda) S_{k-1}(\lambda|i) \lambda_i^2 \ge \frac{1}{n} \sum_{i=1}^{n} S_k^{\frac{1}{k}-1}(\lambda) S_{k-1}(\lambda|i) \lambda_i^2.$$

Lemma 3.8. *Suppose* $\lambda \in \overline{\Gamma}_k$. *Then, for any* $r = 1, \dots, n$ *and* $\varepsilon > 0$,

$$\sum_{i=1}^{n} S_k^{\frac{1}{k}-1}(\lambda) S_{k-1}(\lambda|i) |\lambda_i| \le \varepsilon \sum_{i \ne r} S_k^{\frac{1}{k}-1}(\lambda) S_{k-1}(\lambda|i) \lambda_i^2 + \frac{C}{\varepsilon} \sum_{i=1}^{n} S_k^{\frac{1}{k}-1}(\lambda) S_{k-1}(\lambda|i) + Q(r), \tag{3.10}$$

where $Q(r) = S_k^{\frac{1}{k}}(\lambda) - (C_n^k)^{\frac{1}{k}}$ *if* $\lambda_r \ge 0$ *and* $Q(r) = 0$ *if* $\lambda_r < 0$.

At any boundary point $p \in \partial\Omega$, we may choose coordinates $z_1, \dots, z_n$ with the origin p such that the positive x_n axis is the interior normal direction to $\partial\Omega$ at p. Let ϱ be a defining function of Ω, that is, $\varrho < 0$ in Ω, $\varrho = 0$, $D\nu\varrho = 1$, on $\partial\Omega$, where ν is a unit outer normal to $\partial\Omega$. We may assume that, $\frac{\partial\varrho}{\partial x_j}(0) = 0$ for $1 \le i \le n-1$ and $\frac{\partial\varrho}{\partial y_j}(0) = 0$ for all $1 \le i \le n$. Moreover, around the origin, we can write

$$\varrho(z) = -x_n + \operatorname{Re}\sum_{i,j=1}^{n} \varrho_{ij}(0)z_iz_j + \sum_{i,j=1}^{n} \varrho_{i\bar{j}}(0)z_i\bar{z}_j + Q(z),$$

where $|Q(z)| \le C|z|^3$. Let

$$t_i = y_i, \quad i = 1, \dots, n, \quad t_{n+i} = x_i, \quad i = 1, \dots, n.$$

Let

$$a_\alpha(z) = -\frac{\frac{\partial\varrho}{\partial t_\alpha}}{\frac{\partial\varrho}{\partial x_n}}, \quad 1 \le \alpha \le 2n-1.$$

Then,

$$a_\alpha(0) = 0.$$

So $T = \frac{\partial}{\partial t_\alpha} + a_\alpha \frac{\partial}{\partial x_n}$ is a tangential vector to $\partial\Omega$ near the origin. We write

$$a_\alpha(z) = \sum_{\beta=1}^{2n-1} b_{\alpha\beta}t_\beta + b_\alpha x_n + O(|t|^2 + x_n^2), \quad z \in \overline{\Omega} \text{ near } 0.$$

Let

$$T_\alpha = \frac{\partial}{\partial t_\alpha} + \sum_{\beta=1}^{2n-1} b_{\alpha\beta}t_\beta \frac{\partial}{\partial x_n}.$$

Then,

$$T = T_\alpha + b_\alpha x_n \frac{\partial}{\partial x_n} + O(|z|^2)\frac{\partial}{\partial x_n}.$$

So

$$T_\alpha(u - \underline{u}) = O(|t|^2), \quad \text{on } \partial\Omega.$$

Note that

$$\partial_i t_\beta = \begin{cases} -\frac{\sqrt{-1}}{2}\delta_{i\beta}, & 1 \le \beta \le n, \\ \frac{1}{2}\delta_{i\beta-n}, & \beta > n \end{cases}$$

and

$$\partial_{\bar{j}} t_\beta = \begin{cases} \frac{\sqrt{-1}}{2}\delta_{j\beta}, & 1 \le \beta \le n, \\ \frac{1}{2}\delta_{j\beta-n}, & \beta > n. \end{cases}$$

We then have

$$
\begin{aligned}
LT_\alpha(u-\underline{u}) &:= T_\alpha f - LT_\alpha \underline{u} + \sum_{\beta=1}^{2n-1} b_{\alpha\beta}F^{i\bar{j}}\left(t_{\beta,i}u_{x_n\bar{j}} + t_{\beta,\bar{j}}u_{x_n i}\right)\\
&= T_\alpha f - LT_\alpha \underline{u} + 2\sum_{\beta=1}^{2n-1} b_{\alpha\beta}F^{i\bar{j}}\left(t_{\beta,i}u_{n\bar{j}} + t_{\beta,\bar{j}}u_{\bar{n}i}\right) + \sqrt{-1}\sum_{\beta=1}^{2n-1} b_{\alpha\beta}F^{i\bar{j}}\left(t_{\beta,i}u_{y_n\bar{j}} - t_{\beta,\bar{j}}u_{y_n i}\right)\\
&\geq -C\left(1 + \sum_{i=1}^{n}\frac{S_{k-1}(\lambda|i)}{S_k(\lambda)} + \frac{S_{k-1}(\lambda|i)|\lambda_i|}{S_k(\lambda)}\right) - \frac{1}{4}F^{i\bar{j}}\left(u_{y_n i} - \underline{u}_{y_n i}\right)\left(u_{y_n\bar{j}} - \underline{u}_{y_n\bar{j}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&L\left(\left(u_{y_n} - \underline{u}_{y_n}\right)^2 + \sum_{l=1}^{n-1}|u_l - \underline{u}_l|^2\right)\\
&= 2F^{i\bar{j}}\left(u_{y_n i} - \underline{u}_{y_n i}\right)\left(u_{y_n\bar{j}} - \underline{u}_{y_n\bar{j}}\right) + \sum_{l=1}^{n-1}F^{i\bar{j}}\left(\left(u_{li} - \underline{u}_{li}\right)\left(u_{\bar{l}\bar{j}} - \underline{u}_{\bar{l}\bar{j}}\right) + \left(u_{l\bar{j}} - \underline{u}_{l\bar{j}}\right)\left(u_{\bar{l}i} - \underline{u}_{\bar{l}i}\right)\right)\\
&\quad + 2\left(u_{y_n} - \underline{u}_{y_n}\right)F^{i\bar{j}}\left(u_{y_n i\bar{j}} - \underline{u}_{y_n i\bar{j}}\right) + \sum_{l=1}^{n-1}\left(\left(u_l - \underline{u}_l\right)F^{i\bar{j}}\left(u_{\bar{l}i\bar{j}} - \underline{u}_{\bar{l}i\bar{j}}\right) + \left(u_{\bar{l}} - \underline{u}_{\bar{l}}\right)F^{i\bar{j}}\left(u_{li\bar{j}} - \underline{u}_{li\bar{j}}\right)\right)\\
&\geq 2F^{i\bar{j}}\left(u_{y_n i} - \underline{u}_{y_n i}\right)\left(u_{y_n\bar{j}} - \underline{u}_{y_n\bar{j}}\right) + \sum_{l=1}^{n-1}F^{i\bar{j}}u_{l\bar{j}}u_{\bar{l}i} - C\left(1 + \sum_{i=1}^{n}\frac{S_{k-1}(\lambda|i)}{S_k(\lambda)} + \frac{S_{k-1}(\lambda|i)|\lambda_i|}{S_k(\lambda)}\right).
\end{aligned}
$$

Let

$$\Psi = A_1 v + A_2|z|^2 - A_3\left(\left(u_{y_n} - \underline{u}_{y_n}\right)^2 + \sum_{l=1}^{n-1}|u_l - \underline{u}_l|^2\right).$$

By Lemmas 3.5, 3.6, and 3.8, we see that

$$L(\Psi \pm T_\alpha(u-\underline{u})) \leq 0 \quad \text{in } \Omega_\delta$$

and

$$\Psi \pm T_\alpha(u-\underline{u}) \geq 0 \quad \text{on } \partial\Omega_\delta,$$

when $A_1 \gg A_2 \gg A_3 \gg 1$. Therefore,

$$|u_{t_\alpha x_n}| \leq C.$$

In particular, from (4.30), we know

$$|u_{y_n y_n}| \leq C.$$

Double normal derivative estimates on $\partial\Omega$

For any fixed $p \in \partial\Omega$, we choose the coordinate such that $p = 0$, $\partial\Omega \bigcap B_r(0) = (t', \varphi(t'))$, and $\nabla\varphi(0) = 0$.

Case 1: $x_0 \in \partial\Omega_0$.

Let ρ_0 be a defining function of Ω_0, which is strictly plurisubharmonic in a neighborhood of Ω_0. So

$$\rho_0(t', \varphi(t')) = 0 \quad \text{on } \partial\Omega_0.$$

Then, we have

$$\rho_{0,t_\alpha t_\beta}(0) = -\rho_{0,t_{2n}}(0)\varphi_{t\alpha t_\beta}(0) \quad 1 \leq \alpha, \beta \leq 2n-1.$$

On the other hand, we have

$$u_{\alpha\beta}(0) = -u_{t_{2n}}(0)\varphi_{\alpha\beta}(0) \quad 1 \leq \alpha, \beta \leq 2n-1.$$

Thus,

$$u_{t_\alpha t_\beta}(0) = \frac{u_{t_{2n}}(0)}{\rho_{0,t_{2n}}(0)}\rho_{0,t\alpha t_\beta}(0) \quad 1 \le \alpha, \beta \le 2n-1$$

and

$$u_{i\bar{j}}(0) = \frac{u_{t_{2n}}(0)}{\rho_{0,t_{2n}}(0)}\rho_{0,i\bar{j}}(0) \ge c\rho_{0,i\bar{j}}(0) > 0.$$

Since ρ_0 is strictly plurisubharmonic in $\overline{\Omega}_0$, we have

$$S_{k-1}\Big(\big\{u_{i\bar{j}}(0)\big\}_{1\le i,j\le n-1}\Big) \ge c^{k-1}S_{k-1}\Big(\big\{\rho_{0,i\bar{j}}(0)\big\}_{1\le i,j\le n-1}\Big) \ge c_1 > 0. \tag{3.11}$$

Case 2: $x_0 \in \partial\Omega_1$.

Note that $u \ge \underline{u}$ near $\partial\Omega_1$, $u = \underline{u}$, and $0 < \underline{u}_\nu \le u_\nu$ on $\partial\Omega_1$, ν is the unit outer normal to $\partial\Omega_1$, there exists a smooth function g such that $u = g\underline{u}$ near $\partial\Omega_1$, and $g \ge 1$ outside of Ω nearby $\partial\Omega_1$. So $\forall\ 1 \le i, j \le n-1$,

$$u_{i\bar{j}}(0) = g_{i\bar{j}}(0)\underline{u}(0) + g_i(0)\underline{u}_{\bar{j}}(0) + g_{\bar{j}}(0)\underline{u}_i(0) + g(0)\underline{u}_{i\bar{j}}(0).$$

Note that $\underline{u} = \tau\rho_1$ near $\partial\Omega_1$, where ρ_1 is a given strictly plurisubharmonic function in a neighborhood Ω and τ a constant independent of ε and R as taken in Lemma 2.12. We also have

$$\begin{aligned} S_{k-1}\Big(\big\{u_{i\bar{j}}(0)\big\}_{1\le i,j\le n-1}\Big) &= \tau^{k-1}g^{k-1}(0)S_{k-1}\Big(\big\{\rho_{1,i\bar{j}}(0)\big\}_{1\le i,j\le n-1}\Big) \ge \tau^{k-1}g_0^{k-1}C_n^{k-1}(C_n^k)^{\frac{1-k}{k}}\min_{\partial\Omega} S_k^{\frac{k-1}{k}}(\partial\bar{\partial}\rho_1) \\ &:= c_1 > 0. \end{aligned} \tag{3.12}$$

Let $c_0 = \min\{c_1, c_2\}$ (see (3.11) and (3.12)), we have

$$\begin{aligned} u_{n\bar{n}}(0)c_0 &\le u_{n\bar{n}}(0)S_{k-1}\Big(\big\{u_{i\bar{j}}(0)\big\}_{1\le i,j\le n-1}\Big) \\ &= S_k\Big(\big\{u_{i\bar{j}}(0)\big\}_{1\le i,j\le n}\Big) - S_k\Big(\big\{u_{i\bar{j}}(0)\big\}_{1\le i,j\le n-1}\Big) + \sum_{i=1}^{n-1}|u_{i\bar{n}}|^2 S_{k-2}\Big(\big\{u_{i\bar{j}}(0)\big\}_{1\le i,j\le n-1}\Big) \le C. \end{aligned}$$

Then, we obtain

$$u_{n\bar{n}}(0) \le C,$$

where C is a uniform constant. On the other hand, $u_{n\bar{n}}(0) \ge \sum_{i=1}^{n-1}u_{\alpha\bar{\alpha}}(0) \ge -C$. In conclusion, we have $|u_{n\bar{n}}(0)| \le C$.

In conclusion, we obtain the uniform C^2 estimate. □

3.3 Proof of Theorem 3.1

The uniqueness follows from the comparison principle for k-subharmonic solutions of complex k-Hessian equations in Lemma 2.7 by Blocki [5].

For the existence part, since u^ε is increasing on ε, $u^0 := \lim_{\varepsilon\to 0}u^\varepsilon$ exists. Since $|u^\varepsilon|_{C^2(\overline{\Omega})} \le C$, there exists a subsequence u^{ε_i} that converges to u^0 in $C^{1,\alpha}$ on $\overline{\Omega}$ and $u^0 \in C^{1,1}(\overline{\Omega})$.

4 Solving the approximating equation in $\Sigma_R := B_R\backslash\Omega$

We always assume Ω is a smooth, strongly pseudoconvex domain containing the origin and Ω is holomorphically convex in a ball. Recall that we always assume $B_{r_0} \subset\subset \Omega \subset\subset B_{R_0} \subset\subset B_{S_0}$ and Ω is holomorphically convex in B_{S_0}.

Since the Green function in this case is $-|z|^{2-\frac{2n}{k}}$, we want to solve the following complex k-Hessian equation:

$$\begin{cases} (dd^c u)^k \wedge \omega^{n-k} = 0 & \text{in } \Omega^c := \mathbb{C}^n \backslash \overline{\Omega}, \\ u = -1 & \text{on } \partial\Omega, \\ u(z) \to 0 & \text{as } |z| \to \infty. \end{cases} \tag{4.1}$$

By scaling of z, we consider (4.1) with $B_t \subset\subset \Omega \subset\subset B_1 \subset\subset B_{1+s}$, where $t = \frac{r_0}{R_0}$, $s = \frac{S_0}{R_0} - 1$.

4.1 Construction of the approximating equation

Let w^ε be an approximation of the Green function $-|z|^{2-\frac{2n}{k}}$,

$$w^\varepsilon(z) = -\left(\frac{|z|^2 + \varepsilon^2}{1 + \varepsilon^2}\right)^{1-\frac{n}{k}}.$$

We have

$$f^\varepsilon := H_k(w^\varepsilon) = S_k\left(w^\varepsilon_{i\bar{j}}\right) = C_n^k\left(\frac{n}{k} - 1\right)^k \varepsilon^2(1 + \varepsilon^2)^{n-k}(|z|^2 + \varepsilon^2)^{-n-1}.$$

It is clear that $\rho_0 = |z|^2 - (1 + s)^2$ is a plurisubharmonic defining function of B_{1+s}. Let ρ_1 be a defining function of Ω such that ρ_1 is plurisubharmonic in a neighborhood U of Ω.

By Lemma 2.12, there is a smooth plurisubharmonic function ρ solving

$$\begin{cases} H_k(\rho) \geq \varepsilon_0, & \text{in } B_{1+s} \backslash \overline{\Omega}, \\ \rho = \tau\rho_1, & \text{near } \partial\Omega, \\ \rho = 1 + \mathrm{K}\rho_0, & \text{near } \partial B_{1+s}. \end{cases} \tag{4.2}$$

Let $\varphi = \left(1 - \left(1 + \frac{s^2}{16+s^2}\right)^{1-\frac{n}{k}}\right)\rho - 1$. In $B_{1+s} \backslash B_{1+\frac{s}{2}}$, $\forall\ \varepsilon \leq \varepsilon_0$, $\varepsilon_0 < \frac{s^2}{8}$,

$$w^\varepsilon \geq -\left(\frac{\left(1 + \frac{s}{2}\right)^2}{1 + \varepsilon_0^2}\right)^{1-\frac{n}{k}} > -\left(1 + \frac{s^2}{8 + s^2}\right)^{1-\frac{n}{k}}.$$

So

$$w^\varepsilon - \varphi > \left(1 + \frac{s^2}{16 + s^2}\right)^{1-\frac{n}{k}} - \left(1 + \frac{s^2}{8 + s^2}\right)^{1-\frac{n}{k}} \quad \text{in } B_{1+s} \backslash B_{1+\frac{s}{2}}.$$

Let V be a neighborhood of Ω, $\Omega \subset\subset V$, then

$$w^\varepsilon \leq -1 \quad \text{and} \quad \varphi \geq \left(1 - \left(1 + \frac{s^2}{16 + s^2}\right)^{1-\frac{n}{k}}\right)\inf_{B_1\backslash V}\rho - 1 \quad \text{in } B_1 \backslash V.$$

So

$$w^\varepsilon - \varphi \leq \left(1 - \left(1 + \frac{s^2}{16 + s^2}\right)^{1-\frac{n}{k}}\right)\inf_{B_1\backslash V}\rho \quad \text{in } B_1 \backslash V.$$

Apply Lemma 2.11 with w^ε, φ, and $\delta < \min\left\{\left(1 + \frac{s^2}{16+s^2}\right)^{1-\frac{n}{k}} - \left(1 + \frac{s^2}{8+s^2}\right)^{1-\frac{n}{k}}, \left(1 - \left(1 + \frac{s^2}{16+s^2}\right)^{1-\frac{n}{k}}\right)\inf_{B_1\backslash V}\rho\right\}$, we obtain a smooth k-subharmonic function $\underline{u}^\varepsilon$ such that $\underline{u}^\varepsilon = w^\varepsilon$ in $\mathbb{C}^n \backslash B_{1+\frac{s}{2}}$, $\underline{u}^\varepsilon = \varphi$ in $B_1 \backslash \Omega$, and $\underline{u}^\varepsilon \geq \max\{\varphi, w^\varepsilon\}$ in Ω^c. Moreover, by the concavity of $S_k^{\frac{1}{k}}$,

$$H_k^{\frac{1}{k}}(\underline{u}^\varepsilon) \geq \frac{1+t(z)}{2} H_k^{\frac{1}{k}}(\varphi) + \frac{1-t(z)}{2} H_k^{\frac{1}{k}}(w^\varepsilon) \quad \text{in } \{|\varphi - w^\varepsilon| < \delta\}.$$

If we take $\varepsilon_0 < \min\{1, 2^{k-n} t^{-2(n+1)} (C_n^k)^{-1} \left(\frac{n}{k} - 1\right)^k \left(1 - \left(1 + \frac{s^2}{16+s^2}\right)^{1-\frac{n}{k}}\right) \varepsilon_0\}$, then for any $\varepsilon \leq \varepsilon_0$, $f^\varepsilon < \varepsilon_0$. So we obtain

$$H_k(\underline{u}^\varepsilon) \geq f^\varepsilon \quad \text{in } \Omega^c.$$

In conclusion, for sufficient small ε, we can construct a smooth, strictly k-subharmonic function $\underline{u}^\varepsilon$ as follows:

Lemma 4.1. *For any $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 < \frac{s^2}{8}$, there exists a strictly k-subharmonic function $\underline{u}^\varepsilon \in C^\infty(\mathbb{C}^n \backslash \Omega)$ satisfying*

$$\underline{u}^\varepsilon = \begin{cases} w^\varepsilon & \text{in } \mathbb{C}^n \backslash B_{1+\frac{s}{2}}, \\ \left(1 - \left(1 + \dfrac{s^2}{16+s^2}\right)^{1-\frac{n}{k}}\right)\rho - 1 & \text{in } B_1 \backslash \Omega, \end{cases}$$

$$\underline{u}^\varepsilon \geq \max\{w^\varepsilon, \left(1 - \left(1 + \frac{s^2}{16+s^2}\right)^{1-\frac{n}{k}}\right)\rho - 1\} \quad \text{in } B_{1+\frac{s}{2}} \backslash B_1,$$

and

$$H_k(\underline{u}^\varepsilon) \geq f^\varepsilon \quad \text{in } \Omega^c,$$

where ρ is a function satisfying (4.2).

By the aforementioned preliminaries in this section, we are able to construct the approximation equations for $\varepsilon \in (0, \varepsilon_0)$ and $R > 1 + s$.

$$\begin{cases} H_k(u^{\varepsilon,R}) = f^\varepsilon & \text{in } \Sigma_R := B_R \backslash \Omega, \\ u^{\varepsilon,R} = \underline{u}^\varepsilon & \text{on } \partial\Sigma_R. \end{cases} \tag{4.3}$$

Since $\underline{u}^\varepsilon$ is a subsolution, by Li [30], (4.3) has a strictly k-subharmonic solution $u^{\varepsilon,R} \in C^\infty(\overline{\Sigma}_R)$. Our goal is to establish uniform C^2 estimates of $u^{\varepsilon,R}$, which is independent of ε and R. We prove the following.

Theorem 4.2. *For sufficient small ε and sufficient large R, $u^{\varepsilon,R}$ satisfies*

$$\begin{aligned} C^{-1}|z|^{2-\frac{2n}{k}} &\leq -u^{\varepsilon,R}(z) \leq C|z|^{2-\frac{2n}{k}}, \\ |Du^{\varepsilon,R}(z)| &\leq C|z|^{1-\frac{2n}{k}}, \\ |\partial\bar{\partial} u^{\varepsilon,R}(z)| &\leq C|z|^{-\frac{2n}{k}}, \\ |D^2 u^{\varepsilon,R}(z)| &\leq C, \end{aligned}$$

where C is a uniform constant that is independent of ε and R.

In subsections 4.2, 4.3 and 4.4, we will prove uniform C^2-estimates of solutions to equation (4.3). The key point is that these estimates are independent of ε and R.

4.2 C^0 estimates

Since $\underline{u}^\varepsilon$ is a subsolution to (4.3), we obtain that

$$u^{\varepsilon,R} \geq \underline{u}^\varepsilon \geq -\left(\frac{|z|^2 + \varepsilon^2}{1+\varepsilon^2}\right)^{1-\frac{n}{k}} \geq -(1+\varepsilon_0^2)^{\frac{n}{k}-1} |z|^{2-\frac{2n}{k}}.$$

For any $R' \geq R \geq 1 + s$, let $u^{\varepsilon,R}$ and $u^{\varepsilon,R'}$ be solutions to (4.3) on Σ_R and $\Sigma_{R'}$, respectively. We have

$$u^{\varepsilon,R} = \underline{u}^{\varepsilon} \leq u^{\varepsilon,R'} \quad \text{on } \partial B_R.$$

By Lemma 2.7,

$$u^{\varepsilon,R} \leq u^{\varepsilon,R'}, \quad \text{in } \Sigma_R.$$

On the other hand, choose $R_1 := \max\left\{1 + s, \frac{t\varepsilon_0}{\sqrt{1-t^2}}\right\}$. Then, for any $R \geq R_1$,

$$\begin{cases} H_k(-t^{\frac{2n}{k}-2}|z|^{2-\frac{2n}{k}}) = 0 < f^{\varepsilon} = H_k(u^{\varepsilon,R}) & \text{in } \Sigma_R, \\ u^{\varepsilon,R} = -1 \leq -t^{\frac{2n}{k}-2}|z|^{2-\frac{2n}{k}} & \text{on } \partial\Omega, \\ u^{\varepsilon,R} = -\left(\dfrac{R^2+\varepsilon^2}{1+\varepsilon^2}\right)^{1-\frac{n}{k}} \leq -t^{\frac{2n}{k}-2}R^{2-\frac{2n}{k}} & \text{on } \partial B_R. \end{cases}$$

Using Lemma 2.7 again, we have

$$u^{\varepsilon,R} \leq -t^{\frac{2n}{k}-2}|z|^{2-\frac{2n}{k}} \quad \text{in } \Sigma_R.$$

So we have, for any $R' > R \geq R_1$,

$$-(1+\varepsilon_0^2)^{\frac{n}{k}-1}|z|^{2-\frac{2n}{k}} \leq u^{\varepsilon,R}(z) \leq u^{\varepsilon,R'}(z) \leq -t^{\frac{2n}{k}-2}|z|^{2-\frac{2n}{k}}, \quad z \in \Sigma_R.$$

4.3 Gradient estimates

In this subsection, we prove the global gradient estimate. The key point is that the estimate here does not depend on ε and R. We also prove that the positive lower bound of the gradient of the solution.

4.3.1 Reducing global gradient estimates to boundary gradient estimates

This part is the key part of gradient estimates. The point in here is that the gradient estimate is independent of the approximating process. This estimates is motivated by Guan [15].

Theorem 4.3. *Let u be the solution of the approximating equation* (4.3). *Denote by*

$$P = |Du|^2(-u)^{-\frac{2n-k}{n-k}}. \tag{4.4}$$

Then, we have the following gradient estimate:

$$\max_{\Sigma_R} P \leq \max\left\{\max_{\partial\Sigma_R} P, \left(\frac{2(n-k)}{k(2n-k)}\right)^2(-u)^{-\frac{k}{n-k}}|D\log f^{\varepsilon}|^2\right\}. \tag{4.5}$$

Proof. For simplicity, we use f instead of f^{ε} during the proof.

Let $a = \frac{2n-k}{n-k}$. Select the auxiliary function

$$\varphi = \log P = \log|Du|^2 - a\log(-u).$$

Suppose φ obtain its maximum at $z_0 \in \Sigma_R$. We can choose the holomorphic coordinate such that $\{u_{i\bar{j}}\}(z_0)$ is diagonal. Denote by $\lambda_i = u_{i\bar{i}}(z_0)$. The following computations are at z_0:

$$0 = \varphi_i = \frac{|Du|_i^2}{|Du|^2} - a\frac{u_i}{u} = \frac{u_l u_{\bar{l}i} + u_{li}u_{\bar{l}}}{|Du|^2} - a\frac{u_i}{u} = \frac{u_i\lambda_i + u_{li}u_{\bar{l}}}{|Du|^2} - a\frac{u_i}{u}.$$

Then, we have the observation

$$a\frac{|u_i|^2}{u} = \frac{|u_i|^2\lambda_i}{|Du|^2} + \sum_{l=1}^{n}\frac{u_{l\bar{i}}u_{\bar{l}}u_{i}}{|Du|^2}, \quad \forall i = 1, \dots, n, \tag{4.6}$$

which implies $\sum_{l=1}^{n} u_{li}u_{\bar{l}}u_{\bar{i}}$ is real at z_0. Denote by $F^{i\bar{j}} = \frac{\partial}{\partial u_{i\bar{j}}}S_k(\partial\bar{\partial}u)$. By direct computation, we can obtain

$$\begin{aligned}
0 \ge F^{i\bar{j}}\varphi_{i\bar{j}} &= F^{i\bar{j}} \cdot \left(\frac{|Du|^2_{i\bar{j}}}{|Du|^2} - \frac{|Du|^2_i|Du|^2_{\bar{j}}}{|Du|^4} - a\frac{u_{i\bar{j}}}{u} + a\frac{u_i u_{\bar{j}}}{u^2}\right)\\
&= F^{i\bar{j}} \cdot \left(\frac{|Du|^2_{i\bar{j}}}{|Du|^2} - \left(1 - \frac{1}{a}\right)\frac{|Du|^2_i|Du|^2_{\bar{j}}}{|Du|^4} - a\frac{u_{i\bar{j}}}{u}\right)\\
&= \frac{2\mathrm{Re}\{u_l f_{\bar{l}}\}}{|Du|^2} - akf\frac{|Du|^2}{u} + \sum_{i,l=1}^{n}\frac{S_{k-1}(\lambda|i)|u_{li}|^2}{|Du|^2} + \sum_{i=1}^{n}\frac{S_{k-1}(\lambda|i)\lambda_i^2}{|Du|^2}\\
&\quad - \frac{n}{2n-k}\sum_{i=1}^{n}\frac{|u_i|^2}{|Du|^4}S_{k-1}(\lambda|i)\lambda_i^2 - \frac{n}{2n-k}\sum_{i=1}^{n}S_{k-1}(\lambda|i)\frac{\left|\sum_{l=1}^{n}u_{\bar{l}}u_{li}\right|^2}{|Du|^4} - \frac{2n}{2n-k}\sum_{i=1}^{n}S_{k-1}(\lambda|i)\lambda_i\frac{\sum_{l=1}^{n}u_{li}u_{\bar{l}}u_{\bar{i}}}{|Du|^4}.
\end{aligned}$$

We claim

$$\begin{aligned}
\mathcal{E} := \sum_{i=1}^{n}\Bigg(&\sum_{l=1}^{n}S_{k-1}(\lambda|i)|u_{li}|^2 + S_{k-1}(\lambda|i)\lambda_i^2 - \frac{n}{2n-k}\frac{|u_i|^2}{|Du|^2}S_{k-1}(\lambda|i)\lambda_i^2 - \frac{n}{2n-k}S_{k-1}(\lambda|i)\frac{\left|\sum_{l=1}^{n}u_{\bar{l}}u_{li}\right|^2}{|Du|^2}\\
&- \frac{2n}{2n-k}S_{k-1}(\lambda|i)\lambda_i\frac{\sum_{l=1}^{n}u_{li}u_{\bar{l}}u_{\bar{i}}}{|Du|^2}\Bigg) \ge 0.
\end{aligned}\tag{4.7}$$

Then,

$$0 \ge |Du|^2F^{i\bar{j}}\varphi_{i\bar{j}} \ge 2\mathrm{Re}\left\{u_l f_{\bar{l}}\right\} - akf\frac{|Du|^2}{u} \ge -2|Du||Df| - akf\frac{|Du|^2}{u}.$$

It follows that

$$|Du| \le \frac{2}{ak}(-u)|D\log f| = \frac{2(n-k)}{k(2n-k)}(-u)|D\log f|.$$

Thus,

$$|Du|^2(-u)^{-a} \le \left(\frac{2(n-k)}{k(2n-k)}\right)^2(-u)^{2-a}|D\log f|^2.$$

Now, we prove Claim (4.7). Since

$$\begin{aligned}
\sum_{i=1}^{n}S_{k-1}(\lambda|i)\lambda_i^2 &= S_1 f - (k+1)S_{k+1}\\
&= \sum_{i=1}^{n}\frac{|u_i|^2}{|Du|^2}(S_1 f - (k+1)S_{k+1})\\
&= \sum_{i=1}^{n}f\frac{|u_i|^2}{|Du|^2}\left(\lambda_i + S_1(\lambda|i) - (k+1)\frac{S_k(\lambda|i)}{S_{k-1}(\lambda|i)}\right) + \sum_{i=1}^{n}\frac{|u_i|^2}{|Du|^2}\left((k+1)\frac{S_k^2(\lambda|i)}{S_{k-1}(\lambda|i)} - (k+1)S_{k+1}(\lambda|i)\right),
\end{aligned}$$

we have

$$
\begin{aligned}
\mathcal{E} =& \sum_{i=1}^{n} f \frac{|u_i|^2}{|Du|^2}\left(\lambda_i + S_1(\lambda|i) - (k+1)\frac{S_k(\lambda|i)}{S_{k-1}(\lambda|i)}\right) + \sum_{i=1}^{n} \frac{|u_i|^2}{|Du|^2}\left((k+1)\frac{S_k^2(\lambda|i)}{S_{k-1}(\lambda|i)} - (k+1)S_{k+1}(\lambda|i)\right)\\
&+ \sum_{i,l=1}^{n} S_{k-1}(\lambda|i)|u_{li}|^2 - \frac{n}{2n-k}\sum_{i=1}^{n}\frac{|u_i|^2}{|Du|^2}S_{k-1}(\lambda|i)\lambda_i^2 - \frac{n}{2n-k}S_{k-1}\sum_{i=1}^{n}(\lambda|i)\frac{|\sum_{l=1}^{n} u_{\bar{l}}u_{li}|^2}{|Du|^2}\\
&- \frac{2n}{2n-k}\sum_{i,l=1}^{n} S_{k-1}(\lambda|i)\lambda_i \frac{u_{li}u_{\bar{l}}u_{\bar{i}}}{|Du|^2}\\
:=& \left(\sum_{i\in G} + \sum_{i\in H}\right)\mathrm{T}_i
\end{aligned}
$$

in which

$$
G = \{i|\lambda_i \geq 0\} \quad H = \{i|\lambda_i < 0\},
$$

and

$$
\begin{aligned}
\mathrm{T}_i =& f\frac{|u_i|^2}{|Du|^2}\left(\lambda_i + S_1(\lambda|i) - (k+1)\frac{S_k(\lambda|i)}{S_{k-1}(\lambda|i)}\right) + \frac{|u_i|^2}{|Du|^2}\left((k+1)\frac{S_k^2(\lambda|i)}{S_{k-1}(\lambda|i)} - (k+1)S_{k+1}(\lambda|i)\right)\\
&+ \sum_{l=1}^{n} S_{k-1}(\lambda|i)|u_{li}|^2 - \frac{n}{2n-k}\frac{|u_i|^2}{|Du|^2}S_{k-1}(\lambda|i)\lambda_i^2 - \frac{n}{2n-k}S_{k-1}(\lambda|i)\frac{|\sum_{l=1}^{n} u_{\bar{l}}u_{li}|^2}{|Du|^2} - \frac{2n}{2n-k}\sum_{l=1}^{n} S_{k-1}(\lambda|i)\lambda_i\frac{u_{li}u_{\bar{l}}u_{\bar{i}}}{|Du|^2}.
\end{aligned}
$$

We will prove in the following that $\forall i$, $\mathrm{T}_i \geq 0$.

Case 1. $i \in H$. Let

$$
\mathrm{T}_i = A + B,
$$

where

$$
\begin{aligned}
A :=& f\frac{|u_i|^2}{|Du|^2}\left(\lambda_i + S_1(\lambda|i) - (k+1)\frac{S_k(\lambda|i)}{S_{k-1}(\lambda|i)}\right) + \frac{|u_i|^2}{|Du|^2}\left((k+1)\frac{S_k^2(\lambda|i)}{S_{k-1}(\lambda|i)} - (k+1)S_{k+1}(\lambda|i)\right)\\
&- \frac{n}{n-k}\frac{|u_i|^2}{|Du|^2}S_{k-1}(\lambda|i)\lambda_i^2
\end{aligned}
$$

and

$$
\begin{aligned}
B :=& \left(\frac{n}{n-k} - \frac{n}{2n-k}\right)\frac{|u_i|^2}{|Du|^2}S_{k-1}(\lambda|i)\lambda_i^2 + \sum_{l=1}^{n} S_{k-1}(\lambda|i)|u_{li}|^2 - \frac{n}{2n-k}S_{k-1}(\lambda|i)\frac{|\sum_{l=1}^{n} u_{\bar{l}}u_{li}|^2}{|Du|^2}\\
&- \frac{2n}{2n-k}\sum_{l=1}^{n} S_{k-1}(\lambda|i)\lambda_i\frac{u_{li}u_{\bar{l}}u_{\bar{i}}}{|Du|^2}.
\end{aligned}
$$

Since

$$
f = S_k(\lambda) = S_{k-1}(\lambda|i)\lambda_i + S_k(\lambda|i),
$$

we have

$$
\lambda_i^2 = \frac{f^2}{S_{k-1}^2(\lambda|i)} + \frac{S_k^2(\lambda|i)}{S_{k-1}^2(\lambda|i)} - \frac{2fS_k(\lambda|i)}{S_{k-1}^2(\lambda|i)} = \frac{f\lambda_i}{S_{k-1}(\lambda|i)} + \frac{S_k^2(\lambda|i)}{S_{k-1}^2(\lambda|i)} - \frac{fS_k(\lambda|i)}{S_{k-1}^2(\lambda|i)}.
$$

Then,

$$
-\frac{n}{n-k}\frac{|u_i|^2}{|Du|^2}S_{k-1}(\lambda|i)\lambda_i^2 = f\frac{|u_i|^2}{|Du|^2}\left(-\frac{n}{n-k}\lambda_i + \frac{n}{n-k}\frac{S_k(\lambda|i)}{S_{k-1}(\lambda|i)}\right) + \frac{|u_i|^2}{|Du|^2}\left(-\frac{n}{n-k}\frac{S_k^2(\lambda|i)}{S_{k-1}(\lambda)}\right).
$$

By (a) and (b) of Proposition 2.2, we have

$$\begin{aligned}A &= f\frac{|u_i|^2}{|Du|^2}\left(\lambda_i + S_1(\lambda|i) - (k+1)\frac{S_k(\lambda|i)}{S_{k-1}(\lambda|i)} - \frac{n}{n-k}\lambda_i + \frac{n}{n-k}\frac{S_k(\lambda|i)}{S_{k-1}(\lambda|i)}\right)\\ &\quad + \frac{|u_i|^2}{|Du|^2}\left((k+1)\frac{S_k^2(\lambda|i)}{S_{k-1}(\lambda|i)} - (k+1)S_{k+1}(\lambda|i) - \frac{n}{n-k}\frac{S_k^2(\lambda|i)}{S_{k-1}(\lambda)}\right)\\ &= f\frac{|u_i|^2}{|Du|^2}\left(-\frac{k}{n-k}\lambda_i + S_1(\lambda|i) - (k+1-\frac{n}{n-k})\frac{S_k(\lambda|i)}{S_{k-1}(\lambda|i)}\right)\\ &\quad + \frac{|u_i|^2}{|Du|^2}\left((k+1-\frac{n}{n-k})\frac{S_k^2(\lambda|i)}{S_{k-1}(\lambda|i)} - (k+1)S_{k+1}(\lambda|i)\right)\\ &\geq f\frac{|u_i|^2}{|Du|^2}\left(-\frac{k}{n-k}\lambda_i + \frac{k}{n-1}S_1(\lambda|i)\right) \geq 0,\end{aligned}$$

where the last inequality is due to the assumption of Case 1. Note that

$$\sum_{l=1}^n S_{k-1}(\lambda|i)|u_{li}|^2 - \frac{n}{2n-k}S_{k-1}(\lambda|i)\frac{|\sum_{l=1}^n u_{\bar{l}}u_{li}|^2}{|Du|^2} \geq \frac{n-k}{2n-k}\sum_{l=1}^n S_{k-1}(\lambda|i)|u_{li}|^2$$

and

$$\frac{2n}{2n-k}S_{k-1}(\lambda|i)\lambda_i\frac{\sum_{l=1}^n u_{li}u_{\bar{l}}u_{\bar{i}}}{|Du|^2} \leq \frac{1}{\varepsilon}\frac{n^2}{(2n-k)^2}\frac{|u_i|^2}{|Du|^2}S_{k-1}(\lambda|i)\lambda_i^2 + \varepsilon S_{k-1}(\lambda|i)\frac{\left|\sum_{l=1}^n u_{\bar{l}}u_{li}\right|^2}{|Du|^2}.$$

Take $\varepsilon = \frac{n-k}{2n-k}$, then $\frac{1}{\varepsilon}\frac{n^2}{(2n-k)^2} = \frac{n}{n-k} - \frac{n}{2n-k}$. It follows that $B \geq 0$.

Case 2. $i \in G$. Then, let

$$T_i = E + F,$$

where

$$E \coloneqq f\frac{|u_i|^2}{|Du|^2}\left(\lambda_i + S_1(\lambda|i) - (k+1)\frac{S_k(\lambda|i)}{S_{k-1}(\lambda|i)}\right) + \frac{|u_i|^2}{|Du|^2}\left((k+1)\frac{S_k^2(\lambda|i)}{S_{k-1}(\lambda|i)} - (k+1)S_{k+1}(\lambda|i)\right) - \frac{|u_i|^2}{|Du|^2}S_{k-1}(\lambda|i)\lambda_i^2$$

and

$$\begin{aligned}F &\coloneqq \left(1 - \frac{n}{2n-k}\right)\frac{|u_i|^2}{|Du|^2}S_{k-1}(\lambda|i)\lambda_i^2 + \sum_{l=1}^n S_{k-1}(\lambda|i)|u_{li}|^2 - \frac{n}{2n-k}S_{k-1}(\lambda|i)\frac{|\sum_{l=1}^n u_{\bar{l}}u_{li}|^2}{|Du|^2}\\ &\quad - \frac{2n}{2n-k}\sum_{l=1}^n S_{k-1}(\lambda|i)\lambda_i\frac{u_{li}u_{\bar{l}}u_{\bar{i}}}{|Du|^2}.\end{aligned}$$

Since $i \in G$, we have $\lambda_i \geq 0$, it follows from (4.6) that

$$\sum_{l=1}^n u_{li}u_{\bar{l}}u_{\bar{i}} < 0.$$

Then,

$$F \geq \frac{n-k}{2n-k}\frac{|u_i|^2}{|Du|^2}S_{k-1}(\lambda|i)\lambda_i^2 + \sum_{l=1}^n S_{k-1}(\lambda|i)|u_{li}|^2 - \frac{n}{2n-k}S_{k-1}(\lambda|i)\frac{|\sum_{l=1}^n u_{\bar{l}}u_{li}|^2}{|Du|^2} \geq 0.$$

Using (b) of Proposition 2.2, we obtain

$$\begin{aligned}E &= f\frac{|u_i|^2}{|Du|^2}\left(S_1(\lambda|i) - k\frac{S_k(\lambda|i)}{S_{k-1}(\lambda|i)}\right) + \frac{|u_i|^2}{|Du|^2}\left(k\frac{S_k^2(\lambda|i)}{S_{k-1}(\lambda|i)} - (k+1)S_{k+1}(\lambda|i)\right)\\ &\geq \frac{k-1}{n-1}\frac{|u_i|^2}{|Du|^2}S_1(\lambda|i) + \frac{k}{n-k}\frac{|u_i|^2}{|Du|^2}\frac{S_k^2(\lambda|i)}{S_{k-1}(\lambda|i)} \geq 0.\end{aligned}$$

Hence, we complete the proof of claim (4.7). □

4.3.2 Boundary gradient estimates

We always assume $R >> R_1$. To prove the boundary gradient estimates, we will construct upper barriers on $\partial\Omega$ and ∂B_R, respectively.

Let $h_1 \in C^\infty(\overline{\Sigma}_{R_1})$ be the solution of the following equation:

$$\begin{cases} \Delta h_1 = 0 & \text{in } \Sigma_{R_1}, \\ h_1 = -1 & \text{on } \partial\Omega, \\ h_1 = -t^{\frac{2n}{k}-2}|z|^{2-\frac{2n}{k}} & \text{on } \partial B_{R_1}. \end{cases}$$

$u^{\varepsilon,R}$ is k-subharmonic in Σ_R, thus is subharmonic in Σ_R. Note that

$$h_1 = u^{\varepsilon,R} = -1 \quad \text{on } \partial\Omega \quad \text{and} \quad h_1 = -t^{\frac{2n}{k}-2}R_1^{2-\frac{2n}{k}} \geq u^{\varepsilon,R} \quad \text{on } \partial B_{R_1}.$$

By comparison theorem for the Laplace equation, we obtain

$$u^{\varepsilon,R} \leq h_1 \quad \text{in } \Sigma_{R_1}.$$

Let ν be the unit outer normal to $\partial\Omega$, then

$$\left(1 - (1 + \frac{s^2}{16 + s^2})^{1-\frac{n}{k}}\right)\rho_\nu = \underline{u}^\varepsilon_\nu \leq u^{\varepsilon,R}_\nu \leq h_{1,\nu} \leq C(h_1) = C(\Omega, t, R_1) \quad \text{on } \partial\Omega,$$

where ρ is defined in (4.2). So, there is a constant C independent of ε and R such that

$$|Du^{\varepsilon,R}| \leq C, \quad \text{on } \partial\Omega.$$

Let $h_2 \in C^\infty(\overline{B_R}\backslash B_{\frac{R}{2}})$ be a solution to the following equations:

$$\begin{cases} \Delta h_2 = 0 & \text{in } B_R\backslash\overline{B_{\frac{R}{2}}}, \\ h_2 = \underline{u}^\varepsilon & \text{on } \partial B_R, \\ h_2 = -(2t)^{\frac{2n}{k}-2}|z|^{2-\frac{2n}{k}} & \text{on } \partial B_{\frac{R}{2}}. \end{cases}$$

For any C^2 function g, set

$$\tilde{g} = R^{\frac{2n}{k}-2}g(R\,\cdot).$$

Then, $\tilde{h}_2(z) = R^{\frac{2n}{k}-2}h_2(Rz)$ satisfies

$$\begin{cases} \Delta\tilde{h}_2 = 0 & \text{in } B_1\backslash\overline{B_{\frac{1}{2}}}, \\ \tilde{h}_2 = \underline{\tilde{u}}^\varepsilon = -\left(\dfrac{1 + \frac{\varepsilon^2}{R^2}}{1 + \varepsilon^2}\right)^{1-\frac{n}{k}} & \text{on } \partial B_1, \\ \tilde{h}_2 = -(2t)^{\frac{2n}{k}-2} & \text{on } \partial B_{\frac{1}{2}}. \end{cases}$$

Note that

$$\tilde{h}_2 = \tilde{u}^{\varepsilon,R} \quad \text{on } \partial B_1 \quad \text{and} \quad \tilde{h}_2 \geq \tilde{u}^{\varepsilon,R} \quad \text{on } \partial B_{\frac{1}{2}}.$$

By comparison theorem, we obtain

$$u^{\varepsilon,R} \leq \tilde{h}_2 \quad \text{in } B_1\backslash B_{\frac{1}{2}}.$$

Let ν be the unit outer normal to B_1. Then,

$$\tilde{h}_{2,\nu} \leq \tilde{u}^{\varepsilon,R}_\nu \leq \underline{\tilde{u}}^\varepsilon_\nu \quad \text{on } \partial B_1.$$

Note that

$$-2^{1-\frac{n}{k}} \ge -\left(\frac{1+\frac{\varepsilon^2}{R^2}}{1+\varepsilon^2}\right)^{1-\frac{n}{k}} \ge -\left(\frac{1}{1+\varepsilon_0^2}\right)^{1-\frac{n}{k}},$$

then $\tilde{h}_2$ is uniformly bounded on $\partial B_1 \backslash \overline{B_{\frac{1}{2}}}$. Since the gradient estimate of harmonic function depends only on the domain and C^0 norm of boundary value, there is a positive constant independent of ε and R, such that

$$|\tilde{h}_{2,\nu}| \le C, \quad \text{on } \partial B_1.$$

On the other hand, since

$$\underline{\tilde{u}}^\varepsilon = -\left(\frac{|z|^2+\frac{\varepsilon^2}{R^2}}{1+\varepsilon^2}\right)^{1-\frac{n}{k}}, \quad \text{in a neighborhood of } \partial B_1,$$

we have

$$\underline{\tilde{u}}^\varepsilon_\nu = \left(\frac{n}{k}-1\right)\left(\frac{1+\frac{\varepsilon^2}{R^2}}{1+\varepsilon^2}\right)^{-\frac{n}{k}} \frac{\bar{z}\cdot\nu}{1+\varepsilon^2} \quad \text{on } \partial B_1.$$

Hence,

$$|D\tilde{u}^{\varepsilon,R}| \le C, \quad \text{on } \partial B_1 \quad \text{independent of } \varepsilon \text{ and } R.$$

So, we have the (ε, R)-independent estimate as follows:

$$|Du^{\varepsilon,R}| \le CR^{1-\frac{2n}{k}}, \quad \text{on } \partial B_R.$$

Set $a = \frac{2n-k}{n-k}$, from C^0 estimate, we have

$$(-u^{\varepsilon,R})^{-a} \le (t^{-1}R)^{\frac{4n-2k}{k}}, \quad \text{on } \partial B_R.$$

So we have

$$|Du^{\varepsilon,R}|^2(-u^{\varepsilon,R})^{-a} \le C, \quad \text{on } \partial\Sigma_R,$$

where C is a constant independent of ε and R.

Since

$$(-u^{\varepsilon,R})^{2-a} \le \left((t^{-1}|z|)^{2-\frac{2n}{k}}\right)^{-\frac{k}{n-k}} = t^{-2}|z|^2 \tag{4.8}$$

and

$$D\log f^\varepsilon = -(n+1)\frac{\bar{z}}{|z|^2+\varepsilon^2}. \tag{4.9}$$

We have

$$(-u^{\varepsilon,R})^{2-a}|D\log f^\varepsilon|^2 \le C_n \frac{1}{t^2}\frac{|z|^4}{(|z|^2+\varepsilon^2)^2} \le C(n,t).$$

By Theorem 4.3,

$$|Du^{\varepsilon,R}|^2(-u^{\varepsilon,R})^{-a} \le C,$$

where C is independent of ε and R. Use the C^0 estimate once more, we drive that

$$|Du^{\varepsilon,R}|^2 \le C(-u^{\varepsilon,R})^a \le C|z|^{1-\frac{2n}{k}}.$$

4.4 Second-order estimates

We will prove the second-order estimate of the approximating equations.

4.4.1 The global second-order estimate can be reduced to the boundary second-order estimate

We use the idea of Hou et al. [21] (see also the real case by Chou and Wang [10]) to prove the following estimate.

Theorem 4.4. *Let u be the k-subharmonic solution to (4.3) and consider $H = u_{\xi\bar{\xi}}(-u)^{-\frac{n}{n-k}}\psi(P)$. If $(-u)^{-\frac{k}{n-k}}|D\log f^\varepsilon|^2$ and $(-u)^{-\frac{k}{n-k}}|D^2\log f^\varepsilon|$ are uniformly bounded, which is independent of ε and R, then we have*

$$\max_{\Sigma_R} H \le C + \max_{\partial\Sigma_R} H, \tag{4.10}$$

where $P = |Du|^2(-u)^{-\frac{2n-k}{n-k}}$, $\psi(t) = (M-t)^{-\sigma}$, $\sigma \le \frac{a-1}{8a^2}$, and $M = 2\max_{\Sigma_R}P + 1$, $a = \frac{2n-k}{n-k}$, C is a positive constant depending only on n, k, $\sup_{\Sigma_R}P$, $\sup_{\Sigma_R}(-u)^{-\frac{k}{n-k}}|D\log f^\varepsilon|^2$, and $\sup_{\Sigma_R}(-u)^{-\frac{k}{n-k}}|D^2\log f^\varepsilon|$.

Theorem 4.5. *Let u be the k-subharmonic solution to (4.3). Let $\hat{w} := -\left(\frac{r_0^2 + |z|^2}{1+\varepsilon^2}\right)^{1-\frac{n}{k}}$. Then, for sufficient small ε and b, for any unit vector $\xi \in \mathbb{R}^{2n}$, there holds*

$$\max_{\Sigma_R}(\hat{w} - u + bu_{\xi\xi}) \le \max_{\partial\Sigma_R}(\hat{w} - u + bu_{\xi\xi}).$$

Proof of Theorem 4.4. For simplicity, we write f instead of f^ε during the proof.

Suppose the maximum of H is attained at an interior point $z_0 \in \Sigma_R$ along the direction $\xi_0 = \frac{\partial}{\partial z_1}$. We can choose the holomorphic coordinate such that $\{u_{i\bar{j}}\}$ is diagonal at z_0 and $\lambda_i := u_{i\bar{i}}$ with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$. The following calculations are at z_0. Then, we have

$$0 = \varphi_i = \frac{u_{1\bar{1}i}}{u_{1\bar{1}}} - (a-1)\frac{u_i}{u} + \sigma\frac{P_i}{M-P}.$$

Denote by $F^{i\bar{j}} := \frac{\partial}{\partial u_{i\bar{j}}}\log S_k(\partial\bar{\partial}u) = \frac{S_k^{i\bar{j}}}{S_k}$, and $F^{i\bar{j},r\bar{s}} = \frac{\partial^2}{\partial u_{i\bar{j}}u_{r\bar{s}}}\log S_k(\partial\bar{\partial}u) = \frac{S^{i\bar{j},r\bar{s}}}{S_k} - \frac{S_k^{i\bar{j}}S_k^{r\bar{s}}}{S_k^2}$, $S_k^{i\bar{j}} := \frac{\partial}{\partial u_{i\bar{j}}}S_k(\partial\bar{\partial}u)$, $S_k^{i\bar{j},r\bar{s}} = \frac{\partial^2}{\partial u_{i\bar{j}}\partial u_{r\bar{s}}}S_k(\partial\bar{\partial}u)$. Then, by direct calculation, we have

$$\begin{aligned}
0 &\ge F^{i\bar{j}}\varphi_{i\bar{j}} \\
&= \lambda_1^{-1}F^{i\bar{j}}u_{1\bar{1}i\bar{j}} - \frac{F^{i\bar{i}}|u_{1\bar{1}i}|^2}{u_{1\bar{1}}^2} + (a-1)\frac{F^{i\bar{j}}u_{i\bar{j}}}{(-u)} + (a-1)\frac{F^{i\bar{i}}|u_i|^2}{u^2} + \sigma\frac{F^{i\bar{j}}P_{i\bar{j}}}{M-P} + \sigma\frac{F^{i\bar{i}}|P_i|^2}{(M-P)^2} \\
&= \lambda_1^{-1}F^{i\bar{j}}u_{1\bar{1}i\bar{j}} - \frac{F^{i\bar{i}}|u_{1\bar{1}i}|^2}{u_{1\bar{1}}^2} + \frac{(a-1)k}{(-u)} + (a-1)\frac{F^{i\bar{i}}|u_i|^2}{u^2} + \sigma\frac{F^{i\bar{j}}P_{i\bar{j}}}{M-P} + \sigma\frac{F^{i\bar{i}}|P_i|^2}{(M-P)^2} \\
&:= \mathrm{I} + \mathrm{II} + \cdots + \mathrm{VI}.
\end{aligned} \tag{4.11}$$

Take the first, and second-order derivatives to P, we have

$$P_i = |Du|_i^2(-u)^{-a} + |Du|^2((-u)^{-a})_i$$

and

$$\begin{aligned}
P_{i\bar{j}} &= |Du|_{i\bar{j}}^2(-u)^{-a} + |Du|_i^2((-u)^{-a})_{\bar{j}} + |Du|_{\bar{j}}^2((-u)^{-a})_i + |Du|^2((-u)^{-a})_{i\bar{j}} \\
&= (u_l u_{\bar{l}i\bar{j}} + u_{li\bar{j}}u_{\bar{l}} + u_{li}u_{\bar{l}\bar{j}} + u_{l\bar{j}}u_{\bar{l}i})(-u)^{-a} + a(-u)^{-a-1}((u_l u_{\bar{l}i} + u_{li}u_{\bar{l}})u_{\bar{j}} \\
&\quad + (u_l u_{\bar{l}\bar{j}} + u_{l\bar{j}}u_l)u_i) + a(-u)^{-a-1}|Du|^2 u_{i\bar{j}} + a(a+1)(-u)^{-a-2}|Du|^2 u_i u_{\bar{j}}.
\end{aligned}$$

So,

$$
\begin{aligned}
F^{i\bar{j}}P_{i\bar{j}} = {} & F^{i\bar{j}} \cdot ((u_l u_{\bar{l}i\bar{j}} + u_{li\bar{j}}u_{\bar{l}} + u_{li}u_{\bar{l}\bar{j}} + u_{l\bar{j}}u_{\bar{l}i})(-u)^{-a} + a(-u)^{-a-1}((u_l u_{\bar{l}i} + u_{li}u_{\bar{l}})u_{\bar{j}} + (u_l u_{\bar{l}\bar{j}} + u_{l\bar{j}}u_{\bar{l}})u_i) \\
& + a(-u)^{-a-1}|Du|^2 u_{i\bar{j}} + a(a+1)(-u)^{-a-2}|Du|^2 u_i u_{\bar{j}}) \\
= {} & 2\mathrm{Re}\left\{u_l \tilde{f}_{\bar{l}}\right\}(-u)^{-a} + F^{i\bar{i}}|u_{li}|^2(-u)^{-a} + F^{i\bar{i}}\lambda_i^2(-u)^{-a} + 2a(-u)^{-a-1}F^{i\bar{i}}\lambda_i|u_i|^2 + 2a(-u)^{-a-1}F^{i\bar{i}}u_{li}u_{\bar{l}}u_{\bar{i}} \\
& + ka(-u)^{-a-1}|Du|^2 + a(a+1)(-u)^{-a-2}|Du|^2F^{i\bar{i}}|u_i|^2
\end{aligned}
\tag{4.12}
$$

and

$$
\begin{aligned}
F^{i\bar{j}}P_{i\bar{j}} \geq {} & 2\mathrm{Re}\left\{u_l \tilde{f}_{\bar{l}}\right\}(-u)^{-a} + ka(-u)^{-a-1}|Du|^2 + a(a+1)(-u)^{-a-2}|Du|^2F^{i\bar{i}}|u_i|^2 + \frac{1}{2}F^{i\bar{i}}|u_{li}|^2(-u)^{-a} \\
& + \frac{1}{2}F^{i\bar{i}}\lambda_i^2(-u)^{-a} - 2a^2(-u)^{-a-2}F^{i\bar{i}}|u_i|^2|Du|^2 - 2a^2(-u)^{-a-2}F^{i\bar{i}}|u_i|^4 \\
\coloneqq {} & a_1 + \cdots + a_7.
\end{aligned}
\tag{4.13}
$$

We divide the rest of the computation into two cases: $\lambda_k \geq \delta\lambda_1$ and $\lambda_k < \delta\lambda_1$.

Case 1. $\lambda_k \geq \delta\lambda_1$. Then,

$$
\mathrm{II} \coloneqq -\frac{F^{i\bar{i}}|u_{1\bar{1}i}|^2}{u_{1\bar{1}}^2} = -F^{i\bar{i}}|(a-1)\frac{u_i}{u} - \sigma\frac{P_i}{M-P}|^2 \geq -2F^{i\bar{i}}\left((a-1)^2\frac{|u_i|^2}{u^2} + \sigma^2\frac{|P_i|^2}{(M-P)^2}\right).
$$

So,

$$
\begin{aligned}
\mathrm{II} + \mathrm{IV} + \mathrm{VI} \coloneqq {} & -\frac{F^{i\bar{i}}|u_{1\bar{1}i}|^2}{u_{1\bar{1}}^2} + (a-1)\frac{F^{i\bar{i}}|u_i|^2}{u^2} + \sigma\frac{F^{i\bar{i}}|P_i|^2}{(M-P)^2} \\
\geq {} & ((a-1) - 2(a-1)^2)\frac{F^{i\bar{i}}|u_i|^2}{u^2} + (\sigma - 2\sigma^2)\frac{F^{i\bar{i}}|P_i|^2}{(M-P)^2} \\
\geq {} & ((a-1) - 2(a-1)^2)\frac{F^{i\bar{i}}|u_i|^2}{u^2},
\end{aligned}
$$

where the last inequality holds since $\sigma \leq \frac{1}{2}$.

By the concavity of $S_k^{\frac{1}{k}}$, we have

$$
\mathrm{I} \coloneqq \lambda_1^{-1}F^{i\bar{j}}u_{1\bar{1}i\bar{j}} = \lambda_1^{-1}((\log f)_{1\bar{1}} - F^{i\bar{j},r\bar{s}}u_{i\bar{j}1}u_{r\bar{s}\bar{1}}) \geq \lambda_1^{-1}(\log f)_{1\bar{1}}.
$$

By (2.2), we have

$$
F^{i\bar{i}}\lambda_i^2 \geq F^{k\bar{k}}\lambda_k^2 \geq \theta\mathcal{F}\lambda_k^2 \geq \delta^2\theta\mathcal{F}\lambda_1^2,
\tag{4.14}
$$

where $\mathcal{F} = \sum_{i=1}^n F^{i\bar{i}}$, $\theta = \theta(n,k)$, and we use the assumption of **Case 1** in the last inequality. Based on (4.14), we have the following calculation:

$$
\begin{aligned}
\frac{1}{4}a_5 + a_6 + a_7 \coloneqq {} & \frac{1}{8}F^{i\bar{i}}\lambda_i^2(-u)^{-a} - 2a^2(-u)^{-a-2}F^{i\bar{i}}|u_i|^2|Du|^2 - 2a^2(-u)^{-a-2}F^{i\bar{i}}|u_i|^4 \\
\geq {} & \frac{1}{8}F^{i\bar{i}}\lambda_i^2(-u)^{-a} - 4a^2(-u)^{-a-2}\mathcal{F}|Du|^4 \\
\geq {} & (-u)^{a-2}\mathcal{F}\left(\frac{\delta^2\theta}{8}(\lambda_1(-u)^{-a+1})^2 - 4a^2P^2\right) \\
\geq {} & 0,
\end{aligned}
$$

where the last inequality holds if we suppose

$$
(\lambda_1(-u)^{-a+1})^2 \geq \frac{32a^2}{\delta^2\theta}P^2.
\tag{4.15}
$$

By Newton-MacLaurin inequality, we have

$$
\frac{S_k}{S_{k-1}} \leq \frac{n-k+1}{nk}S_1.
$$

So,

$$\mathcal{F} = \frac{\sum_{i=1}^{n} S_{k-1,i}}{S_k} = (n-k+1)\frac{S_{k-1}}{S_k} \geq nkS_1^{-1} \geq \frac{k}{\lambda_1}.$$

Combined with (4.14), we have

$$F^{i\bar{i}}\lambda_i^2 \geq k\delta^2\theta\lambda_1. \tag{4.16}$$

By (4.16),

$$\begin{aligned}\frac{1}{4}a_5 + a_1 &:= \frac{1}{8}F^{i\bar{i}}\lambda_i^2(-u)^{-a} + 2\mathrm{Re}\{u_l\tilde{f}_l\}(-u)^{-a}\\ &\geq \frac{k\delta^2\theta}{8}\lambda_1(-u)^{-a} - 2|Du||D\tilde{f}|(-u)^{-a}\\ &\geq (-u)^{-1}\left(\frac{k\delta^2\theta}{8}\lambda_1(-u)^{-a+1} - 2P^{\frac{1}{2}}|D\tilde{f}|(-u)^{-\frac{a}{2}+1}\right)\\ &\geq 0,\end{aligned}$$

where last inequality holds if we assume

$$\lambda_1(-u)^{-a+1} \geq \frac{16}{k\delta^2\theta}|D\tilde{f}|(-u)^{-\frac{a}{2}+1}P^{\frac{1}{2}}. \tag{4.17}$$

Note that $a-1=\frac{n}{n-k}>1$, it follows from (4.14) that

$$\begin{aligned}\frac{\sigma}{M-P}\cdot\frac{1}{4}a_5 + \mathrm{II} + \mathrm{IV} + \mathrm{VI} &\geq \frac{\sigma}{M-P}\cdot\frac{1}{8}F^{i\bar{i}}\lambda_i^2(-u)^{-a} + ((a-1)-2(a-1)^2)\frac{F^{i\bar{i}}|u_i|^2}{u^2}\\ &\geq \frac{\sigma}{M-P}\cdot\frac{\delta^2\theta}{8}\mathcal{F}\lambda_1^2(-u)^{-a} + ((a-1)-2(a-1)^2)\mathcal{F}\frac{|Du|^2}{u^2}\\ &= (-u)^{a-2}\mathcal{F}\left(\frac{\sigma}{M-P}\cdot\frac{\delta^2\theta}{8}(\lambda_1(-u)^{-a+1})^2 - (2(a-1)^2-(a-1))P\right)\\ &\geq 0,\end{aligned}$$

where the last inequality holds if we suppose

$$(\lambda_1(-u)^{-a+1})^2 \geq \frac{16M}{\sigma\delta^2\theta}(2(a-1)^2-(a-1))P. \tag{4.18}$$

By (4.16), we have

$$\begin{aligned}\frac{\sigma}{M-P}\cdot\frac{1}{4}a_5 + \mathrm{I} &\geq \frac{\sigma}{M-P}\cdot\frac{\delta^2\theta}{8}\mathcal{F}\lambda_1^2(-u)^{-a} - \lambda_1^{-1}(\log f)_{1\bar{1}}\\ &\geq \frac{\sigma}{M-P}\cdot\frac{k\delta^2\theta}{8}\lambda_1^{-1}(-u)^{-a} - \lambda_1|D^2\log f|\\ &= (-u)^{a-2}\lambda_1^{-1}\left(\frac{\sigma}{M-P}\cdot\frac{k\delta^2\theta}{8}(\lambda_1(-u)^{-a+1})^2 - |D^2\log f|(-u)^{-a+2}\right)\\ &\geq 0,\end{aligned}$$

where the last inequality holds if we suppose

$$(\lambda_1(-u)^{-a+1})^2 \geq \frac{16M}{k\sigma\delta^2\theta}|D^2\log f|(-u)^{-a+2}. \tag{4.19}$$

From assumptions (4.15), (4.17), (4.18), and (4.19), we have

$$0 \geq F^{i\bar{j}}\varphi_{i\bar{j}} > 0,$$

which leads to a contradiction. Since P, $|D\log f|(-u)^{-\frac{a}{2}+1}$ and $|D^2\log f|(-u)^{-a+2}$ are uniformly bounded, we finish the proof of Case 1.

Case 2. $\lambda_k \le \delta\lambda_1$. By the first-order derivatives condition, we have

$$\sigma\sum_{i\ge2}\frac{F^{i\bar i}|P_i|^2}{(M-P)^2} = \frac{1}{\sigma}\sum_{i\ge2}F^{i\bar i}\left|\frac{u_{1\bar 1 i}}{u_{1\bar 1}} - (a-1)\frac{u_i}{u}\right|^2 \ge \frac{\varepsilon}{\sigma}\sum_{i\ge2}F^{i\bar i}\left|\frac{u_{1\bar 1 i}}{u_{1\bar 1}}\right|^2 - \frac{1}{\sigma}\cdot\frac{\varepsilon}{1-\varepsilon}(a-1)^2\sum_{i\ge2}F^{i\bar i}\frac{|u_i|^2}{u^2}.$$

Putting the above inequality into (4.11), we have

$$\begin{aligned}
0 &\ge F^{i\bar j}\varphi_{i\bar j}\\
&= \frac{(a-1)k}{(-u)} + \lambda_1^{-1}F^{i\bar j}u_{1\bar 1 i\bar j} + \sigma\frac{F^{i\bar j}P_{i\bar j}}{M-P} - \frac{F^{1\bar 1}|u_{1\bar 1 1}|^2}{u_{1\bar 1}^2} + (a-1)\frac{F^{1\bar 1}|u_1|^2}{u^2} + \sigma\frac{F^{1\bar 1}|P_1|^2}{(M-P)^2}\\
&\quad - \sum_{i\ge2}\frac{F^{i\bar i}|u_{1\bar 1 i}|^2}{u_{1\bar 1}^2} + (a-1)\sum_{i\ge2}\frac{F^{i\bar i}|u_i|^2}{u^2} + \sigma\sum_{i\ge2}\frac{F^{i\bar i}|P_i|^2}{(M-P)^2}\\
&\ge \frac{(a-1)k}{(-u)} + \lambda_1^{-1}F^{i\bar j}u_{1\bar 1 i\bar j} + \sigma\frac{F^{i\bar j}P_{i\bar j}}{M-P} - \frac{F^{1\bar 1}|u_{1\bar 1 1}|^2}{u_{1\bar 1}^2} + (a-1)\frac{F^{1\bar 1}|u_1|^2}{u^2} + \sigma\frac{F^{1\bar 1}|P_1|^2}{(M-P)^2}\\
&\quad - \sum_{i\ge2}\left(1-\frac{\varepsilon}{\sigma}\right)\frac{F^{i\bar i}|u_{1\bar 1 i}|^2}{u_{1\bar 1}^2} + \left((a-1) - \frac{1}{\sigma}\cdot\frac{\varepsilon}{1-\varepsilon}(a-1)^2\right)\sum_{i\ge2}\frac{F^{i\bar i}|u_i|^2}{u^2}\\
&:= \mathrm{I}' + \mathrm{II}' + \cdots + \mathrm{VIII}'.
\end{aligned}\tag{4.20}$$

We take

$$\varepsilon \le \min\left\{\frac{1}{4}, \frac{3}{8(a-1)}\sigma\right\},\tag{4.21}$$

then

$$\mathrm{VIII}' \ge \frac{a-1}{2}\sum_{i\ge2}\frac{F^{i\bar i}|u_i|^2}{u^2}.$$

Note that

$$\mathrm{IV}' = -\frac{F^{1\bar 1}|u_{1\bar 1 1}|^2}{u_{1\bar 1}^2} = -F^{1\bar 1}\left|(a-1)\frac{u_1}{u} - \sigma\frac{P_1}{M-P}\right|^2 \ge -2(a-1)^2F^{1\bar 1}\frac{|u_1|^2}{u^2} - 2\sigma^2\frac{|P_1|^2}{(M-P)^2}.$$

By the choice of σ, we have

$$\mathrm{IV}' + \mathrm{V}' + \mathrm{VI}' \ge (a-1-2(a-1)^2)F^{1\bar 1}\frac{|u_1|^2}{u^2}.$$

Putting the above inequalities into (4.20),

$$\begin{aligned}
0 &\ge \lambda_1^{-1}F^{i\bar j}u_{1\bar 1 i\bar j} - \sum_{i\ge2}\left(1-\frac{\varepsilon}{\sigma}\right)\frac{F^{i\bar i}|u_{1\bar 1 i}|^2}{u_{1\bar 1}^2} + \frac{(a-1)k}{(-u)} + \sigma\frac{F^{i\bar j}P_{i\bar j}}{M-P} + \left(\frac{a-1}{2} - 2(a-1)^2\right)F^{1\bar 1}\frac{|u_1|^2}{u^2}\\
&\quad + \frac{a-1}{2}\sum_{i\ge2}\frac{F^{i\bar i}|u_i|^2}{u^2} := \mathrm{I}'' + \cdots + \mathrm{VI}''.
\end{aligned}\tag{4.22}$$

We have

$$\begin{aligned}
\mathrm{VI}'' + \frac{\sigma}{M-P}(a_6 + a_7) &:= \frac{a-1}{2}\sum_{i\ge2}\frac{F^{i\bar i}|u_i|^2}{u^2} - \frac{\sigma}{M-P}(2a^2(-u)^{-a-2}F^{i\bar i}|u_i|^2|Du|^2 + 2a^2(-u)^{-a-2}F^{i\bar i}|u_i|^4)\\
&= \sum_{i\ge2}F^{i\bar i}\frac{|u_i|^2}{u^2}\left(\frac{a-1}{2} - \frac{\sigma}{M-P}\cdot 2a^2|Du|^2(-u)^{-a} - \frac{\sigma}{M-P}2a^2|u_i|^2(-u)^{-a}\right)\\
&\quad - \frac{\sigma}{M-P}(2a^2(-u)^{-a-2}F^{1\bar 1}|u_1|^2|Du|^2 + 2a^2(-u)^{-a-2}F^{1\bar 1}|u_1|^4)\\
&\ge \sum_{i\ge2}F^{i\bar i}\frac{|u_i|^2}{u^2}\left(\frac{a-1}{2} - \frac{\sigma P}{M-P}\cdot 4a^2\right) - 4a^2\frac{\sigma P}{M-P}F^{1\bar 1}\frac{|u_1|^2}{u^2}\\
&\ge -(a-1)F^{1\bar 1}\frac{|u_1|^2}{u^2},
\end{aligned}$$

where the last inequality holds if we take $\sigma \le \frac{a-1}{8a^2}$.

$$\begin{aligned}\frac{\sigma}{M-P}\left(\frac{1}{4}a_5 + a_6 + a_7\right) + \mathrm{V}'' + \mathrm{VI}'' &\ge \frac{\sigma}{M-P}\cdot\frac{1}{8}F^{i\bar{i}}\lambda_i^2(-u)^{-a} - \left(\frac{a-1}{2} + 2(a-1)^2\right)F^{1\bar{1}}\frac{|u_1|^2}{u^2}\\ &\ge F^{1\bar{1}}(-u)^{a-2}\left(\frac{\sigma}{M-P}\cdot\frac{1}{8}(\lambda_1(-u)^{-a+1})^2 - (2(a-1)^2 + \frac{a-1}{2})P\right)\\ &\ge 0,\end{aligned}\tag{4.23}$$

where the last inequality holds if we assume

$$(\lambda_1(-u)^{-a+1})^2 \ge \frac{16M}{\sigma}\left(2(a-1)^2 + \frac{a-1}{2}\right)P.\tag{4.24}$$

By

$$S_k^{-1}(\lambda)S_{k-1}(\lambda|i)\lambda_i^2 = S_1(\lambda) - (k+1)\frac{S_{k+1}(\lambda)}{S_k(\lambda)} \ge \frac{k}{n}S_1(\lambda) \ge \frac{k}{n}\lambda_1,$$

we have

$$\begin{aligned}\frac{1}{4}a_5 + a_1 &:= \frac{1}{8}F^{i\bar{i}}\lambda_i^2(-u)^{-a} - 2\mathrm{Re}\{u_l \log f_l\}\\ &\ge \frac{k}{8n}\lambda_1(-u)^{-a} - 2|Du||D\log f|(-u)^{-a}\\ &= (-u)^{-1}\left(\frac{k}{8n}\lambda_1(-u)^{-a+1} - 2P^{\frac{1}{2}}|D\log f|(-u)^{-\frac{a}{2}+1}\right)\\ &\ge 0,\end{aligned}\tag{4.25}$$

where the last inequality holds if we assume

$$\lambda_1(-u)^{-a+1} \ge \frac{16n}{k}P^{\frac{1}{2}}|D\log f|(-u)^{-\frac{a}{2}+1}.\tag{4.26}$$

By Proposition 2.4, when δ is small enough (depending on ε and σ),

$$\begin{aligned}\mathrm{I}'' + \mathrm{II}'' &:= \lambda_1^{-1}F^{i\bar{j}}u_{1\bar{1}i\bar{j}} - \sum_{i\ge 2}\left(1-\frac{\varepsilon}{\sigma}\right)\frac{F^{i\bar{i}}|u_{1\bar{1}i}|^2}{u_{1\bar{1}}^2}\\ &\ge f^{-1}\lambda_1^{-2}\sum_{i\ge 2}|u_{1\bar{1}i}|^2\left(\lambda_1 S_{k-2,1i} - (1-\frac{\varepsilon}{\sigma})S_{k-1,i}\right) + \lambda_1^{-1}(\log f)_{1\bar{1}}\\ &\ge -\lambda_1^{-1}|D^2\log f|,\end{aligned}\tag{4.27}$$

where we use the concavity of $\log S_k$ in the first inequality.

Substituting (4.23), (4.25), and (4.27) into (4.22), we obtain

$$0 \ge \frac{(a-1)k}{-u} - \lambda_1^{-1}|D^2\log f|.\tag{4.28}$$

Then,

$$\lambda_1(-u)^{-a+1} \le (a-1)k|D^2\log f|(-u)^{-a+2}.\tag{4.29}$$

Since P, $|D\log f|(-u)^{-\frac{a}{2}+1}$, and $|D^2\log f|(-u)^{-a+2}$ are uniformly bounded, we finish the proof of Case 2. □

Proof of Theorem 4.5. Observe that the equation is equivalent to

$$F[u] := S_k^{\frac{1}{k}}(\partial\bar{\partial}u) = (f^{\varepsilon})^{\frac{1}{k}}.$$

Denote by $F^{i\bar{j}} = \frac{\partial F[u]}{\partial u_{i\bar{j}}}$ and $F^{i\bar{j},k\bar{l}} = \frac{\partial^2 F[u]}{\partial u_{i\bar{j}}\partial u_{k\bar{l}}}$. Now, we consider any unit vector $\xi \in \mathbb{R}^{2n}$. Differentiating the equation above twice with respect to ξ, we obtain

$$F^{i\bar{j}}u_{\xi\xi i\bar{j}} = D_{\xi\xi}F[u] - F^{i\bar{j},k\bar{l}}u_{i\bar{j}\xi}u_{k\bar{l}\xi} \geq ((f^{\varepsilon})^{\frac{1}{k}})_{\xi\xi} \geq -\frac{2(n+k)}{k}\frac{(f^{\varepsilon})^{\frac{1}{k}}}{\varepsilon^2+|z|^2}.$$

Consider the function

$$\hat{w} := -\left(\frac{r_0^2+|z|^2}{1+\varepsilon^2}\right)^{1-\frac{n}{k}}.$$

By the concavity of $S_k^{\frac{1}{k}}$, we have

$$F^{i\bar{j}}\left(\hat{w}_{i\bar{j}} - u_{i\bar{j}}\right) \geq F[\hat{w}] - F[u] = \left(C_n^k\left(\frac{n}{k}-1\right)^k(1+\varepsilon^2)^{n-k}\right)^{\frac{1}{k}}\left((\mu^2(|z|^2+\mu^2)^{-k-n})^{\frac{1}{k}} - (\varepsilon^2(|z|^2+\varepsilon^2)^{-k-n})^{\frac{1}{k}}\right).$$

If ε and b are sufficiently small, we have

$$F^{i\bar{j}}(\hat{w} - u + bu_{\xi\xi})_{i\bar{j}} \geq 0, \quad \text{in } \Sigma_R.$$

Maximum principle leads that

$$\max_{\Sigma_R}(\hat{w} - u + bu_{\xi\xi}) \leq \max_{\partial\Sigma_R}(\hat{w} - u + bu_{\xi\xi}).$$

□

4.4.2 Second-order estimate on the boundary $\partial\Sigma_R$

Step 1: Tangential derivative estimates

Consider a point $p \in \partial\Omega$. Without loss of generality, let p be the origin. Choose the coordinate $z_1, \dots, z_n$ such that the x_n axis is the inner normal direction to $\partial\Omega$ at 0. Suppose

$$t_1 = y_1, \quad t_2 = y_2, \cdots, \quad t_n = y_n, \quad t_{n+1} = x_1, \quad t_{n+2} = x_2, \cdots, \quad t_{2n} = x_n.$$

Denote by $t' = (t_1, \dots, t_{2n-1})$. Then, around the origin, $\partial\Omega$ can be represented as a graph

$$t_{2n} = x_n = \varphi(t') = B_{\alpha\beta}t_\alpha t_\beta + O(|t'|^3).$$

Since

$$u(t', \varphi(t')) = 0 \quad \text{on } \partial\Omega,$$

we have

$$u_{t_\alpha t_\beta}(0) = -u_{t_{2n}}(0)B_{\alpha\beta}, \quad \alpha, \beta = 1, \dots, 2n-1.$$

It follows that for any $\alpha, \beta = 1, \dots, 2n-1$, we ahve

$$|u_{t_\alpha t_\beta}(0)| \leq C, \quad \text{on } \partial\Omega. \tag{4.30}$$

Note that $u \geq \underline{u}^{\varepsilon}$ near $\partial\Omega$, $u = \underline{u}^{\varepsilon}$, and $0 < \underline{u}_\nu^{\varepsilon} \leq u_\nu$ on $\partial\Omega$, there exists a smooth function g such that $u = g\underline{u}$ near $\partial\Omega$, and $g \geq 1$ outside of Ω nearby $\partial\Omega$. So $\forall\ 1 \leq i, j \leq n-1$,

$$u_{i\bar{j}}(0) = g_{i\bar{j}}(0)\underline{u}^{\varepsilon}(0) + g_i(0)\underline{u}_{\bar{j}}^{\varepsilon}(0) + g_{\bar{j}}(0)\underline{u}_i^{\varepsilon}(0) + g(0)\underline{u}_{i\bar{j}}^{\varepsilon}(0).$$

Note that $\underline{u}^{\varepsilon} = c_0\rho_1$ near $\partial\Omega$, where ρ_1 is a given strictly plurisubharmonic function in a neighborhood Ω, $c_0 = \left(1 - \left(1 + \frac{s^2}{16+s^2}\right)^{1-\frac{n}{k}}\right)\tau$, τ is a constant independent of ε and R as taken in Lemma 2.12. We also have

$$S_{k-1}\left(\left\{u_{i\bar{j}}(0)\right\}_{1\leq i,j\leq n-1}\right) = c_0^{k-1}g^{k-1}(0)S_{k-1}\left(\left\{\rho_{1,i\bar{j}}(0)\right\}_{1\leq i,j\leq n-1}\right) \geq c_0^{k-1}g_0^{k-1}C_n^{k-1}(C_n^k)^{\frac{1-k}{k}}\min_{\partial\Omega}S_k^{\frac{k-1}{k}}(\partial\bar{\partial}\rho_1) > 0. \tag{4.31}$$

Set $R \geq R_2 \geq R_1$, R_2 is to be determined later. Consider a harmonic function h_3, which is a solution to

$$\begin{cases} \Delta h_3 = 0 & \text{in } B_R \backslash \overline{B_2}, \\ h_3 = -(1+\varepsilon^2)^{\frac{n}{k}-1}(R^2+\varepsilon^2)^{1-\frac{n}{k}} & \text{on } \partial B_R, \\ h_3 = -t^{\frac{2n}{k}-2}|z|^{2-\frac{2n}{k}} & \text{on } \partial B_2. \end{cases} \tag{4.32}$$

Set

$$\bar{h}(z) := \tilde{h}_3(z) = R^{\frac{2n}{k}-2}h_3(Rz).$$

By maximum principle, we know,

$$\underline{\tilde{u}}^{\varepsilon} \leq \tilde{u} \leq \bar{h},$$

where $\tilde{u}(z) = R^{\frac{2n}{k}-2}u(Rz)$. Note that

$$\bar{h} = -\left(\frac{1+\frac{\varepsilon^2}{R^2}}{1+\varepsilon^2}\right)^{1-\frac{n}{k}} \quad \text{on } \partial B_1, \quad \text{and} \quad \bar{h} = -\left(\frac{2}{Rt}\right)^{2-\frac{2n}{k}} \quad \text{on } \partial B_{\frac{2}{R}}.$$

If we choose $R^2 \geq (R_2)^2 := \max\{(R_1)^2, 4t^{-2}4(1+\varepsilon_0^2), 16\}$, then

$$\bar{h}\,|_{\partial B_1} \geq \bar{h}\,|_{\partial B_{\frac{2}{R}}}.$$

Similarly, as in gradient estimates, there is a positive constant C, independent of ε and R, such that

$$\bar{h}_\nu \leq \tilde{u}_\nu \leq \underline{\tilde{u}}^{\varepsilon}_\nu \leq C \quad \text{on } \partial B_2.$$

In fact, we can prove that

$$\bar{h}_\nu > c_0 > 0,$$

where c_0 is also independent of ε and R. In fact, we can solve (4.32) as follows:

$$\bar{h} = -\frac{-\left(\frac{1+\frac{\varepsilon^2}{R^2}}{1+\varepsilon^2}\right)^{1-\frac{n}{k}} + \left(\frac{2}{Rt}\right)^{2-\frac{n}{k}}}{\left(\frac{2}{R}\right)^{2-N} - 1}|z|^{2-N} - \left(\frac{2}{Rt}\right)^{2-\frac{n}{k}} + \left(-\left(\frac{1+\frac{\varepsilon^2}{R^2}}{1+\varepsilon^2}\right)^{1-\frac{n}{k}} + \left(\frac{2}{Rt}\right)^{2-\frac{n}{k}}\right)\frac{\left(\frac{2}{R}\right)^{2-N}}{\left(\frac{2}{R}\right)^{2-N} - 1}.$$

Then,

$$\bar{h}_\nu = \left(-\left(\frac{1+\frac{\varepsilon^2}{R^2}}{1+\varepsilon^2}\right)^{1-\frac{n}{k}} + \left(\frac{2}{Rt}\right)^{2-\frac{n}{k}}\right)\left(\left(\frac{2}{R}\right)^{2-N} - 1\right)^{-1}\left(\frac{N}{2}-1\right) \geq \left(\frac{N}{2}-1\right)\left(2^{\frac{n}{k}-1}-1\right)\left(\frac{1}{1+\varepsilon_0^2}\right)^{1-\frac{n}{k}}(2^{N-2}-1)^{-1} > 0.$$

It follows that there exists a (ε, R)-independent constant C, such that

$$C^{-1}R^{1-\frac{2n}{k}} \leq u_\nu \leq CR^{1-\frac{2n}{k}} \quad \text{on } \partial B_2,$$

where ν is the unit outer normal to ∂B_2.

For any $p \in \partial B_R$, we choose the coordinate such that $p = (0, \dots, -R)$. Then, near p, ∂B_R is locally represented by $t_{2n} = x_n = \varphi(t') = -\sqrt{R^2 - \sum_{i=1}^{2n-1} t_i^2}$. Since

$$u(t', \varphi(t')) = -\left(\frac{R^2+\varepsilon^2}{1+\varepsilon^2}\right)^{1-\frac{n}{k}} \quad \text{on } \partial B_R,$$

we have

$$u_{t_\alpha t_\beta}(p) = -u_{t_{2n}}(p)\frac{\partial^2 t_{2n}}{\partial t_\alpha \partial t_\beta} = -R^{-1}u_{t_{2n}}(p)\delta_{\alpha\beta} = R^{-1}u_\nu(p)\delta_{\alpha\beta}.$$

Hence,

$$|u_{t_\alpha t_\beta}| \le CR^{-\frac{2n}{k}}, \quad \alpha, \beta = 1, \dots, 2n-1, \tag{4.33}$$

$$u_{i\bar{j}} = \frac{1}{4}\Big(u_{t_{n+i}t_{n+j}} + u_{t_it_j} - \sqrt{-1}u_{t_it_{n+j}} + \sqrt{-1}u_{t_{n+i}t_j}\Big) \ge CR^{-\frac{2n}{k}}\delta_{ij}, \quad i, j = 1, \dots, n. \tag{4.34}$$

Step 2: Tangential-normal derivative estimates $\partial\Sigma_R$

Follow the approach by Guan in [16], we estimate the tangential-normal derivatives on boundary. We first prove the tangential-normal derivatives estimate on $\partial\Omega$. Suppose $0 \in \partial\Omega$, to estimate $u_{t_\alpha t_n}(0)$ for $\alpha = 1, \dots, 2n-1$, we consider the auxiliary function

$$v = u - \underline{u} + td - \frac{N}{2}d^2$$

on $\Omega_\delta = \Omega \cap B_\delta(0)$ with constant N, t, δ to be determined later. Define a linear operator

$$Lv = F^{i\bar{j}}v_{i\bar{j}},$$

where $F^{i\bar{j}} = \frac{\partial}{\partial u_{i\bar{j}}}S_k^{\frac{1}{k}}(\partial\bar{\partial}u)$. Then,

$$\mathcal{F} = \sum_{i=1}^{n} F^{i\bar{i}} = S_k^{\frac{1}{k}-1}S_{k-1}(\lambda|i) = (n-k+1)S_k^{\frac{1}{k}-1}S_{k-1} \ge C_{n,k} > 0.$$

By Lemma 3.5, for N sufficiently large and t and δ sufficiently small, there holds

$$\begin{cases} Lv \le -\frac{\varepsilon}{4}(1+\mathcal{F}) & \text{in } \Omega_\delta, \\ v \ge 0 & \text{on } \partial\Omega, \end{cases}$$

where $\varepsilon > 0$ is a uniform constant depending only on subsolution $\underline{u}$ restricted in a small neighborhood of $\partial\Omega$.

In our setting, ε can be taken independent of ε and R, since $\underline{u}^\varepsilon = c_0\rho_1$ near $\partial\Omega$, where ρ_1 is a given strictly plurisubharmonic function in a neighborhood Ω, $c_0 = \left(1 - \left(1 + \frac{s^2}{16+s^2}\right)^{1-\frac{n}{k}}\right)\tau$, τ is a constant independent of ε and R as taken in Lemma 2.12.

We use a similar notation as in subsection 3.2. Let

$$\Psi = A_1 v + A_2|z|^2 - A_3\left(\left(u_{y_n} - \underline{u}_{y_n}\right)^2 + \sum_{l=1}^{n-1}|u_l - \underline{u}_l|^2\right).$$

After a similar computation to the boundary tangential-normal derivatives estimate on the pseudoconvex boundary in 3.2, we see that

$$L(\Psi \pm T_\alpha(u - \underline{u})) \le 0 \quad \text{in } \Omega_\delta$$

and

$$\Psi \pm T_\alpha(u - \underline{u}) \ge 0 \quad \text{on } \partial\Omega_\delta,$$

when $A_1 \gg A_2 \gg A_3 \gg 1$. Therefore,

$$|u_{t_\alpha x_n}| \le C \quad \text{on } \partial\Omega. \tag{4.35}$$

Next, we prove the tangential-normal derivatives estimate on ∂B_R. Let

$$\tilde{u}(z) = R^{\frac{2n}{k}-2}u(Rz) \quad \text{and} \quad \tilde{\underline{u}}^\varepsilon(z) = R^{\frac{2n}{k}-2}\underline{u}^\varepsilon(Rz).$$

Consider the boundary tangential-normal derivatives estimate on ∂B_1. Let $p = (0, \dots, -1) \in \partial B_1$. Write a defining function ϱ of B_1 near p by

$$\varrho(z) = -x_n - \left(R^2 - \sum_{i=1}^{n-1}|z_i|^2 - y_n^2\right)^{\frac{1}{2}}.$$

Then,

$$|T_\alpha(\tilde{u} - \underline{\tilde{u}}^\varepsilon)| \le C. \quad \text{in } B_1(0) \cap B_{\frac{1}{2}}(p).$$

Let $w = |z|^2 - 1$, then

$$L(-w) = -\sum_{i=1}^{n} F^{i\bar{i}} \le -C_{n,k}(1 + \mathcal{F}).$$

Let

$$\Phi = -B_1 w + B_2|z - p|^2 - B_3\left(\sum_{l=1}^{n-1}|\tilde{u}_l - \underline{\tilde{u}}_l^\varepsilon|^2 + \left(\tilde{u}_{y_n} - \underline{\tilde{u}}_{y_n}^\varepsilon\right)^2\right).$$

Similarly, we obtain

$$L(\Phi \pm T_\alpha(\tilde{u} - \underline{\tilde{u}}^\varepsilon)) \le 0 \quad \text{in } \Omega_\delta$$

and

$$\Phi \pm T_\alpha(\tilde{u} - \underline{\tilde{u}}^\varepsilon) \ge 0 \quad \text{on } \partial\Omega_\delta,$$

when $B_1 \gg B_2 \gg B_3 \gg 1$. So, we have

$$|\tilde{u}_{t_\alpha x_n}| \le C \quad \text{on } \partial B_1.$$

Therefore,

$$|u_{t_\alpha x_n}| \le CR^{-\frac{2n}{k}} \quad \text{on } \partial B_R. \tag{4.36}$$

Step 3: Double normal derivative estimates $\partial\Sigma_R$

By pure tangential derivative estimates (4.30) and (4.33), we have

$$|u_{y_n y_n}| \le C \quad \text{on } \partial\Omega \quad \text{and} \quad |u_{y_n y_n}| \le CR^{-\frac{2n}{k}} \quad \text{on } \partial B_R.$$

To estimate the double normal derivative $u_{x_n x_n}$, it suffices to estimate $u_{n\bar{n}}$. By rotation of $(z_1, \ldots, z_{n-1})$, we may assume that $\{u_{i\bar{j}}\}_{1\le i,j\le n-1}$ is diagonal. Then,

$$f^\varepsilon = S_k(\partial\bar{\partial}u) = u_{n\bar{n}}S_{k-1}\left(\left\{u_{i\bar{j}}\right\}_{1\le i,j\le n-1}\right) + S_k\left(\left\{u_{i\bar{j}}\right\}_{1\le i,j\le n-1}\right) - \sum_{\beta=1}^{n-1}|u_{\beta n}|^2 S_{k-2}\left(\left\{u_{i\bar{j}}\right\}_{1\le i,j\le n-1}\right).$$

It suffices to give a uniform lower positive bound for $S_{k-1}(\{u_{i\bar{j}}\}_{1\le i,j\le n-1})$.

By (4.30), (4.31), and (4.35), we obtain

$$u_{n\bar{n}}(0) \le C \quad \text{on } \partial\Omega.$$

On the other hand,

$$u_{n\bar{n}}(0) \ge -\sum_{i=1}^{n-1} u_{i\bar{i}} \ge -C.$$

By (4.33), (4.34), and (4.36), we obtain

$$\begin{aligned}
Cu_{n\bar{n}}R^{-\frac{2n(k-1)}{k}} &\le u_{n\bar{n}}(0)S_{k-1}\left(\left\{u_{i\bar{j}}\right\}_{1\le i,j\le n-1}\right) \\
&= S_k(\partial\bar{\partial}u) - S_k\left(\left\{u_{i\bar{j}}(0)\right\}_{1\le i,j\le n-1}\right) + \sum_{\beta=1}^{n-1}|u_{\beta n}(0)|^2 S_{k-2}\left(\left\{u_{i\bar{j}}(0)\right\}_{1\le i,j\le n-1}\right) \\
&\le CR^{-2n}.
\end{aligned}$$

Therefore,

$$|u_{n\bar{n}}(0)| \leq CR^{-\frac{2n}{k}} \quad \text{on } \partial B_R.$$

Step 4: Second-order derivative estimates in Σ_R

As in Theorem 4.4, let $H = Q(M - P)^\sigma$, $Q = u_{i\bar{i}}(-u)^{-a+1}$, and $P = |Du|^2(-u)^{-a}$. Suppose the maximum of H is to obtain at a boundary point $z_0 \in \partial\Sigma_R$. Then,

$$Q = (M-P)^\sigma H \leq M^\sigma H(z_0) \leq M^\sigma Q(z_0)\left(M - \max_{\Sigma_R} P\right)^{-\sigma} \leq M^\sigma \max_{\partial\Sigma_R} Q\left(M - \max_{\Sigma_R} P\right)^{-\sigma}. \tag{4.37}$$

Note that P is bounded (uniformly in ε and R). By (4.8) and (4.9),

$$(-u)^{2-a}|D\log f^\varepsilon|^2 \leq C(n,k,t) \quad \text{and} \quad (-u)^{2-a}|D^2\log f^\varepsilon|^2 \leq C(n,k,t).$$

By Theorem 4.4, if the maximum of H is obtained at a interior point, there is a positive constant C independent of ε and R such that $Q \leq C$. Combined with (4.37), there is a positive constant C independent of ε and R such that

$$Q \leq C \quad \text{in } \Sigma_R.$$

Then, we obtain

$$\Delta u \leq C(-u)^{a-1} \leq C|z|^{-\frac{2n}{k}} \quad \text{in } \Sigma_R. \tag{4.38}$$

By boundary second-order derivative estimates and C^0 estimate, we obtain that for any unit vector $\xi \in \mathbb{R}^{2n}$,

$$\max_{\partial\Sigma_R}(\hat{w} - u + bu_{\xi\xi}) \leq C.$$

Hence,

$$u_{\xi\xi} \leq C, \quad \text{in } \overline{\Sigma}_R.$$

u is subharmonic since u is k-admissible, then

$$-C \leq u_{\xi\xi} \leq C \quad \text{in } \Sigma_R.$$

In conclusion, we obtain

$$|D^2u| \leq C, \quad \text{in } \Sigma_R. \tag{4.39}$$

5 Proof of Theorem 1.1

5.1 Uniqueness

The uniqueness follows from the comparison principle for k-subharmonic solutions of the complex k-Hessian equation in bounded domains in Lemma 2.7 by Blocki [5].

Suppose u and v are two solutions to (1.3). For any $z_0 \in \mathbb{C}^n\backslash\Omega$, there exists R_0 such that $z_0 \in B_{R_0}(0)\backslash\Omega$. Since $u(z) \to 0$, $v(z) \to 0$ as $|z| \to \infty$, $\forall\, \varepsilon > 0$, there exists $R \gg R_0$ such that

$$v - \varepsilon \leq u \leq v + \varepsilon \quad \text{in } \mathbb{C}^n\backslash B_R.$$

By the comparison principle Lemma 2.7,

$$v - \varepsilon \leq u \leq v + \varepsilon \quad \text{in } B_R\backslash\Omega.$$

Note that $z_0 \in B_{R_0}(0)\backslash\Omega \subset B_R(0)\backslash\Omega$, we have

$$v(z_0) - \varepsilon \le u(z_0) \le v(z_0) + \varepsilon.$$

Let $\varepsilon \to 0$, we obtain that $u(z_0) = v(z_0)$. Since z_0 is arbitrary, $u = v$ in $\mathbb{C}^n \backslash \Omega$.

5.2 The existence and $C^{1,1}$-estimates

The existence follows from the uniform C^2-estimates for $u^{\varepsilon,R}$. The proof is similar to that in [15] by Guan.

For any fixed $M_0 > R_2$, for the solution to (4.3), by the C^2 estimates, we have

$$\|u^{\varepsilon,R}\|_{C^2(\bar{\Sigma}_{M_0})} \le C_1 \quad \text{independent of } \varepsilon, R, \text{ and } M_0$$

for all $R \ge M_0$. By the Evans-Krylov theory, we obtain, for $0 < \alpha < 1$,

$$\|u^{\varepsilon,R}\|_{C^{2,\alpha}(\bar{\Sigma}_{M_0})} \le C_2(\varepsilon, M_0) \quad \text{independent of } R.$$

By compactness, we can find a sequence $R_j \to \infty$ such that

$$u^{\varepsilon,R_j} \to u^\varepsilon \quad \text{in } C^2\big(\bar{\Sigma}_{M_0}\big),$$

where u^ε satisfies

$$\begin{cases} H_k(u^\varepsilon) = f^\varepsilon & \text{in } \Sigma_{M_0}, \\ u = -1 & \text{on } \partial\Omega \end{cases}$$

and

$$C^{-1}|z|^{2-\frac{2n}{k}} \le -u^\varepsilon(z) \le C|z|^{2-\frac{2n}{k}}, \quad |Du^\varepsilon(z)| \le C|z|^{1-\frac{2n}{k}}, \quad |\partial\bar{\partial}u^\varepsilon(z)| \le C|z|^{-\frac{2n}{k}}, \quad |D^2u^\varepsilon(z)| \le C.$$

Moreover,

$$\|u^\varepsilon\|_{C^{2,\alpha}(\bar{\Sigma}_{M_0})} \le C_2(\varepsilon, M_0) \quad \text{for any } M_0 > R_2.$$

By the classical Schauder theory, u^ε is smooth.

By the above decay estimates for u^ε, for any sequence $\varepsilon_j \to 0$, there is a subsequence of $\{u^{\varepsilon_j}\}$ converging to a function u in $C^{1,\alpha}$ norm on any compact subset of $\mathbb{C}^n \backslash \Omega$ (for any $0 < \alpha < 1$). Thus, $u \in C^{1,1}(\mathbb{C}^n \backslash \Omega)$ and satisfies the desired estimates (1.4). By the convergence theorem of the complex k-Hessian operator proved by Trudinger and Zhang in [35] (see also Lu [32]), u is a solution to (1.3).

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