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GLOBAL C^2-ESTIMATES FOR SMOOTH SOLUTIONS TO UNIFORMLY PARABOLIC EQUATIONS WITH NEUMANN BOUNDARY CONDITION

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Abstract. In this paper, we establish global C^2 a priori estimates for solutions to the uniformly parabolic equations with Neumann boundary condition on the smooth bounded domain in $\mathbb{R}^n$ by a blow-up argument. As a corollary, we obtain that the solutions converge to ones which move by translation. This generalizes the viscosity results derived before by Da Lio.

1. **Introduction.** In this paper, we consider the large time behavior of smooth solutions to the following uniformly parabolic equations with linear Neumann boundary value condition,

$$\begin{cases} u_t - F(\nabla^2 u) = 0 & \text{in } \Omega \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{on } \overline{\Omega} \times \{0\}, \\ G(x, \nabla u) = 0 & \text{on } \partial\Omega \times (0, \infty), \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, F is a smooth real function defined on $\mathcal{S}^n$, $\mathcal{S}^n$ denotes the space of $n \times n$ real symmetric matrices, $G(x, \nabla u) = u_\nu - \varphi(x)$, ν is the inner unit normal vector of $\partial\Omega$ and $u_0, \varphi \in C^\infty(\overline{\Omega})$ such that $G(x, \nabla u_0) = 0$.

Suppose F satisfies the following structure conditions:

(F1) $\lambda I \le F_r(r), \quad |F(r)| \le \mu_0 |r|$;
(F2) $|F_X(r)| \le \mu_1 |X|$;
(F3) $F_{XX}(r) \le 0$,

for all $x \in \Omega, r \in \mathcal{S}^n, X \in \mathcal{S}^n$, where λ, μ_0, μ_1 are positive constants. In addition, we assume

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(F4) There is a smooth function F_∞, such that

$$s^{-1}F(sr) \to F_\infty(r) \quad \text{locally uniformly in } C^1(\mathcal{S}^n), \quad \text{as } s \to +\infty.$$

Our main result is the following global C^2-estimates.

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Assume F satisfies* (F1)-(F4)*, $\varphi \in C^\infty(\overline{\Omega})$, then we have the uniform (in t) estimate for the solution to* (1)*,*

$$\|u_t(\cdot,t)\|_{C(\overline{\Omega})} + \|\nabla u(\cdot,t)\|_{C(\overline{\Omega})} + \|\nabla^2 u(\cdot,t)\|_{C(\overline{\Omega})} \le C, \tag{2}$$

where C is independent of t.

In [6], Huang and Ye exhibited a convergence result under assumptions on a-priori estimates,

Theorem 1.2 ([6])**.** *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Assume F satisfies* (F1) *and* (F3)*, $\varphi \in C^\infty(\overline{\Omega})$. For any $T > 0$, assume that $u \in C^{4+\alpha,\frac{4+\alpha}{2}}(\overline{\Omega \times (0,T)})$ is a unique solution to the nonlinear parabolic equation* (1) *which satisfies*

$$\|u_t(\cdot,t)\|_{C(\overline{\Omega})} + \|\nabla u(\cdot,t)\|_{C(\overline{\Omega})} + \|\nabla^2 u(\cdot,t)\|_{C(\overline{\Omega})} \le C_1,$$

and

$$\sum_{k=1}^n G_{p_k}(x,\nabla u)\nu_k \ge \frac{1}{C_2}, \tag{3}$$

where C_1 and C_2 are positive constants independent of $t > 1$. Then $u(\cdot,t)$ converges to a function $U + \tau t$ in $C^{1+\zeta}(\overline{\Omega}) \cap C^{4+\alpha'}(\overline{D})$ as $t \to \infty$ for any $D \subset\subset \Omega$, $\zeta < 1$ and $\alpha' < \alpha$, that is

$$\lim_{t\to+\infty} \|u(\cdot,t)-(U(\cdot)+\tau t)\|_{C^{1+\zeta}(\overline{\Omega})} = 0,\ \lim_{t\to+\infty} \|u(\cdot,t)-(U(\cdot)+\tau t)\|_{C^{4+\alpha}(\overline{D})} = 0. \tag{4}$$

Joint with Theorem 1.2, we have the following convergence result.

Corollary 1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Assume F satisfies* (F1)-(F4)*, $u_0, \varphi \in C^\infty(\overline{\Omega})$, such that $G(x,\nabla u_0) = 0$, then the unique smooth solution $u(x,t)$ to equation* (1) *converges to $\tau t + U$ in the sense of* (4)*, where (U,τ) is a suitable solution to*

$$\begin{cases} F(\nabla^2 U) = \tau & \text{in } \Omega, \\ G(x,\nabla U) = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

The constant τ depends only on Ω, φ and F. The solution to (5) *is unique up to a constant.*

In the note, we deduce the estimate (2) for the problem (1).

Our work is motivated firstly by [3], where Da Lio studied the large time behavior as $t \to +\infty$ of the viscosity solution χ to the Neumann boundary value problem

$$\begin{cases} \chi_t + F(x,\nabla\chi,\nabla^2\chi) = \lambda & \text{in } \Omega \times (0,\infty), \\ \chi(x,0) = \chi_0(x) & \text{on } \overline{\Omega} \times \{0\}, \\ L(x,\nabla\chi) = \mu & \text{on } \partial\Omega \times (0,\infty), \end{cases}$$

where F and L are at least continuous functions defined respectively on $\overline{\Omega}\times\mathbb{R}^n\times\mathcal{S}^n$ and $\overline{\Omega}\times\mathbb{R}^n$. The author got a convergence result $\chi\to u_\infty$, as $t\to+\infty$ uniformly in $\overline{\Omega}$, where u_∞ is a solution to

$$\begin{cases} F(x,\nabla u,\nabla^2 u)=\lambda & \text{in } \Omega\times(0,\infty),\\ L(x,\nabla u)=\mu & \text{on } \partial\Omega\times(0,\infty),\end{cases}$$

under the assumption that F satisfies

(*F1) The function F is locally Lipschitz continuous on $\overline{\Omega}\times\mathbb{R}^n\times\mathcal{S}^n$ and there exists a constant $K>0$ such that, for any $x,y\in\overline{\Omega}$, $p,q\in\mathbb{R}^n$, $M,N\in\mathcal{S}^n$,

$$|F(x,p,M)-F(y,q,N)|\le K\{|x-y|(1+|p|+|q|+|M|+|N|)+|p-q|+|M-N|\}.$$

(*F2) There exists $\varepsilon>0$ such that, for any $x\in\overline{\Omega}$, $p\in\mathbb{R}^n$, $M,N\in\mathcal{S}^n$ with $N\ge 0$,

$$F(x,p,M+N)-F(x,p,M)\le-\varepsilon\mathrm{Tr}(N).$$

(*F3) There exists a continuous function F_∞ such that

$$t^{-1}F(x,tp,tM)\to F_\infty(x,p,M)\quad\text{locally uniformly, as } t\to+\infty,$$

and L satisfies

(*L1) There exists $\delta>0$ such that, for every $(x,p)\in\partial\Omega\times\mathbb{R}^n$, and $s>0$, such that

$$L(x,p+sn(x))-L(x,p)\ge\delta s,$$

where $n(x)$ denotes the unit outward normal vector to $\partial\Omega$ at x.

(*L2) There is a constant $K>0$ such that, for all $x,y\in\partial\Omega$, $p,q\in\mathbb{R}^n$, such that

$$|L(x,p)-L(y,q)|\le K[(1+|p|+|q|)|x-y|+|p-q|].$$

(*3) There exists a continuous function L_∞ such that

$$t^{-1}L(x,tp)\to L_\infty\quad\text{locally uniformly, as } t\to+\infty.$$

More relative work can be found in [2].

Another motivation of the paper comes from [7] in which Kahane considered the heat equation subject to a homogeneous Neumann boundary condition on the smooth and convex domain as follows

$$\begin{cases} u_t-\Delta u=0 & \text{in } \Omega\times(0,+\infty),\\ u(x,0)=u_0(x) & \text{on } \Omega\times\{0\},\\ u_\nu=0 & \text{on } \partial\Omega\times(0,+\infty).\end{cases}$$

He got an estimate for the spatial gradient of solutions in terms of the gradient of the initial data. However, the zero Neumann boundary condition and the convexity of the domain play important roles in the proof of the estimate of the gradient. We will in this paper get rid of these two restrictions to obtain the gradient estimate to the solutions of the diffusion equations and generalize the result of [7] to uniformly parabolic differential equations.

For the smooth solutions case, Schnürer [12] studied a class of curvature flow in $\mathbb{R}^{n+1}$ with second boundary condition

$$\begin{cases} \dot{X}=-(\ln F-\ln f)\nu,\\ \nu(M)=\nu(M_0),\\ M\big|_{t=0}=M_0,\end{cases}\tag{6}$$

where X is the embedding vector of a smooth strictly convex hypersurface with boundary, $M=\mathrm{graph}(-u)\mid_\Omega$, $u:\overline{\Omega}\to\mathbb{R}$, ν is the upwards pointing unit normal vector, $\dot{X}$ is its total time derivative and $\Omega \subset \mathbb{R}^n$, $n \geq 2$ is a smooth strictly convex domain, $f : \overline{\Omega} \to \mathbb{R}$ is a given smooth function, and F is a curvature function. Schnürer transformed the curvature flow into some sort of (1) and then he showed the boundary condition is strictly oblique and derived the C^2-a priori estimates. Finally he proved that there is a smooth solution $M(t) = \text{graph}(-u(\cdot,t)) \mid_\Omega$ to (6) for all times $t \geq 0$ which exists for all positive time and converges smoothly to a translating solution $M^\infty = \text{graph}(u^\infty) \mid_\Omega$ of the flow (6). Huang-Ye [6] established a generalization of Schnürer convergence result as we state in Theorem 1.2.

A similar convergence result was obtained by Ma-Wang-Wei [11] for graphic mean curvature flow with Neumann boundary condition.

$$\begin{cases} u_t = \sum\limits_{i,j=1}^{n} (\delta_{ij} - \frac{u_i u_j}{1+|\nabla u|^2}) u_{ij} & \text{in } \Omega \times (0,\infty), \\ u_\nu = \varphi(x) & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{on } \overline{\Omega}, \end{cases}$$

where Ω is a strictly convex bounded domain in $\mathbb{R}^n$ with smooth boundary for $n \geq 2$, u_0 and φ are smooth functions satisfying $u_{0,\nu} = \varphi$ on $\partial\Omega$. They proved that up to a constant the solutions converge to a translating solution $\lambda t + w$. In other words, (w, λ) is a solution to

$$\begin{cases} \sum\limits_{i,j=1}^{n} (\delta_{ij} - \frac{u_i u_j}{1+|\nabla u|^2}) u_{ij} = \lambda & \text{in } \Omega, \\ u_\nu = \varphi(x) & \text{on } \partial\Omega. \end{cases}$$

In fact, [11] provides a good approach to convergence result under assumption of uniform (in t) $\|u_t(\cdot,t)\|_{C(\overline{\Omega})}$, $\|\nabla u(\cdot,t)\|_{C(\overline{\Omega})}$ estimate to quasi-linear equation which is inspired by [1]. However, the strict convexity of the domain plays an essential role in the proof of the result. After we establish the estimate for $\|u_t(\cdot,t)\|_{C(\overline{\Omega})}$, $\|\nabla u(\cdot,t)\|_{C(\overline{\Omega})}$, $\|\nabla^2 u(\cdot,t)\|_{C(\overline{\Omega})}$, we could apply the proceeding in [6] or [11] to obtain the convergence result. Specifically in this paper, we will apply Theorem 1.2 to obtain the Corollary 1 after we obtain the estimate (2).

In the rest of this paper, we use a blow-up technique to bound the oscillation of $u(\cdot,t)$ and then obtain the estimate of $\|\nabla u(\cdot,t)\|_{C(\overline{\Omega})}$ and $\|\nabla^2 u(\cdot,t)\|_{C(\overline{\Omega})}$. All a priori estimates are based on the computation of equations satisfied by some function for the solution and the distance, this kind of auxiliary function method is usually adopted to get the a priori estimate to the solutions, on can refer to [10], [11] etc. In Section 2, we deal with the special case with $F(\nabla^2 u) = \Delta u$. In Section 3, we deal with general F satisfying (F1) – (F4).

2. **Asymptotic behavior for the diffusion equations.**

In this section, we study the asymptotic behavior of the following diffusion equation with Neumann boundary value. As we all know, the diffusion equation is the most typical one among the uniformly parabolic equations and we treat it firstly.

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega \times [0,T), \\ u(x,0) = u_0(x) & \text{on } \overline{\Omega} \times \{0\}, \\ u_\nu = \varphi & \text{on } \partial\Omega \times [0,T), \end{cases} \tag{7}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$, $\varphi, u_0 \in C^\infty(\overline{\Omega})$ and $u_{0,\nu} = \varphi$ on $\partial\Omega$.

Lemma 2.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary, $n \geq 2$. If $u(x,t)$ is a smooth solution to (7), then $\sup_{\Omega\times[0,T)} |u_t|^2 = \sup_{\Omega} |u_t(x,\,0)|^2$, so there is a constant C depending only on $\|\nabla^2 u_0\|_{C^0(\Omega)}$ such that*

$$\|u_t\|_{L^\infty(\Omega\times[0,T))} \leq C.$$

Proof. A direct computation gives

$$(\frac{\partial}{\partial t} - \Delta)(u_t^2) = 2u_t u_{tt} - 2u_t\Delta u_t - 2|\nabla u_t|^2 = -2|\nabla u_t|^2 \leq 0.$$

We apply the weak maximum principle to get

$$\sup_{\Omega\times(0,T)} |u_t|^2 = \sup_{(\Omega\times\{0\})\cup(\partial\Omega\times(0,T))} |u_t|^2.$$

On the other hand, $(u_t^2)_\nu = 2u_t u_{t\nu} = 0$, Hopf Lemma leads that the maximum cannot occur on $\partial\Omega\times(0,T)$, then

$$\sup_{\Omega\times(0,T)} |u_t|^2 = \sup_{\Omega\times\{0\}} |u_t|^2 = \sup_{\Omega} |\Delta u_0|^2.$$

□

Let $v(x,t) = u(x,t) - u(x_0,t)$, $x_0 \in \Omega$ be a fixed point. In the following of this section, we firstly give a time-independent bound for $|v|$ via a blow-up technique. With the help of C^0 estimate of v, we then get the bound for $\|v\|_{C^2}$. Estimates for $|\nabla u|$ and $|\nabla^2 u|$ then follow naturally. Finally we apply the Schnürer convergence result([12], see also[6]) to obtain that the smooth solution converges to a translating solution.

Lemma 2.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary, $n \geq 2$. If $u(x,t)$ is a smooth solution to (7), $v(x,t)$ is defined as above, then there is a constant $A_0 > 0$, independent of T, such that*

$$\|v\|_{C^0(\Omega\times[0,T))} \leq A_0.$$

Proof. Let $A = \|v\|_{C^0(\Omega\times[0,T))}$. Without loss of generality, we suppose $A \geq \delta := \operatorname{osc}(u_0) > 0$. (Otherwise we get a constant solution to (7).) Suppose A is unbounded, that is to say $A \to \infty$ as $T \to \infty$. Let

$$w(x,t) = \frac{v(x,t)}{A}.$$

It is easy to see

$$w(x_0,\,t) = 0 \text{ for } \ t \in [0,\,T) \quad \text{ and } \sup_{\Omega\times[0,T)} |w| = 1.$$

And $w(x,t)$ satisfies

$$\begin{cases} w_t - \Delta w = -\frac{u_t(x_0,t)}{A} & \text{in } \Omega\times[0,T), \\ w(x,0) = \frac{1}{A}(u_0(x) - u_0(x_0)) & \text{on } \overline{\Omega}\times\{0\}, \\ w_\nu(x,t) = \frac{1}{A}\varphi(x) & \text{on } \partial\Omega\times[0,T). \end{cases} \tag{8}$$

To complete the proof, we need the following propositions.

Proposition 1. *Suppose $w \in C^{3,2}(\Omega\times[0,T))$ and satisfies*

$$w_t - \Delta w = f(t) \quad \textit{in } \Omega\times[0,T),$$

for $f \in C([0,T))$, *Then* $\forall\ \Omega' \subset\subset \Omega$,

$$\sup_{\Omega' \times [0,T)} |\nabla w| \le C,$$

where C *is a positive constant depending only on* $\operatorname{dist}(\Omega', \partial\Omega)$, $\|w\|_{L^\infty(\Omega\times[0,T))}$, $\|f\|_{L^\infty([0,T))}$. □

Proof. We will prove that for $0 < T' < T$, we can bound $|\nabla w|$ in $\overline{\Omega'} \times [0, T']$ independent of T' and then take a limit argument. Denoting by $M = \|w\|_{L^\infty(\Omega\times[0,T))}$, $N = \|f\|_{L^\infty([0,T))}$ and $w_0 = \frac{1}{A}(u_0(x) - u_0(x_0))$. For any $x_1 \in \Omega'$, let $\eta = (1 - \frac{|x-x_1|^2}{R^2})^+$, where R is small such that $R < \operatorname{dist}(\Omega', \partial\Omega)$, and

$$H = \eta^2|\nabla w|^2 + Bw^2,$$

where B is a positive constant to be determined later. Suppose H obtains its maximum at $(x_0, t_0) \in \overline{\Omega} \times [0, T']$.

Case 1. $\eta(x_0) = 0$. Then

$$\eta^2|\nabla w|^2(x,\, t) \le H(x,\, t) \le H(x_0,\, t_0) = Bw^2(x_0,\, t_0) \le BM^2.$$

Case 2. $t_0 = 0$. Then

$$\eta^2|\nabla w|^2(x,\, t) \le H(x,\, t) \le H(x_0,\, t_0) \le \|\nabla w_0\|^2_{C^0(\overline{\Omega})} + BM^2.$$

Case 3. $\eta(x_0) \ne 0$ and $t_0 > 0$. At (x_0, t_0), we compute

$$\begin{aligned} 0 &= H_i = (\eta^2)_i|\nabla w|^2 + \eta^2(|\nabla w|^2)_i + B(w^2)_i, \\ 0 &\le H_t = \eta^2(|\nabla w|^2)_t + B(w^2)_t, \end{aligned}$$

and

$$0 \ge H_{ij} = (2\eta\eta_{ij} - 6\eta_i\eta_j)|\nabla w|^2 - \frac{4Bw}{\eta}(w_i\eta_j + w_j\eta_i) + \eta^2(|\nabla w|^2)_{ij} + B(w^2)_{ij}.$$

Hence

$$\begin{aligned} 0 \ge& \eta^2(\Delta H - H_t) \\ =& (2\eta^3\Delta\eta - 6\eta^2|D\eta|^2)|\nabla w|^2 - 8B\eta w\nabla w\nabla\eta \\ &+ \eta^4(\Delta(|\nabla w|^2) - (|\nabla w|^2)_t) + B\eta^2(\Delta(w^2) - (w^2)_t) \\ :=& \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}, \end{aligned}$$

where

$$\begin{aligned} &\mathrm{I} := (2\eta^3\Delta\eta - 6\eta^2|D\eta|^2)|\nabla w|^2, \\ &\mathrm{II} := -8B\eta w \sum_{i,j=1}^{n} \nabla w\nabla\eta, \\ &\mathrm{III} := \eta^4(\Delta(|\nabla w|^2) - (|\nabla w|^2)_t) = 2\eta^4|\nabla^2 w|^2, \\ &\mathrm{IV} := B\eta^2(\Delta(w^2) - (w^2)_t). \end{aligned}$$

By direct computation, we have

$$|\nabla\eta| \le \frac{2}{R}, \quad |\nabla^2\eta| \le \frac{2}{R^2}. \tag{9}$$

By (9), we obtain that

$$\begin{aligned} &|\mathrm{I}| \le \frac{28}{R^2}\eta^2|Dw|^2, \\ &|\mathrm{II}| \le B\eta^2|Dw|^2 + \frac{64}{R^2}BM^2. \end{aligned}$$

Now we deal with the fourth term.

$$\begin{aligned}\mathrm{IV} =&B\eta^2(\Delta(w^2)-(w^2)_t)\\ =&2B\eta^2|\nabla w|^2+2B\eta^2 w\Delta w_{ij}-2B\eta^2 ww_t=2B\eta^2|\nabla w|^2+2B\eta^2 wf\\ \geq&2B\eta^2|\nabla w|^2-2B\eta^2 MN.\end{aligned}$$

Combining with these terms together, we have

$$\begin{aligned}0\geq&\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}\\ \geq&(B-\frac{28}{R^2})\eta^2|Dw|^2-\frac{64}{R^2}BM^2-2BMN.\end{aligned}$$

Taking $B\lambda=\frac{28}{R^2}+1$, we have

$$\eta^2|\nabla w|^2\leq C(M,R,\|Dw_0\|_{C^0(\overline{\Omega})},\|f\|_{C^0([0,T))}).$$

Combining these cases above, we derive the estimate

$$\eta^2|\nabla w|^2\leq C(\lambda,\mu_0,\mu_1,M,R,\|\nabla w_0\|_{C^0(\overline{\Omega})},\|f\|_{C^0([0,T))}).$$

Hence,

$$|\nabla w|^2(x,\,t)\leq C(M,R,\|\nabla w_0\|_{C^0(\overline{\Omega})},\|f\|_{C^0([0,T))}),\ \forall\ x\in\Omega'.$$

Now Proposition 3 is proved. □

Remark 1. If $f=-\frac{u_t(x_0,t)}{A}$, and $\|w\|_{L^\infty(\Omega\times[0,T))}$ we have

$$\begin{aligned}\sup_{\Omega'\times[0,T)}|\nabla w|\leq& C(\mathrm{dist}(\Omega',\partial\Omega),\|f\|_{L^\infty([0,T))})\\ =&C(\mathrm{dist}(\Omega',\partial\Omega),\,\|u_0\|_{C^2(\Omega)},\mathrm{osc}(u_0)).\end{aligned}$$

Suppose Ω is a bounded domain in $\mathbb{R}^n$, $n\geq 2$, $\partial\Omega\in C^3$. Set

$$d(x)=\mathrm{dist}(x,\partial\Omega)$$

and

$$\Omega_\sigma=\{x\in\Omega: d(x)<\sigma\}.$$

Then there exists a positive constant $\sigma_1>0$ such that $\forall\ \sigma\leq\sigma_1, d\in C^3(\overline{\Omega_\sigma})$. As mentioned in Lieberman [10] or Simon-Spruck [13], we can take ∇d in Ω_σ which is a C^2 vector field and $\nu=\nabla d$ on the boundary. As mentioned in the book [5],we also have the following formulas

$$|\nabla\nu|+|\nabla^2\nu|\leq\tilde{C}(n,\Omega),\quad |\nu|=1,\quad \sum_{i=1}^n\nu^i\nabla_i\nu^j=0,\quad\forall\,j=1,\cdots,n\quad\text{in }\Omega_\sigma.\tag{10}$$

Proposition 2. *Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^n$ and $n\geq 2$. Suppose that $w\in C^{3,2}(\overline{\Omega}\times[0,T))$ is a solution to (8) for $u_0\in C^2(\Omega),\varphi\in C^3(\overline{\Omega})$ and $u\in C^{3,2}(\overline{\Omega}\times[0,T))$, which is a solution to (7). Then for $\sigma_\leq\sigma_1$, there holds*

$$\sup_{\Omega_\sigma\times[0,T)}|\nabla w|\leq C,$$

where C is a positive constant depending only on $\Omega,n,\|u_0\|_{C^2(\Omega)},\mathrm{osc}(u_0),\|\varphi\|_{C^3(\overline{\Omega})}$.

Proof. We will prove that for $0<T'<T$, we can bound $|\nabla w|$ on $\partial\Omega\times[0,T']$ independent of T' and then take a limit argument.

Let

$$H=e^{\beta d}|\nabla h|^2+Bw^2,$$

where $h = w - \frac{1}{A}\varphi d$, B and β are positive constants to be determined later. Denote $-\frac{1}{A}\varphi d$ by P. Suppose that H obtains its maximum at $(x_0, t_0) \in \overline{\Omega_\sigma} \times [0, T']$.

Case 1. $x_0 \in \partial\Omega$. We choose a proper coordinate around x_0, such that $\frac{\partial}{\partial x_n} = \nu$. Note that at (x_0, t_0),

$$d_n = 1,\ d_\alpha = 0,\ \text{for } \alpha = 1, \cdots, n-1, \quad \text{and} \quad d_{ni} = 0,\ \text{for } i = 1, \cdots, n.$$

Then we find

$$h_n = 0, \quad h_k = w_k \quad \text{and} \quad h_{nk} = 0,\ \text{for } k = 1, \cdots, n-1.$$

Denoted by b_{ij} the Weingarten matrix of $\partial\Omega$, we then have at (x_0, t_0) ,

$$\begin{aligned} 0 \geq & H_n = \beta d_n |\nabla h|^2 + (|\nabla h|^2)_n + B(w^2)_n \\ =& \beta|\nabla h|^2 + 2\sum_{k=1}^{n-1} h_k h_{kn} + 2h_n h_{nn} + 2Bww_n \\ =& \beta|\nabla h|^2 + 2\sum_{i,k=1}^{n-1} h_k h_i b_{ik} + 2Bw\frac{\varphi}{A} \\ :=& \mathrm{I} + \mathrm{II} + \mathrm{III}, \end{aligned}$$

where

$$\begin{aligned} &\mathrm{I} = \beta|\nabla h|^2 = \beta(|\nabla w|^2 + |\nabla \mathrm{P}|^2 + 2\sum_{i=1}^{n} w_i \mathrm{P}_i) \geq \frac{\beta}{2}|\nabla w|^2 - \beta|\nabla \mathrm{P}|^2, \\ &|\mathrm{II}| = |2\sum_{i,k=1}^{n-1} h_k h_i b_{ik}| = |2\sum_{i,k=1}^{n-1} w_k w_i b_{ik}| \leq 2|b_{ij}||\nabla w|^2, \\ &|\mathrm{III}| = |2Bw\frac{\varphi}{A}| \leq 2B\frac{\|\varphi\|_{C^1(\overline{\Omega})}}{\delta}. \end{aligned}$$

Hence we obtain

$$\begin{aligned} 0 \geq & H_n = \mathrm{I} + \mathrm{II} + \mathrm{III} \\ \geq & \frac{\beta}{2}|\nabla w|^2 - \beta|\nabla \mathrm{P}|^2 - 2|b_{ij}||\nabla w|^2 - 2B\frac{\|\varphi\|_{C^1(\overline{\Omega})}}{\delta}. \end{aligned}$$

Taking $\beta = 4\sup\limits_{\partial\Omega}|b_{ij}| + 2$, we get

$$|\nabla w|^2(x_0, t_0) \leq C(\Omega, n, \|u_0\|_{L^\infty(\Omega)}, \|\varphi\|_{C^1(\overline{\Omega})}).$$

Case 2. $x_0 \in \partial\Omega_\sigma \cap \Omega$. In this case, the estimate follows from the interior gradient estimate.

Case 3. $t_0 = 0$. We have

$$|\nabla w|^2(x_0, 0) \leq C(\Omega, n, \|u_0\|_{C^1(\Omega)}).$$

Case 4. $(x_0, t_0) \in \Omega_\sigma \times (0, T']$. In this case, we have

$$\begin{aligned} 0 \leq & H_t = e^{\beta d}(|\nabla h|^2)_t + B(w^2)_t, \\ 0 =& H_i = \beta d_i e^{\beta d}|\nabla h|^2 + e^{\beta d}(|\nabla h|^2)i + B(w^2)_i. \end{aligned}$$

Since the Hessian of H at (x_0, t_0) is non-negative definite, by the equalities above we have

$$\begin{aligned}0 \geq H_{ij} =& (\beta d_{ij} + \beta^2 d_i d_j) e^{\beta d} |\nabla h|^2 + \beta d_i e^{\beta d} (|\nabla h|^2)_j \\ &+ \beta d_j e^{\beta d} (|\nabla h|^2)_i + e^{\beta d} (|\nabla h|^2)_{ij} + B(w^2)_{ij} \\ =& (\beta d_{ij} + \beta^2 d_i d_j) e^{\beta d} |\nabla h|^2 + e^{\beta d} (|\nabla h|^2)_{ij} \\ &+ B(w^2)_{ij} - \beta d_i \beta d_j e^{\beta d} |\nabla h|^2 - \beta d_i B(w^2)_j - \beta d_i \beta d_j e^{\beta d} |\nabla h|^2 - \beta d_j B(w^2)_i \\ =& (\beta d_{ij} - \beta^2 d_i d_j) e^{\beta d} |\nabla h|^2 - 2B\beta w (d_i w_j + d_j w_i) + e^{\beta d} (|\nabla h|^2)_{ij} + B(w^2)_{ij}\,.\end{aligned}$$

Then

$$\begin{aligned}0 \geq& \Delta H - H_t \\ =& (\beta \Delta d - \beta^2 |\nabla d|^2) e^{\beta d} |\nabla h|^2 - 4B\beta w (\nabla d, \nabla w) + e^{\beta d} \Delta(|\nabla h|^2) + B\Delta(w^2) \\ &- e^{\beta d} (|\nabla h|^2)_t - B(w^2)_t \\ =& (\beta \Delta d - \beta^2 |\nabla d|^2) e^{\beta d} |\nabla h|^2 - 4B\beta w (\nabla d, \nabla w) + B\big(\Delta(w^2) - (w^2)_t\big) \\ &+ e^{\beta d} \big(\Delta(|\nabla h|^2) - (|\nabla h|^2)_t\big) \\ :=& \mathrm{J}_1 + \mathrm{J}_2 + \mathrm{J}_3 + \mathrm{J}_4,\end{aligned}$$

where

$$|\mathrm{J}_1| = |(\beta \Delta d - \beta^2 |\nabla d|^2) e^{\beta d} |\nabla h|^2| \leq (\beta^2 + \beta \tilde{C}) e^{\beta \mathrm{diam}\Omega} (2|\nabla w|^2 + 2|\nabla \mathrm{P}|^2),$$

$$|\mathrm{J}_2| = |4B\beta w (\nabla d, \nabla w)| \leq B|\nabla w|^2 + 4B\beta^2,$$

$$\mathrm{J}_3 = 2Bw\Delta w + 2B|\nabla w|^2 - 2Bww_t = 2B|\nabla w|^2 + 2Bw \frac{u_t(x_0,t)}{A},$$

$$\begin{aligned}\mathrm{J}_4 \geq& 2e^{\beta d} \sum_{k=1}^n h_k (\Delta h_k - h_{kt}) \\ =& 2e^{\beta d} \sum_{k=1}^n (w_k + \mathrm{P}_k)(\Delta w_k + \Delta \mathrm{P}_k - w_{kt}) \\ =& 2e^{\beta d} \sum_{k=1}^n (w_k \Delta \mathrm{P}_k + \mathrm{P}_k \Delta \mathrm{P}_k) \\ \geq& - e^{\beta \mathrm{diam}\Omega} |\nabla w|^2 - C(\Omega, n, \|\varphi\|_{C^3(\overline{\Omega})}, \|u_0\|_{L^\infty(\Omega)}).\end{aligned}$$

Hence,

$$\begin{aligned}0 \geq& \Delta H - H_t = \mathrm{J}_1 + \mathrm{J}_2 + \mathrm{J}_3 + \mathrm{J}_4 \\ \geq& -(\beta^2 + \beta \tilde{C}) e^{\beta \mathrm{diam}\Omega} (2|\nabla w|^2 + 2|\nabla \mathrm{P}|^2) - B|\nabla w|^2 - 4B\beta^2 \\ &+ 2B|\nabla w|^2 + 2Bw \frac{u_t(x_0,t)}{A} - e^{\beta \mathrm{diam}\Omega} |\nabla w|^2 - C(\Omega, n, \|\varphi\|_{C^3(\overline{\Omega})}, \|u_0\|_{L^\infty(\Omega)}).\end{aligned}$$

By taking $B = \big(2(\beta^2 + \beta \tilde{C}) + 1\big) e^{\beta \mathrm{diam}\Omega} + 1$, we have

$$|\nabla w|^2 (x_0, t_0) \leq C(\Omega, n, \|u_0\|_{C^2(\Omega)}, \|\varphi\|_{C^3(\overline{\Omega})}).$$

Since

$$e^{\beta d} |\nabla h|^2 (x,t) \leq H(x,t) \leq H(x_0,t_0) \leq C(\Omega, n, \|u_0\|_{C^2(\Omega)}, \|\varphi\|_{C^3(\overline{\Omega})}),$$

these four cases together with $|\nabla h|^2 \geq \frac{1}{2} |\nabla w|^2 - |\nabla \mathrm{P}|^2$ give

$$|\nabla w|(x,t) \leq C(\Omega, n, \|u_0\|_{C^2(\Omega)}, \mathrm{osc}(u_0), \|\varphi\|_{C^3(\overline{\Omega})}) \quad \text{in } \Omega_\sigma \times [0, T'].$$

Remark that the bound is independent of T', we finish the proof. □

We deduce the estimate of w_t from that of u_t. Joint with Proposition 1 and 2 we then get the uniform $C^{k,\alpha}$ estimate for $k \in \mathrm{Z}^+$ and $0 < \alpha < 1$ by the Schauder theory.

Proof of Lemma 2.2. We continue the proof of Lemma 2.2. For $n \in \mathrm{Z}^+$, denoted by $w_n = w|_{\overline{\Omega}\times[0,\,n]}$ and assume that $A_n = \sup\limits_{\overline{\Omega}\times[0,\,n]} |w_n|$ is attained at the point $(x_n,\, t_n)$. For $(x,\, s) \in \overline{\Omega} \times [0,\, 1]$, we define that $g_n(x,\, s) = w_n(x,\, s + t_n - 1)$. Then $g_n(x,\, s)$ satisfies that

$$\begin{cases} \frac{\partial g_n}{\partial s} - \Delta g_n = -\frac{f(s+t_n-1)}{A_n} & \text{in } \Omega \times [0,1], \\ g_n(x,0) = w_n(x,\, t_n - 1) & \text{on } \overline{\Omega} \times \{0\}, \\ \frac{\partial g_n}{\partial \nu} = \frac{\varphi(x)}{A_n} & \text{on } \partial\Omega \times [0,1]. \end{cases}$$

Since we have derived the uniform spatial C^1 estimate of w_n independent of $t \in [0,\, n]$, so are $g_n(x,\, s)$ for $s \in [0,\, 1]$. Thus we can conclude that the function sequence $g_n(x,\, 0) = w_n(x,\, t_n - 1)$ is uniformly bounded and the derivatives are also uniformly bounded. Arzela-Ascoli theorem then assures that there exists a subsequence of $g_n(x,\, 0)$, without of loss of generality we assume $g_n(x,\, 0)$, converges to a continuous function $g_0(x)$ defined on $\overline{\Omega}$ satisfying $g_0(x_0) = 0$ and $\sup\limits_{x\in\Omega} |g_0(x)| \leq 1$.

The uniform $C^{k,\alpha}$ estimate for g_n on $\overline{\Omega} \times [0,\, 1]$ can also be obtained by the relation between g_n and w_n, thus we can select a subsequence of g_n converges in the sense of $C^{k,\alpha}$ for $k \in \mathrm{Z}^+$ and $0 < \alpha < 1$ to g on $\overline{\Omega} \times [0,\, 1]$. Obviously, we have

$$\begin{cases} \frac{\partial g}{\partial s} - \Delta g = 0 & \text{in } \Omega \times [0,1], \\ g(x,0) = g_0(x) & \text{on } \overline{\Omega} \times \{0\}, \\ \frac{\partial g}{\partial \nu} = 0 & \text{on } \partial\Omega \times [0,1]. \end{cases}$$

By a simple limit argument we can conclude that $g(x_0,\, s) = 0$ for $s \in [0,\, 1]$ and $|g(\bar{x},\, 1)| = 1$ for some $\bar{x} \in \overline{\Omega}$. This is a contradiction with the maximum principle and Hopf Lemma for the parabolic differential equations. Now we finish the proof of Lemma 2.2. □

Theorem 2.3. *For any $T > 0$, if u is a smooth solution to (7), then we have the estimate,*

$$\|u_t(\cdot,t)\|_{C(\overline{\Omega})} + \|\nabla u(\cdot,t)\|_{C(\overline{\Omega})} + \|\nabla^2 u(\cdot,t)\|_{C(\overline{\Omega})} \leq C,\ \ t \in [0,\, T),$$

where C is a constant independent of t and T.

Proof. The equation for v is

$$\begin{cases} v_t - \Delta v = -u_t(x_0,t) & \text{in } \Omega \times [0,T), \\ v_\nu = \varphi & \text{on } \partial\Omega \times [0,T), \\ v(x,0) = u_0(x) - u_0(x_0) & \text{in } \Omega. \end{cases}$$

By Lemma 2.2 we have $|v| \leq A_0$, a similar proceeding as in Proposition 1 and Proposition 2 gives

$$\|\nabla v(\cdot,t)\|_{C(\overline{\Omega})} \leq C.$$

Schauder theory then gives

$$\|\nabla^2 v(\cdot,t)\|_{C(\overline{\Omega})} \leq C.$$

Noting that $v(x,t) = u(x,t) - u(x_0,t)$, we then obtain

$$\|\nabla u(\cdot,t)\|_{C(\overline{\Omega})} + \|\nabla^2 u(\cdot,t)\|_{C(\overline{\Omega})} \le C.$$

Combining this with Lemma 2.1 we complete the proof of Theorem 2.3. □

3. **Asymptotic behavior for the uniformly parabolic equations.** In this section we deal with the general uniformly parabolic equations, here we should complete the estimate of the second derivative of the solutions while only the gradient estimate is needed for the diffusion equations. We in the following focus on

$$\begin{cases} u_t - F(\nabla^2 u) = 0 & \text{in } \Omega \times [0,T), \\ u(x,0) = u_0(x) & \text{on } \overline{\Omega} \times \{0\}, \\ u_\nu = \varphi & \text{on } \partial\Omega \times [0,T), \end{cases} \tag{11}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $\varphi, u_0 \in C^\infty(\overline{\Omega})$ such that $u_{0,\nu} = \varphi$ on $\partial\Omega$. Moreover, we suppose that F satisfies conditions (F1)-(F4).

Lemma 3.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary, $n \ge 2$. If $u(x,t)$ is a smooth solution to (11), then $\sup\limits_{\Omega\times[0,T)} |u_t|^2 = \sup\limits_{\Omega} |u_t(x,0)|^2$, so there is a constant C depending only on μ_0 and $\|\nabla^2 u_0\|_{C^0(\Omega)}$ such that*

$$\|u_t\|_{L^\infty(\Omega\times[0,T))} \le C.$$

Proof. Let us denote $\frac{\partial}{\partial r_{ij}} |_{r=\nabla^2 u} F(r)$ by F_u^{ij} and let $L = F_u^{ij}\partial_{ij} - \partial_t$. By taking derivative of $u_t = F(\nabla^2 u)$ with respect to t, we have

$$u_{tt} = \sum_{i,j=1}^n F_u^{ij} u_{ijt}.$$

Then

$$L(u_t^2) = 2\sum_{i,j=1}^n F_u^{ij} u_{ti}u_{tj} + 2\sum_{i,j=1}^n F_u^{ij} u_t u_{tij} - 2u_t u_{tt} = 2\sum_{i,j=1}^n F_u^{ij} u_{ti}u_{tj} \ge 0.$$

We apply the weak maximum principle to obtain

$$\sup_{\Omega\times(0,T)} |u_t|^2 = \sup_{(\Omega\times\{0\})\cup(\partial\Omega\times(0,T))} |u_t|^2.$$

We explore the possibility that the maximum occurs on $\partial\Omega \times (0,T)$. Since $\partial_\nu u_t^2 = 2u_t u_{t\nu} = 0$, Hopf Lemma tells us this can not occur. Thus

$$\sup_{\Omega\times(0,T)} |u_t|^2 = \sup_{\Omega\times\{0\}} |u_t|^2 = \sup_{\Omega} |F(\nabla^2 u_0)|^2.$$

□

Let $v(x,t) = u(x,t) - u(x_0,t)$ with $x_0 \in \Omega$. Similarly as in Section 2, we firstly give a time-independent bound for $|v|$ via a blow-up technique. With the help of C^0 estimate, we get C^2 estimate for v. Hence estimates for $|\nabla u|$ and $|\nabla^2 u|$ follows. Finally we apply the Schnürer convergence result to obtain that the smooth solution converges to a translating solution.

Lemma 3.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary, $n \ge 2$. If $u(x,t)$ is a smooth solution to (11), $v(x,t)$ is defined as above, then there is a constant $A_0 > 0$, independent of T, such that*

$$\|v\|_{C^0(\Omega\times[0,T))} \le A_0. \tag{12}$$

Proof. Let $A = \|v\|_{C^0(\Omega\times[0,T))}$. Without loss of generality, we suppose $A \geq \delta := \mathrm{osc}(u_0) > 0$. (Otherwise we get a constant solution to (11).) Suppose A is unbounded, that is $A \to \infty$ as $T \to \infty$. Let

$$w(x,t) = \frac{v(x,t)}{A}.$$

Then w satisfies

$$\begin{cases} w_t - \frac{1}{A}F(A\nabla^2 w) = -\frac{u_t(x_0,t)}{A} & \text{in } \Omega \times [0,T), \\ w(x,0) = \frac{1}{A}(u_0(x) - u_0(x_0)) & \text{on } \overline{\Omega} \times \{0\}, \\ w_\nu(x,t) = \frac{1}{A}\varphi(x) & \text{on } \partial\Omega \times [0,T). \end{cases} \tag{13}$$

It is easy to see $w(x_0,\, t) = 0,\ \sup\limits_{\Omega\times(0,T)} |w| = 1$. To complete the proof, we need the following propositions.

Proposition 3. *Suppose* $w \in C^{3,2}(\Omega \times [0,T))$

$$\begin{cases} w_t - \frac{1}{A}F(A\nabla^2 w) = f(t) & \text{in } \ \Omega \times [0,T), \\ w(x,0) = w_0(x) & \text{in } \ \Omega, \end{cases}$$

for $f \in C([0,T))$, $w_0 \in C^1(\Omega)$. *Then there holds* $\forall\ \Omega' \subset\subset \Omega$,

$$\sup_{\Omega'\times[0,\,T)} |\nabla w| \leq C,$$

where C *is a positive constant depending only on* λ, μ_0, μ_1, $\mathrm{dist}(\Omega', \partial\Omega)$, $\|w\|_{L^\infty(\Omega\times[0,T))}$, $\|f\|_{L^\infty([0,T))}$, $\|w_0\|_{L^\infty(\Omega)}$. □

Proof. We will prove that for $0 < T' < T$, we can bound $|\nabla w|$ in $\overline{\Omega'}\times[0,T']$ independent of T' and then take a limit argument. Denoting by $M = \|w\|_{L^\infty(\Omega\times[0,T))}$, $N = \|f\|_{L^\infty([0,T))}$, and $w_0 = \frac{1}{A}(u_0(x) - u_0(x_0))$. For any $x_1 \in \Omega'$, let $\eta = (1 - \frac{|x-x_1|^2}{R^2})^+$, where R is small such that $R < \mathrm{dist}(\Omega', \partial\Omega)$, and

$$H = \eta^2|\nabla w|^2 + Bw^2,$$

where B is a positive constant to be determined later. Suppose H obtains its maximum at $(x_0, t_0) \in \overline{\Omega} \times [0, T']$.

Case 1. $\eta(x_0) = 0$. Then

$$\eta^2|\nabla w|^2(x,\, t) \leq H(x,\, t) \leq H(x_0,\, t_0) = Bw^2(x_0,\, t_0) \leq BM^2.$$

Case 2. $t_0 = 0$. Then

$$\eta^2|\nabla w|^2(x,\, t) \leq H(x,\, t) \leq H(x_0,\, t_0) \leq \|\nabla w_0\|^2_{C^0(\overline{\Omega})} + BM^2.$$

Case 3. $\eta(x_0) \neq 0$ and $t_0 > 0$. At (x_0, t_0), we compute

$$0 = H_i = (\eta^2)_i|\nabla w|^2 + \eta^2(|\nabla w|^2)_i + B(w^2)_i,$$
$$0 \leq H_t = \eta^2(|\nabla w|^2)_t + B(w^2)_t,$$

and

$$0 \geq H_{ij} = (2\eta\eta_{ij} - 6\eta_i\eta_j)|\nabla w|^2 - \frac{4Bw}{\eta}(w_i\eta_j + w_j\eta_i) + \eta^2(|\nabla w|^2)_{ij} + B(w^2)_{ij}.$$

We denote $\frac{\partial}{\partial r_{ij}}\ |_{r=A\nabla^2 w}\ F(r)$ by F^{ij}. It follows from the structure conditions (F1) and (F2) that

$$\lambda\delta_{ij} \leq F^{ij} \leq \mu_1\delta_{ij} \quad \text{and} \quad |F^{ij}| \leq \mu_1. \tag{14}$$

Hence

$$\begin{aligned}
0 \geq& \eta^2(\sum_{i,j=1}^{n} F^{ij}H_{ij} - H_t)\\
=& \sum_{i,j=1}^{n} F^{ij}(2\eta^3\eta_{ij} - 6\eta^2\eta_i\eta_j)|\nabla w|^2 - 8B\eta w \sum_{i,j=1}^{n} F^{ij} w_i\eta_j\\
&+ \eta^4(\sum_{i,j=1}^{n} F^{ij}(|\nabla w|^2)_{ij} - (|\nabla w|^2)_t) + B\eta^2(\sum_{i,j=1}^{n} F^{ij}(w^2)_{ij} - (w^2)_t)\\
:=& \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV},
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{I} &:= \sum_{i,j=1}^{n} F^{ij}(2\eta^3\eta_{ij} - 6\eta^2\eta_i\eta_j)|\nabla w|^2,\\
\mathrm{II} &:= -8B\eta w \sum_{i,j=1}^{n} F^{ij} w_i\eta_j,\\
\mathrm{III} &:= \eta^4(\sum_{i,j=1}^{n} F^{ij}(|\nabla w|^2)_{ij} - (|\nabla w|^2)_t),\\
\mathrm{IV} &:= B\eta^2(\sum_{i,j=1}^{n} F^{ij}(w^2)_{ij} - (w^2)_t).
\end{aligned}$$

By (14) and (9), we obtain that

$$\begin{aligned}
|\mathrm{I}| &\leq \frac{28\mu_1}{R^2}\eta^2|\nabla w|^2,\\
|\mathrm{II}| &\leq \frac{16}{R}B\eta M\mu_1|\nabla w| \leq \lambda B\eta^2|\nabla w|^2 + \frac{64}{\lambda R^2}BM^2\mu_1^2.
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{III} =& 2\eta^4 \sum_{i,j,k=1}^{n} F^{ij}w_{ki}w_{kj} + 2\eta^4 \sum_{i,j,k=1}^{n} F^{ij}w_k w_{kij} - 2\eta^4 \sum_{k=1}^{n} w_k w_{kt}\\
=& 2\eta^4 \sum_{i,j,k=1}^{n} F^{ij}w_{ki}w_{kj} \geq 2\eta^4\lambda|\nabla^2 w|^2.
\end{aligned}$$

where the second equality is due to taking derivative along the direction of x_k on both sides of the equation $w_t - \frac{1}{A}F(A\nabla^2 w) = f(t)$.

Now we deal with the fourth term.

$$\mathrm{IV} = 2B\eta^2 \sum_{i,j=1}^{n} F^{ij}w_i w_j + 2B\eta^2 w \sum_{i,j=1}^{n} F^{ij}w_{ij} - 2B\eta^2 w w_t := \mathrm{IV}_1 + \mathrm{IV}_2 + \mathrm{IV}_3.$$

By (14), we have

$$\begin{aligned}
\mathrm{IV}_1 &= 2B\eta^2 \sum_{i,j=1}^{n} F^{ij}w_i w_j \geq 2B\eta^2\lambda|\nabla w|^2,\\
|\mathrm{IV}_2| &= |2B\eta^2 w \sum_{i,j=1}^{n} F^{ij}w_{ij}| \leq 2BM\eta^2\mu_1|\nabla^2 w| \leq \lambda\eta^4|\nabla^2 w|^2 + \frac{1}{\lambda}B^2M^2\mu_1^2,
\end{aligned}$$

$$|\mathrm{IV}_3| = |2B\eta^2 w(f(t) + \frac{1}{A}F(A\nabla^2 w))| \le 2BMN + 2BM\eta^2\mu_0|\nabla^2 w|$$
$$\le 2BMN + \lambda\eta^4|\nabla^2 w|^2 + \frac{1}{\lambda}B^2M^2\mu_0^2.$$

Combining these terms together, we have

$$0 \ge \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}$$
$$\ge (B\lambda - \frac{28\mu_1}{R^2})\eta^2|\nabla w|^2 - \frac{64}{\lambda R^2}BM^2\mu_1^2 - \frac{1}{\lambda}B^2M^2(\mu_0^2 + \mu_1^2) - 2BMN.$$

Taking $B\lambda = \frac{28\mu_1}{R^2} + 1$, we have

$$\eta^2|\nabla w|^2 \le C(\lambda, \mu_0, \mu_1, M, R, \|f\|_{C^0([0,T))}).$$

Combining these cases above, we derive the estimate

$$\eta^2|\nabla w|^2 \le C(\lambda, \mu_0, \mu_1, M, R, \|\nabla w_0\|_{C^0(\overline{\Omega})}, \|f\|_{C^0([0,T))}).$$

Hence,

$$|\nabla w|^2(x,\, t) \le C(\lambda, \mu_0, \mu_1, M, \mathrm{dist}(\Omega', \partial\Omega), \|\nabla w_0\|_{C^0(\overline{\Omega})}, \|f\|_{C^0([0,T))}),\ \forall\ x \in \Omega'.$$

Now Proposition 3 is proved. □

Remark 2. Note that $f = -\frac{u_t(x_0,t)}{A}$, $M = 1$ in (13), we have

$$\sup_{\Omega'\times[0,T)} |\nabla w| \le C(\lambda, \mu_0, \mu_1, \mathrm{dist}(\Omega', \partial\Omega), \|u_0\|_{C^2(\Omega)}, \mathrm{osc}(u_0)).$$

Proposition 4. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary, $n \ge 2$. Suppose that $w \in C^{3,2}(\Omega \times [0,T))$ is a solution to (13) for $u_0 \in C^2(\Omega)$, $\varphi \in C^3(\overline{\Omega})$ and $u \in C^{3,2}(\Omega \times [0,T))$ is a solution to (11). Then $\forall\ \sigma \le \sigma_1$, there holds*

$$\sup_{\Omega_\sigma\times[0,\,T)} |\nabla w| \le C,$$

where σ_1 is defined in Section 2, and C is a positive constant depending only on Ω, n, λ, μ_0, μ_1, $\|u_0\|_{C^2(\Omega)}$, $\mathrm{osc}(u_0)$, $\|\varphi\|_{C^3(\Omega)}$.

Proof. We will prove that for $0 < T' < T$, $|\nabla w|$ can be bounded on $\Omega_\sigma \times [0, T']$ independent of T' and then take a limit argument.

Denoting by $M = \|w\|_{L^\infty(\Omega\times[0,T))}$ as in Proposition 3, and $N = \|\frac{u_t(x_0,t)}{A}\|_{L^\infty([0,T))}$. Let

$$H = e^{\beta d}|\nabla h|^2 + Bw^2,$$

where $h = w - \phi d$, $\phi = \frac{\varphi}{A}$, B and β are positive constants to be determined later. Denote $-\phi d$ by P. Suppose H obtains its maximum at $(x_0, t_0) \in \overline{\Omega_\sigma} \times [0, T']$.

Case 1. $x_0 \in \partial\Omega$. The same proceeding as the case 1 in Proposition 2 and also taking $\beta = 4\sup_{\partial\Omega}|b_{ij}| + 2$, we obtain

$$|\nabla w|^2(x_0, t_0) \le C(\Omega, n, \|u_0\|_{L^\infty(\Omega)}, \|\varphi\|_{C^1(\overline{\Omega})}).$$

Case 2. $x_0 \in \partial\Omega_\sigma \cap \Omega$. In this case, the estimate follows from the interior gradient estimate in Proposition 3.

Case 3. $t_0 = 0$. We have

$$|\nabla w|^2(x_0, 0) \le C(\Omega, n, u_0).$$

Case 4. $(x_0,t_0)\in\Omega_\sigma\times(0,T']$. In this case, we compute

$$0=H_i=\beta d_i e^{\beta d}|\nabla h|^2+e^{\beta d}(|\nabla h|^2)i+B(w^2)_i,$$
$$0\le H_t=e^{\beta d}(|\nabla h|^2)_t+B(w^2)_t,$$

and

$$\begin{aligned}0\ \ge&H_{ij}\\ =&(\beta d_{ij}-\beta^2 d_i d_j)e^{\beta d}|\nabla h|^2-2B\beta w(d_i w_j+d_j w_i)+e^{\beta d}(|\nabla h|^2)_{ij}+B(w^2)_{ij}.\end{aligned}$$

Then

$$\begin{aligned}0\ge&\sum_{i,j=1}^n F^{ij}H_{ij}-H_t\\ =&\sum_{i,j=1}^n F^{ij}(\beta d_{ij}-\beta^2 d_i d_j)e^{\beta d}|\nabla h|^2-2B\beta w\sum_{i,j=1}^n F^{ij}(d_i w_j+d_j w_i)\\ &+B(\sum_{i,j=1}^n F^{ij}(w^2)_{ij}-(w^2)_t)+e^{\beta d}(\sum_{i,j=1}^n F^{ij}(|\nabla h|^2)_{ij}-(|\nabla h|^2)_t)\\ :=&\mathrm{J}_1+\mathrm{J}_2+\mathrm{J}_3+\mathrm{J}_4.\end{aligned}$$

Use (14) again, we have

$$|\mathrm{J}_1|=|\sum_{i,j=1}^n F^{ij}(\beta d_{ij}-\beta^2 d_i d_j)e^{\beta d}|\nabla h|^2|\le\mu_1(\beta^2+\beta\tilde{C}^2)e^{\beta \mathrm{diam}(\Omega)}(2|\nabla w|^2+2|\nabla \mathrm{P}|^2),$$

$$|\mathrm{J}_2|=|2B\beta w\sum_{i,j=1}^n F^{ij}(d_i w_j+d_j w_i)|\le 4B\beta M\mu_1|\nabla w|\le B\lambda|\nabla w|^2+\frac{4}{\lambda}\beta^2M^2\mu_1^2.$$

For the third term, we have

$$\mathrm{J}_3=2B\sum_{i,j=1}^n F^{ij}w_i w_j+2B\sum_{i,j=1}^n F^{ij}ww_{ij}-2Bww_t:=\mathrm{J}_{31}+\mathrm{J}_{32}+\mathrm{J}_{33},$$

where

$$\begin{aligned}\mathrm{J}_{31}\ \ge&2B\lambda|\nabla w|^2,\\ \mathrm{J}_{32}\ \le&2B\mu_1|\nabla^2 w|\le\frac{\lambda}{2}|\nabla^2 w|^2+\frac{2B^2\mu_1^2}{\lambda},\\ \mathrm{J}_{33}\ \le&2B\mu_0 M|\nabla^2 w|+2BNM\le\frac{\lambda}{2}|\nabla^2 w|^2+\frac{2B^2\mu_0^2M^2}{\lambda}+2BNM.\end{aligned}$$

For the fourth term, we have

$$\mathrm{J}_4=2e^{\beta d}\sum_{i,j,k=1}^n F^{ij}h_{ki}h_{kj}+2e^{\beta d}\sum_{k=1}^n h_k(\sum_{i,j=1}^n F^{ij}h_{kij}-h_{kt}):=\mathrm{J}_{41}+\mathrm{J}_{42},$$

where

$$\begin{aligned}\mathrm{J}_{41}=&2e^{\beta d}\lambda|\nabla^2 h|^2\\ \ge&2e^{\beta d}\lambda(|\nabla^2 w|^2+|\nabla^2\mathrm{P}|^2+2\sum_{i,k=1}^n w_{ik}\mathrm{P}_{ik})\\ \ge&\lambda|\nabla^2 w|^2-2e^{\beta \mathrm{diam}(\Omega)}\lambda|\nabla^2\mathrm{P}|^2,\end{aligned}$$

and

$$\begin{aligned}
\mathrm{J}_{42} =& 2e^{\beta d}\sum_{k=1}^{n} h_k(\sum_{i,j=1}^{n} F^{ij}h_{kij} - h_{kt})\\
=& 2e^{\beta d}\sum_{k=1}^{n}(w_k+\mathrm{P}_k)(\sum_{i,j=1}^{n} F^{ij}(w_{ijk}+\mathrm{P}_{ijk}) - w_{kt})\\
=& 2e^{\beta d}\sum_{i,j,k=1}^{n} w_kF^{ij}\mathrm{P}_{ijk} + e^{\beta d}\sum_{i,j,k=1}^{n}\mathrm{P}_kF^{ij}\mathrm{P}_{ijk}\\
\geq& -e^{\beta\mathrm{diam}(\Omega)}|\nabla w|^2 - C(\lambda,\mu_1,\Omega,n,\|u_0\|_{L^\infty(\Omega)},\|\varphi\|_{C^3(\overline{\Omega})}).
\end{aligned}$$

Hence

$$\begin{aligned}
0 \geq& \mathrm{J}_1+\mathrm{J}_2+\mathrm{J}_3+\mathrm{J}_4\\
\geq& \big(B\lambda - 2\mu_1(\beta^2+\beta\tilde{C}^2)e^{\beta\mathrm{diam}(\Omega)} - e^{\beta\mathrm{diam}(\Omega)}\big)|\nabla w|^2\\
&- C(\lambda,\mu_0,\mu_1,\Omega,n,\|u_0\|_{L^\infty(\Omega)},N,\|\varphi\|_{C^3(\overline{\Omega})}).
\end{aligned}$$

Let $B\lambda = 2\mu_1(\beta^2+\beta\tilde{C}^2)e^{\beta\mathrm{diam}(\Omega)} + e^{\beta\mathrm{diam}(\Omega)} + 1$, we have

$$|\nabla w|^2(x_0,t_0) \leq C(\lambda,\mu_0,\mu_1,\Omega,n,\|u_0\|_{L^\infty(\Omega)},N,\|\varphi\|_{C^3(\overline{\Omega})}).$$

Combining these cases together, we have,

$$|\nabla w|^2 \leq C(\lambda,\mu_0,\mu_1,\|u_0\|_{C^2(\Omega)},\mathrm{osc}(u_0),\|\varphi\|_{(C^3(\overline{\Omega}))},n,\Omega) \quad \text{in} \quad \Omega_\sigma\times[0,\,T'].$$

Since the bound is independent of the choice of T', we complete the proof of Proposition 4. □

In the following, we will give the global bound for the second derivatives. First of all, we give the interior estimate for the second derivative of w.

Proposition 5. *Suppose $w\in C^{4,2}(\Omega\times[0,T))$ is a solution to*

$$\begin{cases} w_t - \frac{1}{A}F(A\nabla^2 w) = f(t) & \text{in } \Omega\times[0,T),\\ w(x,0) = w_0(x) & \text{in } \Omega,\end{cases}$$

for $f\in C([0,T))$ and $w_0\in C^2(\overline{\Omega})$. Then $\forall\ \Omega'\subset\subset\Omega$, then holds

$$\sup_{\Omega'\times[0,T)}|\nabla^2 w| \leq C,$$

where C is a positive constant depending only on λ, μ_0, μ_1, $\mathrm{dist}(\Omega',\partial\Omega)$, $\|w_0\|_{C^2(\Omega)}$, $\|f\|_{L^\infty([0,T))}$, $\|w\|_{C^1(\Omega\times[0,T))}$.

Proof. Firstly, we remark that, to bound the second derivatives, it is sufficient to give an upper bound for $u_{\tau\tau},\forall\ \tau\in S^{n-1}$, due to structure conditions (F1) and (F2) as well as the boundedness of $\|w_t\|_{C^0(\Omega\times[0,T))}$ and $\|f\|_{C^0([0,T))}$.

As before, for any $0<T'<T$, we will bound $|\nabla^2 w|$ on $\Omega'\times[0,T']$ independent of T'. Denoting by $M_1 = \|w\|_{C^1(\Omega\times[0,T))}$, M,N are defined as in Proposition 4. For any $x_1\in\Omega'$, let $\eta = (1-\frac{|x-x_1|^2}{R^2})^+$, where R is small such that $R<\mathrm{dist}(\Omega',\partial\Omega)$, and let

$$H = \eta^2(w_{\zeta\zeta} + B|\nabla w|^2),$$

where $\zeta\in S^{n-1}$, and B is a positive constant to be determined later. Suppose H obtains its maximum at $(x_0,t_0)\in\overline{\Omega}\times[0,T']$.

Case 1. $x_0\in\Omega\cap\{x:\eta(x)=0\}$. Then

$$\eta^2 w_{\zeta\zeta}(x) \leq H(x,t) \leq H(x_0,t_0) = 0.$$

Case 2. $t_0 = 0$. Then

$$\eta^2 w_{\zeta\zeta} \le H(x,t) \le H(x_0, 0) \le \|\nabla^2 w_0\|_{C^0(\Omega\times(0,T))} + B\|\nabla w_0\|^2_{C^0(\Omega\times(0,T))}.$$

Case 3. $x_0 \in \Omega \cap \{x : \eta(x) > 0\}, t_0 > 0$. At (x_0, t_0), we have

$$0 = H_i = (\eta^2)_i(w_{\zeta\zeta} + B|\nabla w|^2) + \eta^2(w_{\zeta\zeta} + B|\nabla w|^2)_i,$$
$$0 \le H_t = \eta^2(w_{\zeta\zeta} + B|\nabla w|^2)_t,$$

and

$$0 \ge H_{ij} = (2\eta\eta_{ij} - 6\eta_i\eta_j)(w_{\zeta\zeta} + B|\nabla w|^2) + \eta^2(w_{\zeta\zeta} + B|\nabla w|^2)_{ij}.$$

We denote $\frac{\partial}{\partial r_{ij}}\,|_{r=A\nabla^2 w}\, F(r)$ by F^{ij}. Hence

$$\begin{aligned}
0 \ge&\eta^2\Big(\sum_{i,j=1}^n F^{ij}H_{ij} - H_t\Big)\\
=&\sum_{i,j=1}^n F^{ij}(2\eta^3\eta_{ij} - 6\eta^2\eta_i\eta_j)(w_{\zeta\zeta} + B|\nabla w|^2) + \eta^4\Big(\sum_{i,j=1}^n F^{ij}w_{\zeta\zeta ij} - w_{\zeta\zeta t}\Big)\\
&+ B\eta^4\sum_{i,j=1}^n F^{ij}((|\nabla w|^2)_{ij} - (|\nabla w|^2)_t)\\
=&\mathrm{I} + \mathrm{II} + \mathrm{III},
\end{aligned}$$

where

$$\begin{aligned}
|\mathrm{I}| \le&\frac{28\mu_1}{R^2}\eta^2(|w_{\zeta\zeta}| + B|\nabla w|^2) \le \frac{14\mu_1}{R^2}\eta^4|w_{\zeta\zeta}|^2 + \frac{28\mu_1}{R^2}BM_1^2 + \frac{14\mu_1}{R^2},\\
\mathrm{III} =&2B\eta^4\sum_{i,j,k=1}^n F^{ij}(w_k(w_{kij} - w_k w_{kt}) + w_{ki}w_{kj}) \ge 2B\eta^4\lambda|\nabla^2 w|^2.
\end{aligned}$$

Remark that from the concave structure condition (F3), we have $\mathrm{II} \ge 0$.

Therefore we have

$$0 \ge \mathrm{I} + \mathrm{II} + \mathrm{III} \ge 2B\eta^4\lambda|\nabla^2 w|^2 - \frac{14\mu_1}{R^2}\eta^4|w_{\zeta\zeta}|^2 - \frac{28\mu_1}{R^2}BM_1^2 - \frac{14\mu_1}{R^2}.$$

Taking $B = \frac{14\mu_1 + R^2}{2\lambda R^2}$, we have

$$\eta^4|w_{\zeta\zeta}|^2 \le C(\lambda, \mu_1, M_1, R).$$

Combining these cases together, we have

$$\eta^2|w_{\zeta\zeta}| \le C(\lambda, \mu_1, \mathrm{dist}(\Omega, \partial\Omega), M_1, \|w_0\|_{C^2(\Omega)}).$$

Hence

$$\sup_{\Omega'\times[0,T']}|\nabla^2 w| \le C(\lambda, \mu_0, \mu_1, \mathrm{dist}(\Omega', \partial\Omega), M_1, \|w_0\|_{C^2(\Omega)}, \|f\|_{L^\infty([0,T))}).$$

Noting that some quantities in the bracket above are due to the process when we bound $|\nabla^2 w|$ by the positive part of $\nabla^2 w$. Thus Proposition 5 now is proved. □

Now, we are in position to estimate the second derivative of w near boundary.

Proposition 6. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary, $n \ge 2$. Suppose $w \in C^{4,2}(\Omega \times [0,T))$ is a solution to (13) for u, u_0, φ described as in Proposition 4. Then there is a constant $C = C(\Omega, n, u_0, \varphi, \lambda, \mu_1)$ such that for $\sigma \le \sigma_1$,*

$$\sup_{\Omega_\sigma\times[0,T)}|\nabla^2 w| \le C(1 + \sup_{\partial\Omega\times[0,T)}|w_{\nu\nu}|),$$

where σ_1 is defined in Section 2.

Proof. We will prove that for any $0 < T' < T$, we can bounded $\nabla^2 w$ on $\Omega_\sigma \times [0, T']$ independent of T'.

Let

$$H(x,\, t,\, \xi) = e^{\alpha d}(w_{\xi\xi} + B{w_\xi}^2),$$

where α and B are positive constants to be determined later and $\xi \in S^{n-1}$ is a fixed unit vector. We may assume that $|w_{\xi\xi}| \geq 1$, otherwise there is nothing needed to do. We firstly show the following differential equality

$$\sum_{i,j=1}^{n} F^{ij} H_{ij} - H_t \geq 0 \mod \nabla H \quad \text{on} \;\; \Omega_\sigma \times (0,\, T']. \tag{15}$$

In fact,

$$\begin{aligned}
&0 = H_i = \alpha d_i H + e^{\alpha d}(w_{\xi\xi i} + B({w_\xi}^2)_i),\\
&H_t = e^{\alpha d}(w_{\xi\xi t} + B({w_\xi}^2)_t),\\
&H_{ij} = (\alpha d_{ij} - \alpha^2 d_i d_j)H + e^{\alpha d}(w_{\xi\xi ij} + B({w_\xi}^2)_{ij}).
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{i,j=1}^{n} F^{ij} H_{ij} - H_t\\
=&\sum_{i,j=1}^{n} F^{ij}(\alpha d_{ij} - \alpha^2 d_i d_j)H + e^{\alpha d}(\sum_{i,j=1}^{n} F^{ij} w_{\xi\xi ij} - w_{\xi\xi t})\\
&+ Be^{\alpha d}(\sum_{i,j=1}^{n} F^{ij}(w_\xi^2)_{ij} - (w_\xi^2)_t)\\
:=&\mathrm{I} + \mathrm{II} + \mathrm{III},
\end{aligned}$$

where $\mathrm{II} \geq 0$ by (F3) and

$$\begin{aligned}
|\mathrm{I}| &\leq \mu_1(\alpha\tilde{C}^2 + \alpha^2)e^{\alpha d}|w_{\xi\xi}| + C_0(\alpha, \mu_1, n, \Omega),\\
\mathrm{III} &= 2Be^{\alpha d}\sum_{i,j=1}^{n} F^{ij} w_{\xi i} w_{\xi j} + 2Be^{\alpha d} w_\xi \sum_{i,j=1}^{n} F^{ij}(w_{\xi ij} - w_{\xi t})\\
&\geq 2Be^{\alpha d}\lambda \sum_{i=1}^{n} |w_{\xi i}|^2.
\end{aligned}$$

According to Cauchy inequality that $|w_{\xi\xi}|^2 = |\sum_{i=1}^{n} w_{\xi i}\xi^i|^2 \leq \sum_{i=1}^{n} w_{\xi i}^2$, we have by the assumption $|w_{\xi\xi}| \geq 1$ that

$$\mathrm{III} \geq 2Be^{\alpha d}\lambda |w_{\xi\xi}|.$$

Now we take $B = \frac{1}{2\lambda}\big(\mu_1(\alpha\tilde{C} + \alpha^2) + C_0\big)$ and then conclude that (15) is valid.

By the maximum principle, the maximum point of H, denoted by $(x_0,\, t_0,\, \xi_0)$, must occur on $\Omega_\sigma \times \{0\} \times S^{n-1}$, $\big(\partial\Omega_\sigma \cap \Omega\big) \times [0,\, T'] \times S^{n-1}$ or $\partial\Omega \times [0,\, T'] \times S^{n-1}$. In the following, we deal with these three cases one by one.

Case 1. $(x_0,\, t_0,\, \xi_0) \in \Omega \times \{0\} \times S^{n-1}$. In this case, we have

$$w_{\xi_0\xi_0}(x_0,\, t_0) \leq \max\{H(x_0,\, 0,\, \xi_0),\; 0\} \leq C(u_0,\, \Omega).$$

Case 2. $(x_0, t_0, \xi_0) \in \big(\partial\Omega_\sigma \cap \Omega\big) \times [0, T'] \times S^{n-1}$. In this case, we can get the estimate of the second derivative of w according to the interior estimate derived in Proposition 5.

Case 3. $(x_0, t_0, \xi_0) \in \partial\Omega \times [0, T'] \times S^{n-1}$. Then we have in this case

$$0 \geq H_\nu = \alpha(w_{\xi_0\xi_0} + Bw_{\xi_0}^2) + w_{\xi_0\xi_0\nu} + 2Bw_{\xi_0}w_{\xi_0\nu}.$$

Now we firstly assume that $\xi_0 \perp \nu$.

We denote $\sum\limits_{i,j=1}^n w_{ij}\tau^i\mu^j$ by $w_{\tau\mu}$. Then by taking tangential derivatives of the boundary condition $w_\nu = \phi$, where $\phi = \frac{\varphi}{A}$, we have

$$\sum_{p,q=1}^n C^{pq}\sum_{k=1}^n (w_k\nu^k)_p\xi_0^q = \sum_{p,q=1}^n C^{pq}\phi_p\xi_0^q,$$

where $C^{pq} = \delta_{pq} - \nu^p\nu^q = \delta_{pq} - d_pd_q$ in Ω_σ, as defined in [10]. Then

$$w_{\xi_0\nu} = \phi_{\xi_0} - \sum_{k=1}^n w_k\nu^k_{,q}\xi_0^q, \tag{16}$$

it follows that a constant $\Lambda = \Lambda(\varphi,\, \tilde{C},\, \|\nabla w\|_{C^0(\overline{\Omega}\times[0,\,T))})$ can be found such that

$$|w_{\xi_0\nu}| \leq \Lambda. \tag{17}$$

Taking double tangential derivatives of the boundary condition, we obtain

$$\sum_{i,j,k,p,q=1}^n C^{jq}(C^{ip}(w_k\nu^k)_p)_q\xi_0^i\xi_0^j = \sum_{i,j,p,q=1}^n C^{jq}(C^{ip}\phi_p)_q\xi_0^i\xi_0^j,$$

then

$$\begin{aligned} w_{\xi_0\xi_0\nu} = &\sum_{i,j,p,q=1}^n C^{jq}C^{ip}_{,q}\phi_p\xi_0^i\xi_0^j + \phi_{\xi_0\xi_0} - \sum_{p,q,k=1}^n \xi_0^p\xi_0^q(w_{kp}\nu^k_q + w_{kq}\nu^k_p + w_k\nu^k_{pq}) \\ &- \sum_{i,p,q,k=1}^n \xi_0^q C^{ip}_{,q}\xi_0^i(w_k\nu^k)_p. \end{aligned}$$

Hence,

$$|w_{\xi_0\xi_0\nu} + 2Bw_{\xi_0}w_{\xi_0\nu}| \leq 2\tilde{C}|\nabla^2 w| + C(\|\phi\|_{C^2(\overline{\Omega})}, \tilde{C}, \|\nabla w\|_{C^0(\overline{\Omega}\times[0,\,T))}, B).$$

Since w_t is bounded, $F(r)$ is uniformly elliptic, we have for any $(x,t) \in \Omega_\sigma \times [0,T']$,

$$|\nabla^2 w(x_0,t_0)| \leq C_0(\lambda,\, \mu_1,\, u_0)\big(1 + \sup_{\gamma\in S^{n-1}} w^+_{\gamma\gamma}(x_0,t_0)\big).$$

Without loss of generality, we suppose that $\sup\limits_{\gamma\in S^{n-1}} w^+_{\gamma\gamma}(x_0,t_0) = w_{\zeta\zeta} > 0$. Denoted by $\zeta^\top$ the tangential part of ζ to $\partial\Omega$ and $\zeta^\perp$ the perpendicular part of ζ to $\partial\Omega$. we then have by (17)

$$\begin{aligned} |\nabla^2 w(x_0,t_0)| &\leq C_0(1 + w_{\zeta\zeta}(x_0,t_0)) \\ &\leq C_0(1 + w_{\zeta^\top\zeta^\top} + 2w_{\zeta^\top\zeta^\perp} + w_{\zeta^\perp\zeta^\perp}) \\ &\leq C_0(1 + 2\Lambda + w_{\zeta^\top\zeta^\top} + |w_{\nu\nu}|) \\ &\leq C_0(1 + 2\Lambda + H(x_0,\, t_0,\, \xi_0) + |w_{\nu\nu}|) \\ &\leq C_0(1 + 2\Lambda + w_{\xi_0\xi_0} + B\|\nabla w\|^2_{C^0(\overline{\Omega}\times[0,\,T))} + |w_{\nu\nu}|). \end{aligned}$$

Therefore,

$$\begin{aligned}&|w_{\xi_0\xi_0\nu}+2Bw_{\xi_0}w_{\xi_0\nu}|\\ \le\ &2C_0\tilde{C}\big(1+w_{\xi_0\xi_0}+|w_{\nu\nu}|\big)+C(\|\phi\|_{C^2(\overline{\Omega})},\tilde{C},\ \|\nabla w\|_{C^0(\overline{\Omega}\times[0,\,T))},\ B).\end{aligned}$$

Plug this inequality into (3), we then derive by taking $\alpha = 2C_0\tilde{C}+1$ that

$$w_{\xi_0\xi_0}(x_0,t_0) \le C(1+\sup_{\partial\Omega\times[0,T)}|w_{\nu\nu}|), \tag{18}$$

where $C = C(\lambda,\ \mu_1,\ \|u_0\|_{C^2(\Omega)},\ \|\phi\|_{C^2(\overline{\Omega})},\ \tilde{C},\ \|\nabla w\|_{C^0(\overline{\Omega}\times[0,\,T))},\ B)$.

If ξ_0 is not a tangential vector, we can also bound it now. Similar as the discussion above, we denoted by $\xi_0^\top$ the tangential part of ξ_0 and $\xi_0^\perp$ the perpendicular part of ξ_0 to $\partial\Omega$. we then have by (16) and (18) that

$$\begin{aligned}w_{\xi_0\xi_0} =&w_{\xi_0^\top\xi_0^\top}+2w_{\xi_0^\top\xi_0^\perp}+w_{\xi_0^\perp\xi_0^\perp}\\ \le&\ C(1+|w_{\nu\nu}|)\end{aligned}$$

Combining these cases together, we derive that

$$\sup_{\Omega_\sigma\times[0,T')}|\nabla^2 w| \le C(1+\sup_{\partial\Omega\times[0,T)}|w_{\nu\nu}|),$$

where $C = C(\lambda,\mu_1,\Omega,n,\varphi,u_0,\|\nabla w\|_{C^0(\overline{\Omega}\times[0,\,T))})$, but independent of T'. So we have completed the proof of Proposition 6. $\square$

Proposition 7. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary, $n \ge 2$. Suppose $w \in C^{4,2}(\Omega\times[0,T))$ is a solution to (13) for u, u_0, φ described as in Proposition 4. Then there holds*

$$\sup_{\partial\Omega\times[0,T)}|w_{\nu\nu}| \le C,$$

where C is a positive constant depending only on Ω, n, λ, μ_0, μ_1, $\|u_0\|_{C^2(\Omega)}$, $\mathrm{osc}(u_0)$, *$\|\varphi\|_{C^2(\Omega)}$.*

Proof. We will give a T' independent bound for $|w_{\nu\nu}|$ on $\partial\Omega\times[0,T']$ for any $0 < T' < T$, and then take a limit argument. Now we give an estimate for $w_{\nu\nu}$ via barrier function argument. Let

$$M_2 = \sup_{\Omega\times[0,T)}|\nabla^2 w|.$$

As before, we consider $G(x,\,t) = \sum_{i=1}^n w_i\nu^i - \frac{\varphi}{A}$ as a function defined on $\Omega_\sigma\times[0,\,T']$.

Remark that $|G| < C(\|\nabla w\|_{C^0(\Omega\times[0,T))}, u_0, \|\varphi\|_{C^0(\Omega)}) := \hat{C}$.

Let the barrier function be

$$H(x,\,t) = 4\hat{C}K(d-Kd^2)\pm G,$$

where

$$K \ge \frac{1}{2\sigma_1} \tag{19}$$

is a positive constant to be determined later.

It is obvious that

$$H = 0 \quad \text{on } \partial\Omega\times[0,T']. \tag{20}$$

Observing that if $K\sigma = \frac{1}{2}$, we have

$$H > 0 \quad \text{on } \big(\partial\Omega_\sigma\cap\Omega\big)\times[0,T']. \tag{21}$$

On $\Omega_\sigma \times \{0\}$, remark that $G(x,\,0)$ is a function only dependent of $u_0(x)$ and we may assume

$$K \geq \tilde{C} + \sqrt{\frac{\max\limits_{\overline{\Omega}} |\Delta G(x,\,0)|}{4\hat{C}}}, \tag{22}$$

where $\tilde{C}$ is from (10).

We now compute $\Delta H(x,\,0)$ on $\Omega_\sigma \times \{0\}$. Since $K\sigma = \frac{1}{2}$, we have

$$\begin{aligned} \Delta H(x,\,0) &= 4\hat{C}K(\Delta d - 2Kd\Delta d - 2K) \pm \Delta G \\ &\leq 4\hat{C}K(\tilde{C} - 2K) \pm \Delta G \\ &\leq -4\hat{C}K^2 \pm \Delta G \leq 0. \end{aligned}$$

Joint with the fact $H(x,\,0) \geq 0$ on $\partial\Omega_\sigma$ by (20) and (21), we deduce that

$$H > 0 \quad \text{on} \quad \Omega_\sigma \times \{0\}.$$

We now set out to consider the function $H(x,\,t)$ on the domain $\Omega_\sigma \times (0,\,T']$. Denoted by $F^{ij} = \frac{\partial}{\partial r_{ij}}\,|_{r=A\nabla^2 w}\,F(r)$, then on $\Omega_\sigma \times (0,T']$,

$$\begin{aligned} &\sum_{i,j=1}^{n} F^{ij}G_{ij} - G_t \\ =&\sum_{i,j,k=1}^{n} F^{ij}w_{ijk}\nu^k - \sum_{k=1}^{n} w_{kt}\nu^k + \sum_{i,j,k=1}^{n} F^{ij}(w_{ik}\nu_j^k + w_{jk}\nu_i^k) - \sum_{i,j=1}^{n}\frac{1}{A}F^{ij}\varphi_{ij} \\ =&\sum_{i,j,k=1}^{n} F^{ij}(w_{ik}\nu_j^k + w_{jk}\nu_i^k) - \frac{1}{A}\sum_{i,j=1}^{n} F^{ij}\varphi_{ij}, \end{aligned}$$

Hence,

$$|\sum_{i,j=1}^{n} F^{ij}G_{ij} - G_t| \leq C_2(\mu_1, \Omega, n, \|u_0\|_{C^2(\Omega)}, \|\varphi\|_{C^2(\Omega)})(1 + M_2).$$

It follows that on $\Omega_\sigma \times (0,\,T']$

$$\begin{aligned} &\sum_{i,j=1}^{n} F^{ij}H_{ij} - H_t \\ =&4\hat{C}K\sum_{i,j=1}^{n} F^{ij}(d_{ij} - 2Kd_id_j - 2Kdd_{ij}) \pm (\sum_{i,j=1}^{n} F^{ij}G_{ij} - G_t) \\ \leq&4\hat{C}K(\mu_1\tilde{C} - 2K\lambda) + C_2(1 + M_2), \\ \leq&-4\hat{C}\lambda K^2 + C_2(1 + M_2), \\ \leq&0 \end{aligned}$$

provided

$$K \geq \frac{\mu_1\tilde{C}}{\lambda} + \sqrt{\frac{C_2(1 + M_2)}{4\lambda\hat{C}}}. \tag{23}$$

Combing (19), (22) and (23), we set

$$K=\frac{1}{2\sigma_1}+\frac{\mu_1\tilde{C}}{\lambda}+\tilde{C}+\sqrt{\frac{C_2(1+M_2)}{4\lambda\hat{C}}}+\sqrt{\frac{\max\limits_{\overline{\Omega}}|\Delta G(x,0)|}{4\hat{C}}}$$

and

$$\sigma=\frac{1}{2K},$$

then

$$H_\nu\geq 0\quad\text{on } \partial\Omega\times[0,T'].$$

On the other hand, we have

$$\begin{aligned}H_\nu=&4\hat{C}K\pm G_\nu\\=&4\hat{C}K\pm(w_{kl}\nu^k\nu^l+w_k\nu^k_l\nu^l-\frac{1}{A}\varphi_l\nu^l).\end{aligned}$$

Hence we obtain by Proposition 6 that for any $(x,t)\in\Omega_\sigma\times[0,T']$,

$$|w_{\nu\nu}|\leq C\sqrt{1+M_2}\leq C\sqrt{1+|w_{\nu\nu}|},$$

Thus,

$$|w_{\nu\nu}|\leq C.$$

and we complete the proof of Proposition 7. □

Proof of Lemma 3.2. We now can continue the proof of Lemma 3.2. Almost the same proceeding as the final part of the proof of Lemma 2.2, we can derive by (F1), (F2) and (F4) the following uniformly parabolic differential equation

$$\begin{cases}\frac{\partial g}{\partial s}-F_\infty(\nabla^2 g)=0 & \text{in } \Omega\times[0,1],\\ g(x,0)=g_0(x) & \text{on } \overline{\Omega}\times\{0\},\\ \frac{\partial g}{\partial\nu}=0 & \text{on } \partial\Omega\times[0,1],\end{cases}\tag{24}$$

where $g_0(x)$ is a continuous function defined on $\overline{\Omega}$ with $g_0(x_0)=0$ and $\sup\limits_{x\in\Omega}|g_0(x)|\leq 1$.

It follows from $F_\infty(0)=0$ which is a corollary of (F1) that (24) can also be expressed as

$$\begin{cases}\frac{\partial g}{\partial s}-\sum_{i,j=1}^n\int_0^1 F^{ij}_\infty(t\nabla^2 g)\mathrm{d}t\cdot g_{ij}=0 & \text{in } \Omega\times[0,1],\\ g(x,0)=g_0(x) & \text{on } \overline{\Omega}\times\{0\},\\ \frac{\partial g}{\partial\nu}=0 & \text{on } \partial\Omega\times[0,1].\end{cases}$$

However, just as the proof of Lemma 2.2 we have that $g(x_0,\,s)=0$ for $s\in[0,\,1]$ and $|g(\bar{x},\,1)|=1$ for some $\bar{x}\in\overline{\Omega}$. This also violates the maximum principle and Hopf Lemma for the parabolic differential equations. Hence we derive (12) and complete the proof of Lemma 3.2. □

Theorem 3.3. *Suppose u is a smooth solution to (11), then we have the estimate*

$$\|u_t(\cdot,t)\|_{C(\overline{\Omega})}+\|\nabla u(\cdot,t)\|_{C(\overline{\Omega})}+\|\nabla^2 u(\cdot,t)\|_{C(\overline{\Omega})}\leq C,$$

where C is a constant independent of t and T.

Proof. The equation for v is

$$\begin{cases} v_t - F(\nabla^2 v) = -u_t(x_0, t) & \text{in } \Omega \times (0, T), \\ v_\nu = \varphi & \text{on } \partial\Omega \times (0, T), \\ v(x, 0) = u_0(x) - u_0(x_0) & \text{in } \Omega. \end{cases}$$

By Lemma 3.2 we have $|v| \leq A_0$, a similar proceeding as Proposition 3 and Proposition 4 gives

$$\|v\|_{C^2(\Omega\times[0,T))} \leq C.$$

Combining this with Lemma 3.1 gives that

$$\|u_t(\cdot, t)\|_{C(\overline{\Omega})} + \|\nabla u(\cdot, t)\|_{C(\overline{\Omega})} + \|\nabla^2 u(\cdot, t)\|_{C(\overline{\Omega})} \leq C, \quad t \in [0, T).$$

This finishes the proof of Theorem 3.3. □

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