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Shakedown analysis of shape memory alloy structures

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Abstract

Phase transformational shakedown of a structure refers to a status that plastic strains cease developing after a finite number of loading cycles, and subsequently the structure undergoes only elastic deformation and alternating phase transformations with limited magnitudes. Due to the intrinsic complexity in the constitutive relations of shape memory alloys (SMA), there is as yet a lack of effective methods for modeling the mechanical responses of SMA structures, especially when they develop both phase transformation and plastic deformation. This paper is devoted to present an algorithm for analyzing shakedown of SMA structures subjected to cyclic or varying loads within specified domains. Based on the phase transformation and plastic yield criteria of von Mises-type and their associated flow rules, a simplified three-dimensional phenomenological constitutive model is first formulated accounting for different regimes of elastic–plastic deformation and phase transformation. Different responses possible for SMA bodies exposed to varying loads are discussed. The classical Melan shakedown theorem is extended to determine a lower bound of loads for transformational shakedown of SMA bodies without necessity of a step-by-step analysis along the loading history. Finally, a simple example is given to illustrate the application of the present theory as well as some basic features of shakedown of SMA structures. It is interesting to find that phase transformation may either increase or decrease the load-bearing capacity of a structure, depending upon its constitutive relations, geometries and the loading mode.
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1. Introduction

Shape memory alloys (SMAs) such as NiTi and CuZnAl possess the ability to recover their original shapes upon heating or unloading even after severe deformation, i.e., the properties of shape memory and superelasticity (or pseudoelasticity). The underlying mechanisms of these observed phenomena are either the crystalline structural transformation from the austenite to the martensite phase and the reverse process or the reorientations of different martensite variants. Owing to these unique and remarkable characteristics, SMAs have been applied in a number of different fields such as medical engineering and smart structures (Seelecke and Müller, 2004). Some micromechanical and phenomenological models have been developed for simulating the complex constitutive responses under mechanical and/or thermal loadings (Sun and Hwang, 1993a,b; Brinson, 1993; Shaw and Kyriakides, 1995; Birman, 1997; Qidwai and Lagoudas, 2000; Lagoudas and Entchev, 2004; Liu et al., 2006; Müller and Bruhns, in press). When subjected to an applied stress higher than the yield stress, an SMA may also develop plastic deformation, besides elastic and transformational deformations. Considerable interest has been attracted to investigate theoretically and experimentally the plastic constitutive relations of SMAs (Fischer et al., 1996; Levitas, 1998, 2002; Iwamoto, 2004; Qian et al., 2005). However, there is as yet a lack of effective methods for analyzing the coupled behavior of elasticity, plasticity and phase transformation of SMA structures, especially in the case of cyclic or complex loading.

For elastic–plastic structures or components subjected to mechanical and/or thermal loads varying with time, shakedown presents as a necessary condition of safety assessment (Koiter, 1960; König, 1987). A structure in a non-shakedown or inadaptation condition under varying loads may fail by one of the two failure modes, namely alternating plasticity (or low cycle fatigue) and incremental plastic collapse (or ratcheting). Both alternating and incremental plastic deformations cause successive accumulation of damage associated with nucleation, growth and coalescence of microcracks or microvoids. Therefore, of special interest is to evaluate from the viewpoint of damage mechanics whether a structure will shake down or not under a given load domain (Hachemi and Weichert, 1992; Feng and Yu, 1994, 1995; Druyanov and Roman, 1998, 2004; Weichert and Hachemi, 1998; Polizzotto et al., 2001). The shakedown of a body indicates that the damage stops evolving after a finite number of loading cycles.

The methods of shakedown analysis came into existence in the 1930s. Melan (1938) and Koiter (1956) proved the two crucial shakedown theorems, namely the static shakedown theorem (also referred to as Melan or lower bound shakedown theorem) and the kinematic shakedown theorem (also referred to as Koiter or upper bound shakedown theorem), which constitute the backbone of shakedown theory of elastoplastic structures. Accordingly, the numerous methods of shakedown analysis developed thereafter can be divided into two classes, static and kinematic. In the past decades, the shakedown theory has developed rapidly, especially in the following aspects. Firstly, the classical shakedown theorems, originally proved under the simplifying assumptions of geometric linearity and elastic–perfectly plastic constitutive relations obeying the associated flow law, have been extended to

broader classes of problems accounting for the effects of high temperature, strain- or work-hardening, nonlinear geometry, dynamics, damage and non-associated plastic constitutive relations, among others (König and Maier, 1981; König, 1987; Polizzotto, 1982; Gross-Weege, 1990; Pham, 1992, 2001; Pycko and Maier, 1995; Feng and Liu, 1996; Feng and Gross, 1999; Weichert and Maier, 2000; Bousshine et al., 2003; Nguyen, 2003). Secondly, various theoretical and numerical shakedown analysis methods have been established for solving technologically important problems, among which finite element and boundary element methods often play a significant role (e.g., Gross-Weege, 1997; Stein et al., 1993; Zouain et al., 2002; Vu et al., 2004; Abdel-Karim, 2005; Liu et al., 2005). Recently, considerable attention has been paid on damage and shakedown behavior of heterogeneous materials or composites using macro/micro or multiscale numerical approaches (Derrien et al., 1999; Zouain and Silveira, 1999; Li et al., 2003; Magoariec et al., 2004). Thirdly, the shakedown theory has been applied with success in a number of engineering problems such as the construction of nuclear reactors, highways and railways, and employed as one of the tools of structural design and safety assessment in some design standards, rules and regulations (Weichert and Maier, 2000; Maier, 2001). Shakedown theory also provides an effective tool for understanding the physical mechanisms of friction and fretting wear of materials (Anderson and Collins, 1995; Ambrico and Begley, 2000).

SMA structures used in engineering are usually subjected to loads that are cyclic or varying within prescribed bounds. In order to prevent failure of such a structure, it is obviously necessary to judge whether it will shake down or inadapt within these bounds, or in other words, to determine a load domain within which the structure will be safe. Therefore, the present paper is aimed to develop a method for analyzing the behavior of SMA bodies under varying loads with time. The static shakedown analysis method is used here because it leads to a lower bound (a conservative estimate) of the shakedown limit load. The outline of the paper is as follows. Presented in Section 2 is a simplified phenomenological constitutive model that provides a description of the most essential features for SMAs. The associated flow rule is assumed for both phase transformation and plastic strains. In Section 3, possible mechanical responses of SMA bodies subjected to varying loads are discussed. In Section 4, the classical Melan static shakedown theorem is extended to SMA structures with alternating phase transformations. Finally, an illustrative example is given to show the application of the present theory and to examine the influence of phase transformation on shakedown behavior.

2. Constitutive relation of SMAs

Constitutive relations of SMAs, especially those of polycrystals used in most real applications, are intrinsically complicated. Though some constitutive models of great theoretical interest have been established based on micromechanics or non-equilibrium thermodynamics (Sun and Hwang, 1993a,b; Brinson, 1993; Shaw and Kyriakides, 1995; Birman, 1997; Qidwai and Lagoudas, 2000; Lagoudas and Entchev, 2004), they are generally difficult to be applied in analysis of actual engineering structures. Therefore, some approximate simplifications are necessary for implementing them in simulation of mechanical responses of SMA structures, especially for cases of complex loading. Here, we present a simplified three-dimensional elastic-transformational–plastic constitutive relation of SMAs, which is tractable from both analytical and computational viewpoints. Consider an initially isotropic polycrystalline SMA material, where all the grains are

completely randomly oriented at the initial state. The characteristic sizes of grains are so small that a material point at the macroscopic scale can be considered as an ensemble of many grains. The deformation process is assumed quasi-static and isothermal so the effect of transformation induced temperature change and the dynamic effect can be neglected. Assume further that the magnitudes of deformation and strains are relatively small, and then we employ the symmetric Cauchy stress tensor σ_{ij} and the strain tensor $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, where u_i is the displacement vector. A comma preceding an index i stands for the partial differentiation with respect to the coordinate x_i in a Cartesian coordinate system. Small strain formulations allow us to easily understand key ideas without unnecessary formal complications and to derive simpler analytical or numerical solutions with rather good accuracy for many engineering structures. The small strain formulations can be extended to the case of finite strains along the lines presented by, e.g., Levitas (1998, 2002), among others.

2.1. Uniaxial stress–strain relations

To illustrate some basic features of the simplified constitutive model adopted here, the uniaxially tensile stress–strain relation of an SMA undergoing phase transformation and elastic–plastic deformation is first formulated. The loading–unloading curves are schematized in Fig. 1(a) and (b), without and with plastic deformation, respectively. The constitutive relation under monotonically proportional loading in Fig. 1(b) contains four stages, including elasticity (OA), transformation from the austenitic to the martensitic phase
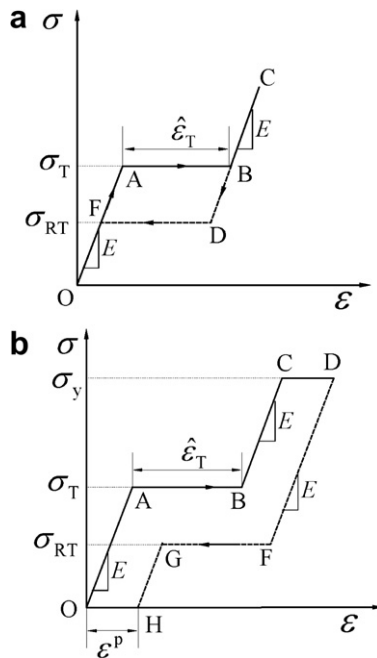

Fig. 1. Simplified stress–strain relation of SMAs under uniaxial tension: (a) without and (b) with plastic deformation.

(AB), post-transformational elasticity (BC), and plasticity (CD). In the case of uniaxial tension, the stress–strain relations of these four regimes are written as

$$\sigma = \begin{cases} E\varepsilon & \text{for } \varepsilon \leqslant \sigma_\text{T}/E, \\ \sigma_\text{T} & \text{for } \sigma_\text{T}/E \leqslant \varepsilon \leqslant \sigma_\text{T}/E + \hat{\varepsilon}_\text{T}, \\ E(\varepsilon - \hat{\varepsilon}_\text{T}) & \text{for } \sigma_\text{T}/E + \hat{\varepsilon}_\text{T} \leqslant \varepsilon \leqslant \sigma_\text{y}/E + \hat{\varepsilon}_\text{T}, \\ \sigma_\text{y} & \text{for } \varepsilon \geqslant \sigma_\text{y}/E + \hat{\varepsilon}_\text{T}, \end{cases} \tag{1}$$

where E denotes the Young's modulus, σ_T the critical stress of forward phase transformation onset, σ_y the plastic yield stress, and $\hat{\varepsilon}_\text{T}$ the saturation value of the transformation strain.

The unloading constitutive responses of an SMA include three regimes, corresponding to DF, FG and GH in Fig. 1(b), respectively. Similarly to Eq. (1), the corresponding stress–strain relations are expressed as

$$\sigma = \begin{cases} E(\varepsilon - \hat{\varepsilon}_\text{T} - \varepsilon_\text{p}) & \text{for } \varepsilon \geqslant \sigma_\text{RT}/E + \varepsilon_\text{p} + \hat{\varepsilon}_\text{T}, \\ \sigma_\text{RT} & \text{for } \sigma_\text{RT}/E + \varepsilon_\text{p} \leqslant \varepsilon \leqslant \sigma_\text{RT}/E + \varepsilon_\text{p} + \hat{\varepsilon}_\text{T}, \\ E(\varepsilon - \varepsilon_\text{p}) & \text{for } \varepsilon_\text{p} \leqslant \varepsilon \leqslant \sigma_\text{RT}/E + \varepsilon_\text{p}, \end{cases} \tag{2}$$

where σ_RT stands for the critical stress of onset of reverse phase transformation, and ε_p the plastic strain developed before unloading. The critical stresses σ_T and σ_RT show more pronounced dependences on temperature than σ_y (Shaw and Kyriakides, 1995). Without loss of generality, it might be assumed that $\sigma_\text{y} > \sigma_\text{T} > \sigma_\text{RT} \geqslant 0$, that is, the SMA has the property of superelasticity under the considered thermomechanical condition. We also assume that an SMA cannot develop plastic deformation until the phase transformation process has been fully completed and the transformation strain has become saturated. In the case of isothermal or quasi-static mechanical loading, all the three parameters, σ_y, σ_T and σ_RT, may be considered as material constants depending upon the service temperature of the considered structure.

In the following subsections, the above constitutive model will be reformulated in a three-dimensional incremental form.

2.2. Elastic and plastic strains

The total strain tensor ε_{ij} of an SMA material is additively decomposed as

$$\varepsilon_{ij} = \varepsilon_{ij}^\text{e} + \varepsilon_{ij}^\text{p} + \varepsilon_{ij}^\text{tr}, \tag{3}$$

where $\varepsilon_{ij}^\text{e}$, $\varepsilon_{ij}^\text{p}$ and $\varepsilon_{ij}^\text{tr}$ denote the elastic, plastic and transformation strains, respectively.

The elastic strain rates $\dot{\varepsilon}_{ij}^\text{e}$ are related to the stress rates $\dot{\sigma}_{ij}$ by Hooke's law, i.e., $\dot{\varepsilon}_{ij}^\text{e} = S_{ijkl}^\text{e}\dot{\sigma}_{kl}$, where S_{ijkl}^e is the elastic compliance tensor. It is generally reasonable to assume that neither phase transformation nor plastic deformation affects S_{ijkl}^e and hence S_{ijkl}^e remain constants for the whole deformation process.

The onset of plastic deformation is determined by the von Mises yield condition defined in terms of the equivalent stress $\sigma_\text{eq} = (3J_2)^{1/2} = \left(\frac{3}{2}s_{ij}s_{ij}\right)^{1/2}$ as

$$\psi_\text{p}(\sigma_{ij}) = \sigma_\text{eq} = (3J_2)^{1/2} = \sigma_\text{y}, \tag{4}$$

where σ_y is the plastic yield stress of Fig. 1, $s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$ is the deviatoric stress tensor with δ_{ij} being the Kronecker delta, and $J_2 = \frac{1}{2}s_{ij}s_{ij}$ is the second invariant of s_{ij}. The von Mises yield surface is shown in the π-plane in Fig. 2. Assuming that the SMA is perfectly plastic and obeys the associated flow rule, the plastic strain rate is given by

$$\dot{\varepsilon}_{ij}^{\mathrm{p}} = \lambda_{\mathrm{p}}\frac{\partial\psi_{\mathrm{p}}}{\partial\sigma_{ij}} = \lambda_{\mathrm{p}}\frac{3s_{ij}}{2\sigma_{\mathrm{eq}}}, \tag{5}$$

where λ_{p} is the multiplier of plastic strains (Koiter, 1960).

2.3. Phase transformation strains

A macroscopic criterion of phase transformation onset for an SMA plays the same role as that of the plastic yield criterion for an elastoplastic material, and describes the domain boundary of elastic deformation, within which no phase transformation occurs. For an isotropic material, the criterion of forward phase transformation (A → M) onset is dependent upon only three independent invariants of the stress tensor σ_{ij} (Raniecki and Lexcellent, 1998), which can be chosen, for example, as

$$J_1(\sigma_{ij}) = \sigma_{\mathrm{m}} = \frac{1}{3}\sigma_{ii}, \quad J_2(\sigma_{ij}) = \frac{1}{3}\sigma_{\mathrm{eq}}^2 = \frac{1}{2}s_{ij}s_{ij}, \quad J_3(\sigma_{ij}) = \det(s_{ij}), \tag{6}$$

where σ_{m} is the hydrostatic stress. Generally, the hydrostatic stress has little influence on the occurrence of phase transformation in SMAs and plastic deformation. In addition, some experimental observations demonstrate that J_2 plays a dominant role in onset of phase transformation in comparison with J_3 (e.g., Raniecki and Lexcellent, 1998). Therefore, a J_2-based (von Mises-type) criterion of forward phase transformation is adopted here as

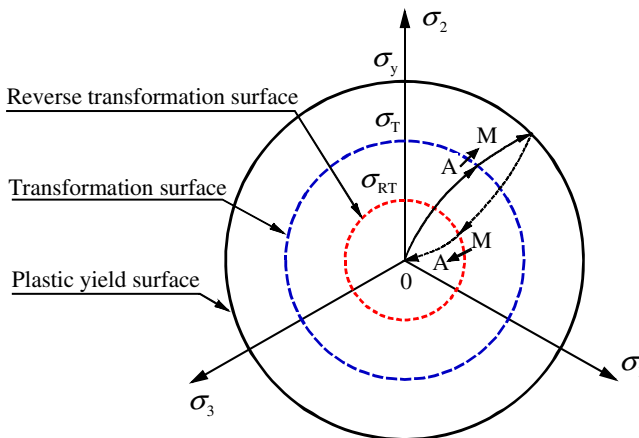$$\psi_{\mathrm{T}}(\sigma_{ij}) = \sigma_{\mathrm{eq}} = (3J_2)^{1/2} = \sigma_{\mathrm{T}}, \tag{7}$$

Fig. 2. Forward and reverse phase transformation surfaces and plastic yield surface in the principal stress space (π-plane).

where σ_T denotes the critical value of von Mises equivalent stress for A → M transformation to occur. The smooth transformation surface in Eq. (7), which satisfies the requirement of strict convexity, is schematized in the isoclinal π-plane of the principal stress space in Fig. 2. The presented formulations can readily be extended to other types of phase transformation surfaces such as those with corners.

No strain hardening during phase transformation is considered here for simplicity. Similar to the classical theory of plasticity, the transformation strain rates follow the normality rule, i.e., they are normal to the transformation surface in the stress space as

$$\dot{\varepsilon}_{ij}^{\mathrm{tr}} = \lambda_T \frac{\partial \psi_T}{\partial \sigma_{ij}}, \tag{8}$$

where the flow multiplier λ_T satisfies

$$\lambda_T = 0 \quad \text{if (i) } \psi_T < \sigma_T, \text{ or (ii) } \psi_T = \sigma_T \text{ and } \dot{\psi}_T = \frac{\partial \psi_T}{\partial \sigma_{ij}} \dot{\sigma}_{ij} < 0,$$

$$\lambda_T \geqslant 0 \quad \text{if } \psi_T = \sigma_T \text{ and } \dot{\psi}_T = 0. \tag{9}$$

In the latter case in Eq. (9), the value of λ_T should be determined by the consistency condition.

Substituting Eq. (7) into (8) leads to

$$\dot{\varepsilon}_{ij}^{\mathrm{tr}} = \lambda_T \frac{3 s_{ij}}{2 \sigma_{\mathrm{eq}}}. \tag{10}$$

It is also seen from Eq. (10) that the phase transformation of an SMA is volume-invariant, as is consistent with experiment results (see, e.g., Orgéas and Favier, 1998; Bouvet et al., 2004).

Different from traditional plasticity, phase transformations of SMAs result from the diffusionless change in crystallographic structures. Once the transformation process has been completed, the transformation strains will reach its saturation value ($\approx 5\%$ for NiTi SMA) though forward and reverse phase transformations can be repeated for many times.

An equivalent transformation strain and an equivalent transformation strain rate are defined as

$$\varepsilon_{\mathrm{eq}}^{\mathrm{tr}} = \left(\frac{2}{3} \varepsilon_{ij}^{\mathrm{tr}} \varepsilon_{ij}^{\mathrm{tr}} \right)^{\frac{1}{2}} = \left(\frac{2}{3} \varepsilon_{ij}'^{\mathrm{tr}} \varepsilon_{ij}'^{\mathrm{tr}} \right)^{\frac{1}{2}}, \tag{11}$$

$$\dot{\varepsilon}_{\mathrm{eq}}^{\mathrm{tr}} = \left(\frac{2}{3} \dot{\varepsilon}_{ij}^{\mathrm{tr}} \dot{\varepsilon}_{ij}^{\mathrm{tr}} \right)^{\frac{1}{2}} = \left(\frac{2}{3} \dot{\varepsilon}_{ij}'^{\mathrm{tr}} \dot{\varepsilon}_{ij}'^{\mathrm{tr}} \right)^{\frac{1}{2}}, \tag{12}$$

where $\varepsilon_{ij}' = \varepsilon_{ij} - \frac{1}{3} \varepsilon_{kk} \delta_{ij}$ is the deviatoric strain tensor. Then the transformation strain is limited by

$$\varepsilon_{\mathrm{eq}}^{\mathrm{tr}} \leqslant \hat{\varepsilon}_T, \tag{13}$$

where $\hat{\varepsilon}_T$ is the maximum equivalent transformation strain. $\hat{\varepsilon}_T$ is considered to be a material constant ($\sim 5\%$ for NiTi SMA) and can be obtained from experiment such as the uniaxial tensile stress–strain curve shown in Fig. 1. Substituting Eq. (10) into (12), we immediately see that

$$\lambda_T = \dot{\varepsilon}_{\mathrm{eq}}^{\mathrm{tr}}. \tag{14}$$

In addition, it is easy to understand that analogously to the constitutive theory of perfectly plastic materials, for an SMA with perfectly plastic type transformation behavior as we studied in this paper, if $\dot{\sigma}_{ij}$ are the stress rates corresponding to the transformation strain rates $\dot{\varepsilon}_{ij}^{\mathrm{tr}}$, one always has

$$\dot{\sigma}_{ij}\dot{\varepsilon}_{ij}^{\mathrm{tr}} = 0. \tag{15}$$

After the forward $(\mathrm{A} \to \mathrm{M})$ transformation is fully completed, the material can bear a further increase in the applied stress. In the case of proportional loading, the SMA with saturated $\mathrm{A} \to \mathrm{M}$ transformation strain exhibits again a linearly elastic stress–strain response, corresponding to the BC regime in Fig. 1 under uniaxial tension.

When non-proportional loads are applied to the SMA, however, the martensite phase in some grains with certain orientations may deform by converting one variant to another which is energetically more favorable under the given stress state. Such conversions among different martensite variants result in the reorientation of transformation strains at the macroscopic scale. Since the characteristic sizes of grains are assumed to be sufficiently small, the reorientation process of the martensite phases in the SMA with saturated phase transformation is considered to be continuous. The transformation strain tensor in parallel with the deviatoric stress tensor is the most favorable energetically. Based on these considerations, we assume that the components of the transformation strain tensor are proportional to the deviatoric stresses, i.e.,

$$\varepsilon_{ij}^{\mathrm{tr}} = \varepsilon_{ij}^{\prime\mathrm{tr}} = \alpha \frac{3s_{ij}}{2\sigma_{\mathrm{eq}}}. \tag{16}$$

Considering the saturation value of the equivalent transformation strain $\varepsilon_{\mathrm{eq}}^{\mathrm{tr}} = \varepsilon_{\mathrm{T}}$, the coefficient α in the BC regime is determined as

$$\alpha = \varepsilon_{\mathrm{T}}. \tag{17}$$

This relation holds for both the loading and unloading regimes provided that the forward $(\mathrm{A} \to \mathrm{M})$ phase transformation has been fully completed and the reverse $(\mathrm{M} \to \mathrm{A})$ phase transformation has not started.

2.4. Reverse phase transformation

During unloading, the material will experience first elastic deformation and, probably, reorientations of martensite variants. The change in the transformation strains induced by reorientations of martensite variants is still described by Eqs. (16) and (17).

When the stress is unloaded to the reverse transformation surface, the martensitic phase will transform to the austenitic phase. Similar to Eq. (7), the reverse transformation surface is assumed as

$$\psi_{\mathrm{RT}}(\sigma_{ij}) = \sigma_{\mathrm{eq}}(\sigma_{ij}) = \sigma_{\mathrm{RT}}, \tag{18}$$

where σ_{RT} stands for the critical von Mises stress for the onset of reverse phase transformation $(\mathrm{M} \to \mathrm{A})$. As a result of the hysteresis effect, the critical stress σ_{RT} of reverse transformation is smaller than that of forward transformation, σ_{T}.

The reverse transformation strain rate tensor is also required by the associated flow rule to be normal to the surface defined by Eq. (18). Different from the forward transformation

and plastic strains, the reverse transformation strains are directed inward to the transformation surface. Therefore, one has

$$\dot{\varepsilon}_{ij}^{\mathrm{tr}} = -\lambda_{\mathrm{RT}} \frac{\partial \psi_{\mathrm{RT}}}{\partial \sigma_{ij}}. \tag{19}$$

The flow multiplier λ_{RT} is determined from the following relations as well as the consistency condition

$$\lambda_{\mathrm{RT}} = 0 \quad \text{if (i) } \psi_{\mathrm{RT}} > \sigma_{\mathrm{RT}}, \text{ or (ii) } \psi_{\mathrm{RT}} = \sigma_{\mathrm{RT}} \text{ and } \dot{\psi}_{\mathrm{RT}} = \frac{\partial \psi_{\mathrm{RT}}}{\partial \sigma_{ij}} \dot{\sigma}_{ij} > 0,$$

$$\lambda_{\mathrm{RT}} \geqslant 0 \quad \text{if } \psi_{\mathrm{RT}} = \sigma_{\mathrm{RT}} \text{ and } \dot{\psi}_{\mathrm{RT}} = 0. \tag{20}$$

3. Response of SMA structures under variable loading

Consider a three-dimensional SMA structure occupying the volume V surrounded by the surface S and subjected to quasi-static loads varying within prescribed domains. Throughout the present paper, it is assumed that there will be no stress singularity in the structure, which may be caused by, e.g., cracks or sharp notches. The boundary conditions of the structure are expressed as

$$\sigma_{ij}(\mathbf{x}, t) n_j = T_i(\mathbf{x}, t) \quad \text{for } \mathbf{x} \in S_{\mathrm{T}}, \tag{21}$$
$$u_i(\mathbf{x}, t) = \bar{u}_i(\mathbf{x}, t) \quad \text{for } \mathbf{x} \in S_u, \tag{22}$$

where the vector $\mathbf{x}$ denotes the coordinate, $T_i(\mathbf{x}, t)$ the tractions specified on S_{T}, and $\bar{u}_i(\mathbf{x}, t)$ the displacements specified on S_u, with $S_{\mathrm{T}} \cup S_u = S$.

For an SMA structure undergoing elastic, plastic and transformational deformations, the total stress field $\sigma_{ij}(\mathbf{x}, t)$ can be decomposed as

$$\sigma_{ij}(\mathbf{x}, t) = \sigma_{ij}^{\mathrm{e}}(\mathbf{x}, t) + \rho_{ij}^{\mathrm{p}}(\mathbf{x}, t) + \rho_{ij}^{\mathrm{tr}}(\mathbf{x}, t), \tag{23}$$

where $\sigma_{ij}^{\mathrm{e}}(\mathbf{x}, t)$ denotes the elastic stress field induced by the applied loads in a fictitious linear elastic structure with the same geometry as the actual one, $\rho_{ij}^{\mathrm{p}}(\mathbf{x}, t)$ and $\rho_{ij}^{\mathrm{tr}}(\mathbf{x}, t)$ stands for the stress changes (or residual stresses) due to plastic deformation and phase transformation, respectively. The total stresses $\sigma_{ij}(\mathbf{x}, t)$ satisfy the equilibrium equation

$$\sigma_{ij,j}(\mathbf{x}, t) + f_i(\mathbf{x}, t) = 0, \tag{24}$$

throughout the structure, where $f_i(\mathbf{x}, t)$ denote the body forces. The stress fields $\rho_{ij}^{\mathrm{p}}(\mathbf{x}, t)$ and $\rho_{ij}^{\mathrm{tr}}(\mathbf{x}, t)$ are self-equilibrating, that is,

$$\rho_{ij,j}^{\mathrm{p}}(\mathbf{x}, t) = 0, \quad \rho_{ij,j}^{\mathrm{tr}}(\mathbf{x}, t) = 0 \quad \text{for all } \mathbf{x} \in V, \tag{25}$$
$$\rho_{ij}^{\mathrm{p}}(\mathbf{x}, t) n_j = 0, \quad \rho_{ij}^{\mathrm{tr}}(\mathbf{x}, t) n_j = 0 \quad \text{for all } \mathbf{x} \in S_{\mathrm{T}}. \tag{26}$$

The varying loads are applied to the structure via the quasi-static change of $f_i(\mathbf{x}, t)$, $T_i(\mathbf{x}, t)$ and $\bar{u}_i(\mathbf{x}, t)$ with time t.

The presence of phase transformation and plastic deformation makes the mechanical responses of SMA structures more complicated than elastoplastic structures, especially when they are subjected to cyclic or stochastically varying loads. Generally, the following six different responses of deformation and failure are possible for an SMA structure.

(i) *Purely elastic response.* If the loads vary within sufficient small domains, the deformation of the structure will be purely elastic, and neither phase transformation nor plastic strain will occur. Such purely elastic response is the most desirable in design of most engineering structures.

(ii) *Elastic shakedown.* If both phase transformation and plastic deformation cease developing further after some initial loading cycles, the structure will behave as a completely elastic one in the subsequent cycles. Such constitutive response with stabilized plastic deformation and phase transformation is called elastic shakedown or adaptation, as shown in Fig. 3(a) and (b), where P and Δ designate a load and the corresponding displacement, respectively. However, it will be shown in Section 4 that due to reverse phase transformation, elastic shakedown seldom occurs in SMA structures with $\sigma_{\mathrm{RT}} \geqslant 0$ as long as the applied elastic stresses violate the criterion of forward phase transformation.

(iii) *Alternating phase transformation.* A typical response for SMA structures under cyclic loading is that both forward and reverse phase transformations happen in every loading cycle and they tend to cancel each other out. Thus, the total transformation strains remain small though phase transformation does not stop evolving. Fig. 3(c) schematizes a cyclic load–displacement curve of an alternating phase transformation process, where no plastic deformation occurs in the structure within the given load domains.

Alternating transformation may also happen in a body if it has developed some plastic deformation in a finite number of initial loading cycles, as shown in Fig. 3(d). The plastic
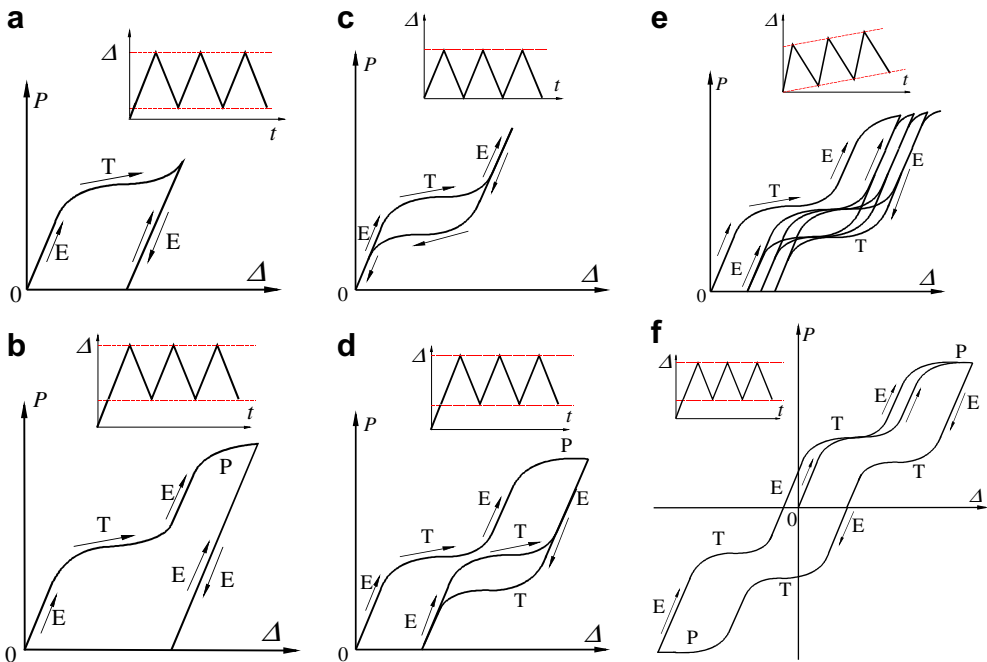

Fig. 3. Different responses of an SMA structure under varying loads: (a) elastic shakedown without plastic deformation, (b) elastic shakedown with plastic deformation, (c) alternating phase transformation without plastic deformation, (d) alternating phase transformation with plastic deformation, (e) incremental plastic collapse (ratcheting), (f) alternating plastic failure (low cycle fatigue).

deformation induces a residual stress field $\rho_{ij}^{\mathrm{p}}(\mathbf{x})$, which will not change further during the subsequent alternating transformation process. In this situation, the self-equilibrating field $\rho_{ij}^{\mathrm{tr}}(\mathbf{x},t)$ varies with time while $\rho_{ij}^{\mathrm{p}}(\mathbf{x})$ does not. The plastic deformation makes the distribution of the total stresses more uniform such that the condition

$$\psi_{\mathrm{p}}[\sigma_{ij}(\mathbf{x},t)] = \psi[\sigma_{ij}^{\mathrm{e}}(\mathbf{x},t) + \rho_{ij}^{\mathrm{p}}(\mathbf{x}) + \rho_{ij}^{\mathrm{tr}}(\mathbf{x},t)] \leqslant \sigma_{\mathrm{y}}, \tag{27}$$

is satisfied throughout the structure and for the whole load domain specified. Thus during the subsequent varying loading, the structure will behave as if it has an infinitely elastic-phase transformational constitutive relation, i.e., $\sigma_{\mathrm{y}} \to \infty$.

It is worthy of mentioning again that in contrary to plastic strains, which can be accumulated to very large values and cause incremental plastic collapse (ratcheting) of a structure, the amount of transformation strains is bounded by the saturation value, $\hat{\varepsilon}_{\mathrm{T}}$. Therefore, the phase transformation strains can evolve only in an alternating manner even if they cannot stop developing with the varying loading. Such a behavior with alternating and limited transformation strains and stabilized plastic strains may be referred to as phase transformational shakedown.

In the case of alternating transformation, the strains $\varepsilon_{ij}^{\mathrm{tr}}$ due to forward and reverse phase transformations over a complete loading cycle from time t_0 to $t_0 + T$ always cancel each other out. Then both the residual stress field ρ_{ij}^{tr} and the corresponding elastic strain field $\varepsilon_{ij}^{\mathrm{tr(e)}} = S_{ijkl}^{\mathrm{e}}\rho_{kl}^{\mathrm{tr}}$ return to their original distributions at the beginning of the cycle. Therefore, it is evident that both the works of any time-independent stress field, denoted by $\bar{\sigma}_{ij}(\mathbf{x})$, to $\dot{\varepsilon}_{ij}^{\mathrm{tr}}$ and $\dot{\varepsilon}_{ij}^{\mathrm{tr(e)}}$ over a complete loading cycle must vanish. That is,

$$\int_{t_0}^{t_0+T} \int_V \bar{\sigma}_{ij}(\mathbf{x})\dot{\varepsilon}_{ij}^{\mathrm{tr}}(\mathbf{x},t)\,\mathrm{d}V\,\mathrm{d}t = 0, \tag{28}$$

$$\int_{t_0}^{t_0+T} \int_V \bar{\sigma}_{ij}(\mathbf{x})\dot{\varepsilon}_{ij}^{\mathrm{tr(e)}}(\mathbf{x},t)\,\mathrm{d}V\,\mathrm{d}t = 0. \tag{29}$$

For most SMAs, phase transformation does not cause evident degradation in their mechanical properties (e.g., stiffness and strength). In other words, damage evolution with transformation process is generally slow. SMA structures can generally bear a very large number of phase transformation cycles without failure, as required for most engineering designs. Therefore, both the elastic shakedown and the above defined transformational shakedown are usually considered to be safe for SMA structures.

(iv) *Incremental plastic collapse*. If the plastic strain increments in each loading cycle are of the same direction, the total plastic deformation will become larger and larger with successive loading cycles, as shown in Fig. 3(e). The continuous increase in plastic strains will cause excessive deformation and damage accumulation associated with nucleation, growth and coalescence of microcracks or microvoids, and the final failure of the system. This is called incremental plastic collapse or ratcheting (König, 1987), during which alternating phase transformation usually happens simultaneously.

(v) *Alternating plasticity*. If the plastic strains change their signs and cancel each other out in each cycle of loading (Fig. 3f), the total plastic strain increments tend to be zero. Then after some initial loading cycles during which a residual stress field may form in the body as a result of plastic deformation, the stress and strain fields become cyclic. In other words, the total stresses and strains after a complete cycle of loading always return to their values (generally not zero) at the beginning of this cycle. Although the total

deformation increments tend to vanish, the amount of plastic work or dissipation accumulates infinitely with consecutive loading cycles and leads to progressive damage of materials, associated mainly with evolution of microcracks. The structure will finally fail in a rather brittle manner after a relatively small number of loading cycles. This phenomenon is referred to as alternating plastic failure, low cycle fatigue, or sometimes plastic shakedown in the literature.

(vi) *Instantaneous failure*. This happens if the applied loads are equal to or higher than the instantaneous loading carrying capacity (i.e., the plastic limit load) of a structure. When the loads reach the carrying capacity of a structure made of an elastic-transformational–perfectly plastic SMA material, plastic collapse occurs in the sense that plastic deformation may increase infinitely under constant loads and the structure cannot support any further increase in the external loads. We will not pay much attention in this paper on determination of plastic limit loads since the limit analysis may be considered as a special, simplified case of shakedown analysis discussed below.

4. Lower bound shakedown theorem

4.1. Elastic shakedown

Due to the complexity in the mechanical responses of SMA structures, which involve forward and reverse phase transformations as well as plastic deformation, an extension of the classical shakedown theorems for elastoplastic structures to SMA structures is not straightforward even when the simplified constitutive relation in Section 2 is adopted.

At first, we examine the condition of elastic shakedown. As aforementioned, two mechanisms may contribute to the residual stress field after the applied loads are completely removed. One is plastic deformation, and the other phase transformation. If an SMA body can shake down elastically with stabilized plastic deformation and transformation strains, then of course two time-independent residual stress fields $\rho_{ij}^{p}(\mathbf{x})$ and $\rho_{ij}^{tr}(\mathbf{x})$, corresponding respectively to plastic deformation and phase transformation developed in the previous loading history, must have formed such that the response of the structures becomes purely elastic in the consecutive loading cycles. Let $\bar{\varepsilon}_{ij}^{p}(\mathbf{x})$ and $\bar{\varepsilon}_{ij}^{tr}(\mathbf{x})$ denotes respectively the plastic strain and transformation strain fields corresponding to $\rho_{ij}^{p}(\mathbf{x})$ and $\rho_{ij}^{tr}(\mathbf{x})$. Clearly, the elastic shakedown requires that: (1) the criterion for plastic deformation

$$\psi_{p}[\sigma_{ij}(\mathbf{x}, t)] = \psi_{p}[\sigma_{ij}^{e}(\mathbf{x}, t) + \rho_{ij}^{p}(\mathbf{x}) + \rho_{ij}^{tr}(\mathbf{x})] \leqslant \sigma_{y} \tag{30}$$

is violated nowhere in the structure for any loading within the specified load domain, (2) the criterion for the occurrence of forward phase transformation

$$\psi_{T}[\sigma_{ij}(\mathbf{x}, t)] = \psi_{T}[\sigma_{ij}^{e}(\mathbf{x}, t) + \rho_{ij}^{p}(\mathbf{x}) + \rho_{ij}^{tr}(\mathbf{x})] \leqslant \sigma_{T} \tag{31}$$

is satisfied for all $\mathbf{x}$ where $\bar{\varepsilon}_{ij}^{tr}(\mathbf{x}) = 0$, and (3) the criterion for reverse phase transformation onset

$$\psi_{RT}[\sigma_{ij}(\mathbf{x}, t)] = \psi_{RT}[\sigma_{ij}^{e}(\mathbf{x}, t) + \rho_{ij}^{p}(\mathbf{x}) + \rho_{ij}^{tr}(\mathbf{x})] \geqslant \sigma_{RT} \tag{32}$$

is satisfied for all $\mathbf{x}$ where $\bar{\varepsilon}_{ij}^{tr}(\mathbf{x}) \neq 0$.

However, the above necessary conditions for elastic shakedown can seldom be satisfied for SMA bodies with $\sigma_{RT} \geqslant 0$ provided that the "elastic" stresses $\sigma_{ij}^{e}(\mathbf{x}, t)$ are beyond the surface of forward phase transformation, i.e.,

$$\psi_{T}[\sigma_{ij}^{e}(\mathbf{x}, t)] > \sigma_{T}, \tag{33}$$

anywhere in the structure, no matter whether plastic deformation has happened.

To illustrate this statement, for instance, we consider such simple systems as bars, beams and plane frames, which are of extensive engineering interest. The stress states at any points in such systems are approximately uniaxial. Assume that the applied elastic stress at a point $\mathbf{x}_0$ changes repeatedly, for instance, between 0 and $\sigma_{max}^{e} > \sigma_T > 0$ under the given changing domains of loads. To meet the forward transformation criterion (31), there must exist a compressive residual stress $\rho = \rho^{p} + \rho^{tr} \leqslant -(\sigma_{max}^{e} - \sigma_T)$ at this point. Then during unloading, the stress $\sigma(\mathbf{x}_0)$ will decrease from $\sigma_{max}^{e} + \rho \leqslant \sigma_T$ to $\rho \leqslant 0$. Thus, a necessary condition for non-occurrence of alternating phase transformation at $\mathbf{x}_0$ is that the critical stress σ_{RT} of reverse phase transformation is compressive and $\sigma_{RT} \leqslant -(\sigma_{max}^{e} - \sigma_T) < 0$. Similarly, if the maximum stress $\sigma_{max}^{e}(\mathbf{x}_0)$ at $\mathbf{x}_0$ is higher than σ_y, elastic shakedown requires that $\sigma_{RT} \leqslant -(\sigma_{max}^{e} - \sigma_y) < 0$. Therefore, for a structure made of a superelastic SMA with $\sigma_{RT} > 0$, one cannot find two stable fields $\rho_{ij}^{p}(\mathbf{x})$ and $\rho_{ij}^{tr}(\mathbf{x})$ such that all the three conditions in (30)–(32) are met simultaneously.

A three-dimensional demonstration can be made similarly. According to the above analysis, therefore, a residual stress field, induced either by plastic deformation or phase transformation, can seldom prevent the occurrence of alternating phase transformation and make the system shake down elastically if the applied cyclic elastic stresses violate either the criterion of phase transformation or that of plastic yielding. The occurrence of elastic shakedown with stabilized plastic deformation and phase transformation strains requires that the threshold value σ_{RT} of reverse phase transformation is compressive and sufficiently low.

4.2. Extended Melan theorem for transformational shakedown

As aforementioned, both elastic shakedown and phase transformational shakedown may be considered to be safe states for SMA structures, and the former can seldom occur except under special thermal, mechanical and structural conditions. Therefore, our emphasis is placed here on the condition of transformational shakedown of an SMA structure.

The extended lower bound or Melan shakedown theorem for phase transformational shakedown is expressed by the following two statements:

(i) *An SMA structure cannot transformationally shake down if one cannot find a time-independent residual stress field $\bar{\rho}_{ij}^{p}(\mathbf{x})$ induced by plastic deformation and a time-dependent residual stress field $\bar{\rho}_{ij}^{tr}(\mathbf{x}, t)$ induced by phase transformation such that*

$$\psi_{p}[\sigma_{ij}(\mathbf{x}, t)] = \psi[\sigma_{ij}^{e}(\mathbf{x}, t) + \bar{\rho}_{ij}^{tr}(\mathbf{x}, t) + \bar{\rho}_{ij}^{p}(\mathbf{x})] \leqslant \sigma_{y} \tag{34}$$

is satisfied throughout the structure and for the whole specified load domain.

(ii) *If one can find a time-independent residual stress field $\bar{\rho}_{ij}^{p}(\mathbf{x})$ induced by plastic deformation such that*

$$\psi_{\mathrm{p}}[\sigma_{ij}(\mathbf{x},t)] = \psi[\sigma_{ij}^{\mathrm{e}}(\mathbf{x},t) + \bar{\rho}_{ij}^{\mathrm{tr}}(\mathbf{x},t) + \bar{\rho}_{ij}^{\mathrm{p}}(\mathbf{x})] < \sigma_{\mathrm{y}} \tag{35}$$

holds throughout the structure and for the whole specified load domain, then transformational shakedown will occur in the SMA body, where $\bar{\rho}_{ij}^{\mathrm{tr}}(\mathbf{x},t)$ denotes a time-dependent residual stress field induced by phase transformation on the basis of $\bar{\rho}_{ij}^{\mathrm{p}}(\mathbf{x})$ and is calculated from the infinitely elastic-phase transformational constitutive relation with $\sigma_{y} \rightarrow \infty$.

The truth of the necessary condition for transformational shakedown, i.e., statement (i), is of course nearly self-evident. If there do not exist a time-independent residual stress field $\bar{\rho}_{ij}^{\mathrm{p}}(\mathbf{x})$ and a time-dependent residual stress field $\bar{\rho}_{ij}^{\mathrm{tr}}(\mathbf{x},t)$ such that their summation with the "elastic" stress field $\sigma_{ij}^{\mathrm{e}}(\mathbf{x},t)$ is always located in the plastic yield surface for the given load domain, then plastic deformation cannot stop developing in the consecutive loading cycles, and therefore the body will fail either by alternating plasticity or by incremental plastic collapse. The first statement can also be described as: *A necessary condition for transformational shakedown of an SMA structure is that there exists a time-independent residual stress field $\bar{\rho}_{ij}^{\mathrm{p}}(\mathbf{x})$ on the basis of which the body behaves as an infinitely elastic-transformational one with $\sigma_{\mathrm{y}} \rightarrow \infty$.*

Now we prove the second statement of the lower bound theorem, i.e., the sufficient condition of transformational shakedown. Assume that a time-independent $\bar{\rho}_{ij}^{\mathrm{p}}(\mathbf{x})$ has been found with the property that its summation with the "elastic" stress $\sigma_{ij}^{\mathrm{e}}(\mathbf{x},t)$ and the phase transformation-induced residual stress $\bar{\rho}_{ij}^{\mathrm{tr}}(\mathbf{x},t)$ satisfies the inequality (35) throughout the body and for the whole specified load domain. Let $\rho_{ij}^{\mathrm{p}}(\mathbf{x},t)$ and $\rho_{ij}^{\mathrm{tr}}(\mathbf{x},t)$ denote the actual residual stress fields associated with plastic deformation and phase transformation, respectively. Introduce a function

$$W = \int_{V} \frac{1}{2} S_{ijkl}^{\mathrm{e}} (\rho_{ij}^{\mathrm{p}} + \rho_{ij}^{\mathrm{tr}} - \bar{\rho}_{ij}^{\mathrm{p}} - \bar{\rho}_{ij}^{\mathrm{tr}})(\rho_{kl}^{\mathrm{p}} + \rho_{kl}^{\mathrm{tr}} - \bar{\rho}_{kl}^{\mathrm{p}} - \bar{\rho}_{kl}^{\mathrm{tr}}) \, \mathrm{d}V, \tag{36}$$

which is the elastic strain energy corresponding to the difference in the two residual stress fields $(\rho_{ij}^{\mathrm{p}} + \rho_{ij}^{\mathrm{tr}})$ and $(\bar{\rho}_{ij}^{\mathrm{p}} + \bar{\rho}_{ij}^{\mathrm{tr}})$ over the body. Obviously, W is always non-negative, and it equals to zero only when $\bar{\rho}_{ij}^{\mathrm{p}} + \bar{\rho}_{ij}^{\mathrm{tr}} = \rho_{ij}^{\mathrm{p}} + \rho_{ij}^{\mathrm{tr}}$. The derivative of W with respect to time t is

$$\dot{W} = \int_{V} S_{ijkl}^{\mathrm{e}} (\rho_{ij}^{\mathrm{p}} + \rho_{ij}^{\mathrm{tr}} - \bar{\rho}_{ij}^{\mathrm{p}} - \bar{\rho}_{ij}^{\mathrm{tr}})(\dot{\rho}_{kl}^{\mathrm{p}} + \dot{\rho}_{kl}^{\mathrm{tr}} - \dot{\bar{\rho}}_{kl}^{\mathrm{tr}}) \, \mathrm{d}V. \tag{37}$$

The actual stresses and strains in the body can be decomposed as

$$\sigma_{ij} = \sigma_{ij}^{\mathrm{e}} + \rho_{ij}^{\mathrm{p}} + \rho_{ij}^{\mathrm{tr}}, \tag{38}$$

$$\varepsilon_{ij} = \varepsilon_{ij}^{\mathrm{e}} + \varepsilon_{ij}^{\mathrm{p}} + \varepsilon_{ij}^{\mathrm{tr}} = \varepsilon_{ij}^{(\mathrm{e})} + \varepsilon_{ij}^{\mathrm{p(e)}} + \varepsilon_{ij}^{\mathrm{tr(e)}} + \varepsilon_{ij}^{\mathrm{p}} + \varepsilon_{ij}^{\mathrm{tr}}, \tag{39}$$

where $\varepsilon_{ij}^{\mathrm{e}}$ is the total elastic strain tensor, $\varepsilon_{ij}^{(\mathrm{e})}$, $\varepsilon_{ij}^{\mathrm{p(e)}}$ and $\varepsilon_{ij}^{\mathrm{tr(e)}}$ are the elastic strains corresponding to the stresses σ_{ij}^{e}, ρ_{ij}^{p} and ρ_{ij}^{tr}, respectively. That is,

$$\varepsilon_{ij}^{\mathrm{e}} = \varepsilon_{ij}^{(\mathrm{e})} + \varepsilon_{ij}^{\mathrm{p(e)}} + \varepsilon_{ij}^{\mathrm{tr(e)}} = S_{ijkl}^{\mathrm{e}} \sigma_{kl}, \tag{40}$$

$$\varepsilon_{ij}^{(\mathrm{e})} = S_{ijkl}^{\mathrm{e}} \sigma_{kl}^{\mathrm{e}}, \varepsilon_{ij}^{\mathrm{p(e)}} = S_{ijkl}^{\mathrm{e}} \rho_{kl}^{\mathrm{p}}, \varepsilon_{ij}^{\mathrm{tr(e)}} = S_{ijkl}^{\mathrm{e}} \rho_{kl}^{\mathrm{tr}}. \tag{41}$$

Making use of Eqs. (37)–(41), one has

$$
\dot{W} = \int_V (\rho^{\mathrm{p}}_{ij} + \rho^{\mathrm{tr}}_{ij} - \bar{\rho}^{\mathrm{p}}_{ij} - \bar{\rho}^{\mathrm{tr}}_{ij})(\dot{\varepsilon}^{\mathrm{p(e)}}_{ij} + \dot{\varepsilon}^{\mathrm{tr(e)}}_{ij} - \dot{\bar{\varepsilon}}^{\mathrm{tr(e)}}_{ij})\,\mathrm{d}V
$$

$$
= \int_V (\rho^{\mathrm{p}}_{ij} + \rho^{\mathrm{tr}}_{ij} - \bar{\rho}^{\mathrm{p}}_{ij} - \bar{\rho}^{\mathrm{tr}}_{ij})[(\dot{\varepsilon}_{ij} - \dot{\varepsilon}^{(\mathrm{e})}_{ij} - \dot{\varepsilon}^{\mathrm{p}}_{ij} - \dot{\varepsilon}^{\mathrm{tr}}_{ij}) - (\dot{\bar{\varepsilon}}_{ij} - \dot{\varepsilon}^{(\mathrm{e})}_{ij} - \dot{\bar{\varepsilon}}^{\mathrm{tr}}_{ij})]\,\mathrm{d}V
$$

$$
= \int_V (\rho^{\mathrm{p}}_{ij} + \rho^{\mathrm{tr}}_{ij} - \bar{\rho}^{\mathrm{p}}_{ij} - \bar{\rho}^{\mathrm{tr}}_{ij})[(\dot{\varepsilon}_{ij} - \dot{\bar{\varepsilon}}_{ij}) - \dot{\varepsilon}^{\mathrm{p}}_{ij} - (\dot{\varepsilon}^{\mathrm{tr}}_{ij} - \dot{\bar{\varepsilon}}^{\mathrm{tr}}_{ij})]\,\mathrm{d}V. \tag{42}
$$

The residual stress field $\rho^{\mathrm{p}}_{ij} + \rho^{\mathrm{tr}}_{ij} - \bar{\rho}^{\mathrm{p}}_{ij} - \bar{\rho}^{\mathrm{tr}}_{ij}$ is self-equilibrating. The strain rate field $\dot{\varepsilon}_{ij} - \dot{\bar{\varepsilon}}_{ij}$ is kinetically admissible since it is the difference of the two kinetically admissible strain rates fields. Thus, the virtual work principle requires that

$$
\int_V (\rho^{\mathrm{p}}_{ij} + \rho^{\mathrm{tr}}_{ij} - \bar{\rho}^{\mathrm{p}}_{ij} - \bar{\rho}^{\mathrm{tr}}_{ij})(\dot{\varepsilon}_{ij} - \dot{\bar{\varepsilon}}_{ij})\,\mathrm{d}V = 0. \tag{43}
$$

Then Eq. (42) is recast as

$$
\dot{W} = - \int_V (\rho^{\mathrm{p}}_{ij} + \rho^{\mathrm{tr}}_{ij} - \bar{\rho}^{\mathrm{p}}_{ij} - \bar{\rho}^{\mathrm{tr}}_{ij})[\dot{\varepsilon}^{\mathrm{p}}_{ij} + (\dot{\varepsilon}^{\mathrm{tr}}_{ij} - \dot{\bar{\varepsilon}}^{\mathrm{tr}}_{ij})]\,\mathrm{d}V \equiv \dot{W}_1 + \dot{W}_2, \tag{44}
$$

where

$$
\dot{W}_1 = - \int_V (\rho^{\mathrm{p}}_{ij} + \rho^{\mathrm{tr}}_{ij} - \bar{\rho}^{\mathrm{p}}_{ij} - \bar{\rho}^{\mathrm{tr}}_{ij})\dot{\varepsilon}^{\mathrm{p}}_{ij}\,\mathrm{d}V, \tag{45}
$$

$$
\dot{W}_2 = - \int_V (\rho^{\mathrm{p}}_{ij} + \rho^{\mathrm{tr}}_{ij} - \bar{\rho}^{\mathrm{p}}_{ij} - \bar{\rho}^{\mathrm{tr}}_{ij})(\dot{\varepsilon}^{\mathrm{tr}}_{ij} - \dot{\bar{\varepsilon}}^{\mathrm{tr}}_{ij})\,\mathrm{d}V. \tag{46}
$$

$\dot{W}_1$ and $\dot{W}_2$ correspond to the work rates which $\rho^{\mathrm{p}}_{ij} + \rho^{\mathrm{tr}}_{ij} - \bar{\rho}^{\mathrm{p}}_{ij} - \bar{\rho}^{\mathrm{tr}}_{ij}$ performs on the plastic strain rates $\dot{\varepsilon}^{\mathrm{p}}_{ij}$ and the difference of the transformation strain rates, $\dot{\varepsilon}^{\mathrm{tr}}_{ij} - \dot{\bar{\varepsilon}}^{\mathrm{tr}}_{ij}$, respectively.

Using Eq. (38) and denoting $\bar{\sigma}_{ij} = \sigma^{(\mathrm{e})}_{ij} + \bar{\rho}^{\mathrm{p}}_{ij} + \bar{\rho}^{\mathrm{tr}}_{ij}$, $\dot{W}_1$ can be rewritten as

$$
\dot{W}_1 = - \int_V (\sigma_{ij} - \bar{\sigma}_{ij})\dot{\varepsilon}^{\mathrm{p}}_{ij}\,\mathrm{d}V. \tag{47}
$$

According to Eq. (38), $\bar{\sigma}_{ij} = \sigma^{\mathrm{e}}_{ij} + \bar{\rho}^{\mathrm{p}}_{ij} + \bar{\rho}^{\mathrm{tr}}_{ij}$ is a safe stress field in the sense that it is located in the plastic yield surface throughout the body. Then it follows from the associated flow rule of plastic strains (or from Drucker's postulate) that $\dot{W}_1$ is always negative.

However, the same conclusion is not correct for $\dot{W}_2$ which has two possibilities in the case of alternating phase transformation. Firstly, $\dot{W}_2$ equals to zero if $\dot{\varepsilon}^{\mathrm{tr}}_{ij} - \dot{\bar{\varepsilon}}^{\mathrm{tr}}_{ij} = 0$ or $\rho^{\mathrm{p}}_{ij} + \rho^{\mathrm{tr}}_{ij} - \bar{\rho}^{\mathrm{p}}_{ij} - \bar{\rho}^{\mathrm{tr}}_{ij} = 0$. Secondly, $\dot{W}_2$ may change its sign in each loading cycle due to alternating phase transformation. However, it is significant to note that the strains induced by forward and reverse phase transformations at a material point will cancel each other out over a loading cycle and they are limited by the same saturation value, $\hat{\varepsilon}_{\mathrm{T}}$. According to the discussion in Section 3, it is clear that in spite of that $\dot{W}_2$ changes its sign in each cycle, its integral over a complete loading cycle must be zero except in the initial cycles. Therefore, the alternating change in sign of $\dot{W}_2$ corresponding to the alternating phase transformation does not affect the changing tendency of W.

Because W is always non-negative, $\dot{W}_1$ must tend to be zero and plastic deformation cannot evolve infinitely. That is, the structure will shake down ultimately to a stabilized plastic strain field in the sense that only elastic deformation and alternating phase transformation will happen. Once the plastic strains have stabilized, there will be a one-to-one

correspondence relationship between the total strains and the stresses, or in other words, both the stresses and the strains will change repeatedly during the subsequent loading cycles. Thus, the sufficient condition for transformational shakedown in the second statement has been proved for a general case.

5. Mathematical programming formulation of shakedown analysis

5.1. Mathematical programming method

Often, shakedown theory considers two classes of engineering problems. One is to judge whether a structure can shake down or not within the prescribed load domains, and the other to determine the possible maximum domains for the varying loads within which a structure can shake down. The extended Melan theorem in Section 4.2 allows determination of whether a structure will be safe in the sense that neither low cyclic fatigue nor incremental plasticity happens. Based on the same theorem, we formulate in this subsection the mathematical programming method for determining a lower bound of the shakedown load limit.

Consider an SMA structure subjected to n varying mechanical loads, $P_\alpha(t) = \xi_\alpha(t)P_{\alpha 0}$ with $\alpha = 1, 2, \ldots, n$, where $P_{\alpha 0}$ and $\xi_\alpha(t)$ denote the αth reference load and the corresponding load factor. Assume, for instance, that the load factors $\xi_\alpha(t)$ vary in the ranges of

$$\xi_{\alpha 1} \leqslant \xi_\alpha(t) \leqslant m\xi_{\alpha 2} \quad (\alpha = 1, 2, \ldots, n), \tag{48}$$

where $\xi_{\alpha 1}$ and $\xi_{\alpha 2}$ are prescribed loading factors. The possible maximum value $m_{\max}$ of the parameter m under which the structure can shake down defines the load limits of transformational shakedown.

To obtain a good estimate of $m_{\max}$ from Melan theorem, one may choose a residual stress fields $\bar{\rho}_{ij}^{p}(\mathbf{x})$ containing several adjustable parameters, ς_β, i.e.,

$$\bar{\rho}_{ij}^{p}(\mathbf{x}) = \bar{\rho}_{ij}^{p}(\mathbf{x}, \varsigma_\beta) \quad (\beta = 1, 2, \ldots, K). \tag{49}$$

Some methods and examples for choosing residual stress fields in conventional shakedown theory can be found in König's (1987) monograph and the references therein. For example, one may assume a temperature distribution in the structure and adopt the induced thermal elastic stress field as the residual stress field, which may contain several adjustable parameters defining the spatial positions and relative magnitudes of the temperature field.

Then the "elastic" stresses the loads cause in the structure are expressed as

$$\sigma_{ij}^{(e)}(\mathbf{x}, t, \xi_\alpha) = \sum_{\alpha=1}^{n} \xi_\alpha(t)\sigma_{ij}^{\alpha 0}(\mathbf{x}), \tag{50}$$

where $\sigma_{ij}^{\alpha 0}(\mathbf{x})$ denotes the elastic stress field corresponding to $P_{\alpha 0}$. The phase transformation-induced residual stress field $\bar{\rho}_{ij}^{tr}(\mathbf{x}, t)$ can be determined from a step-by-step analysis using the infinitely elastic-phase transformational constitutive relation, where no plastic deformation needs to be considered. In the case of alternating phase transformation, the field $\bar{\rho}_{ij}^{tr}(\mathbf{x}, t)$ varies with time but returns after a complete loading cycle to its initial value at the beginning of the cycle. $\bar{\rho}_{ij}^{tr}(\mathbf{x}, t)$ depends both on the applied loadings and the residual stress $\bar{\rho}_{ij}^{p}(\mathbf{x})$, that is,

$$\bar{\rho}_{ij}^{tr}(\mathbf{x}, t) = \bar{\rho}_{ij}^{tr}(\mathbf{x}, t, \varsigma_\beta) \quad (\beta = 1, 2, \ldots, K). \tag{51}$$

Therefore, one cannot assume arbitrarily the phase transformation-induced residual stress field $\bar{\rho}_{ij}^{\mathrm{tr}}(\mathbf{x}, t)$ without considering the field $\bar{\rho}_{ij}^{\mathrm{p}}(\mathbf{x})$. An inappropriate specification of $\bar{\rho}_{ij}^{\mathrm{tr}}(\mathbf{x}, t)$ will lead to an incorrect and non-conservative estimate of the shakedown load.

Thus, the mathematical programming problem for determining the non-dimensional shakedown load factor $m_{\max}$ is formulated as

$$m_{\max} = \max_{\xi_\alpha, \varsigma_\beta} m, \tag{52}$$

subject to the constraints

$$\psi_{\mathrm{p}}[\sigma_{ij}(\mathbf{x}, t, \xi_\alpha, \varsigma_\beta)] = \psi[\sigma_{ij}^{\mathrm{e}}(\mathbf{x}, t, \xi_\alpha) + \rho_{ij}^{\mathrm{tr}}(\mathbf{x}, t, \varsigma_\beta) + \rho_{ij}^{\mathrm{p}}(\mathbf{x}, \varsigma_\beta)] \leqslant \sigma_{\mathrm{y}},$$

$$\sigma_{ij}^{(\mathrm{e})}(\mathbf{x}, t, \xi_\alpha) = \sum_{\alpha=1}^{n} \xi_\alpha(t) \sigma_{ij}^{\alpha 0}(\mathbf{x}),$$

$$\xi_{\alpha 1} \leqslant \xi_\alpha(t) \leqslant m \xi_{\alpha 2},$$

$$\alpha = 1, 2, \ldots, n; \quad \beta = 1, 2, \ldots, K. \tag{53}$$

For other loading modes different from Eq. (48), the mathematical programming problem for determining the shakedown load domain can be formulated similarly. An example will be given in Section 6 to illustrate the application of this method.

5.2. Simplifications of the method

In the preceding subsection, a general mathematical programming method has been formulated for phase transformational shakedown of SMA bodies, which requires a step-by-step analysis to determine $\sigma_{ij}^{(\mathrm{e})}(\mathbf{x}, t)$ and $\bar{\rho}_{ij}^{\mathrm{tr}}(\mathbf{x}, t)$ based on the assumption of $\bar{\rho}_{ij}^{\mathrm{p}}(\mathbf{x})$. On one hand, however, such an analysis is generally cumbersome and time-consuming though plastic deformation has been excluded from the analysis according to the extended Melan theorem in Section 4.2. On the other hand, the actual future history of the loads of an engineering structure is often very complicated or even unknown explicitly except their changing range, Ω, shown in Fig. 4.
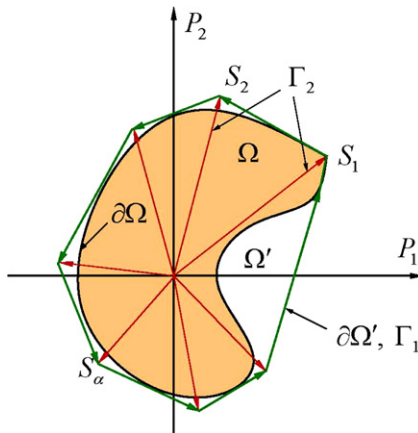

Fig. 4. A specified load domain Ω and two simplified loading paths, Γ_1 and Γ_2.

Fortunately, some simplifications can be made resorting to some arguments in the classical shakedown theory (König, 1987), which are valid for any convex yielding surfaces such as the von Mises yield surface adopted here. For shakedown of SMA structures in the sense of alternating phase transformation with stabilized plastic deformation, four conclusions, among others, are given as follows to make the shakedown analysis simpler:

(i) Conclusion 1. *If an SMA structure shakes down over any loading path within the boundary $\partial\Omega$ of a given and bounded load domain Ω, then it shakes down also over any loading path contained with Ω.*
According to this conclusion, one may derive an estimate of shakedown load by choosing a loading path within the boundary $\partial\Omega$.

(ii) Conclusion 2. *If an SMA structure shakes down over any loading path contained within a specified load domain Ω, then it also shakes down over any loading path contained within the convex hull of Ω (i.e., within the smallest convex set containing Ω).*
This means that a sufficient and necessary condition for a structure to shake down over Ω is that the structure shakes down over its convex hull.

(iii) Conclusion 3. *If an SMA structure shakes down over a loading path within the edge boundary $\partial\Omega'$ of a closed convex super-polyhedral load domain Ω' containing the given load domain Ω, as shown in Fig. 4, then it also shakes down over any loading path contained with Ω.*
Thus, one may obtain a lower bound of shakedown load by specifying a rather simple loading path in which all the loads change in linear functions along $\partial\Omega'$.

(iv) Conclusion 4. *If an SMA structure shakes down over any loading path which contains all the vertices of a closed convex super-polyhedral load domain Ω' embedding the given load domain Ω, then it also shakes down over any loading path contained with Ω.*

The proofs of these conclusions can be made easily for convex yield surfaces, analogously to those in the classical shakedown theory (König, 1987). In order to judge the shakedown or inadaptation of a structure or to determine a lower bound of the shakedown load domain, thus, one does not need to perform a step-by-step full analysis along the loading history. Instead, one may choose a simple loading path according to the above arguments to calculate a conservative estimate or a lower bound of the shakedown load.

For instance, consider a body subjected to a given load domain Ω of two loads, P_1 and P_2, as shown in Fig. 4, without knowing their concrete loading history. To obtain a lower bound of the shakedown limit load, one may choose a closed convex polyhedral load domain Ω' containing Ω. Only an analysis along a simple loading path containing all the vertices of Ω' is necessary. Two examples of simplified loading paths are: (a) Γ_1 along the boundary $\partial\Omega'$ of Ω', i.e., $S_1 \to S_2 \to \cdots \to S_\alpha \to \cdots \to S_1$, and (b) Γ_2 containing n proportionally loading and unloading steps from the original point to all the vertices of Ω', i.e., $0 \to S_1 \to 0 \to S_2 \to 0 \cdots \to S_\alpha \to \cdots \to S_n \to 0$, as shown in Fig. 4. From either the loading path Γ_1 or Γ_2, one can obtain a conservative estimate of the shakedown load limit by solving the mathematical programming problem corresponding to the much simplified loading path, instead of the actual complicated loading history.

6. Example

To illustrate the application of the above method and to reveal the effect of phase transformation on shakedown of SMA structures, a three-bar structure subjected to a varying load P, as shown in Fig. 5, is a simple and good example. Assume that the three bars are made of the same SMA material and have the same cross-section of area A. Their lengths are $l_1 = l$ and $l_2 = l_3 = l/\cos\theta$, respectively, where θ denotes the angle between the first and the second (or the third) bars. To show some basic features of SMA structures under varying loads, the exact analytical solution is given below without detailed derivations.

In what follows, denote the load by $p = P/A$ for conciseness. Since there is only one applied force in this example, we will formulate the solution in terms of the load p directly, instead of the non-dimensional load factor m as in Eqs. (52) and (53). Assume that p changes randomly between $p_{\min}$ and $p_{\max}$ with $|p_{\min}| \leqslant p_{\max}$. The elastic stresses in the three bars are

$$\sigma_1^e(p) = \frac{p}{1 + 2\cos^3\theta}, \quad \sigma_2^e(p) = \sigma_3^e(p) = \frac{p\cos^2\theta}{1 + 2\cos^3\theta}. \tag{54}$$

The critical load corresponding to the onset of forward phase transformation in the first bar is

$$p_T^{SMA} = \sigma_T(1 + 2\cos^3\theta). \tag{55}$$

The tensile elastic-transformational limit load, which corresponds to the beginning of plastic deformation in the body, is derived as

$$p_{EL}^{SMA} = \begin{cases} \sigma_y(1 + 2\cos^3\theta) + 2E\varepsilon_T\cos^3\theta & \text{for } \frac{1}{2}\arccos\left(\frac{2\sigma_T - \sigma_y - E\varepsilon_T}{\sigma_y + E\varepsilon_T}\right) \leqslant \theta \leqslant \frac{\pi}{2} \\ \sigma_y + 2\sigma_T\cos\theta & \text{for } \frac{1}{2}\arccos\left(\frac{2\sigma_T - \sigma_y + E\varepsilon_T}{\sigma_y + E\varepsilon_T}\right) \leqslant \theta \leqslant \frac{1}{2}\arccos\left(\frac{2\sigma_T - \sigma_y - E\varepsilon_T}{\sigma_y + E\varepsilon_T}\right) \\ \sigma_y(1 + 2\cos^3\theta) - 2E\varepsilon_T\sin^2\theta\cos\theta & \text{for } 0 \leqslant \theta \leqslant \frac{1}{2}\arccos\left(\frac{2\sigma_T - \sigma_y - E\varepsilon_T}{\sigma_y + E\varepsilon_T}\right). \end{cases} \tag{56}$$
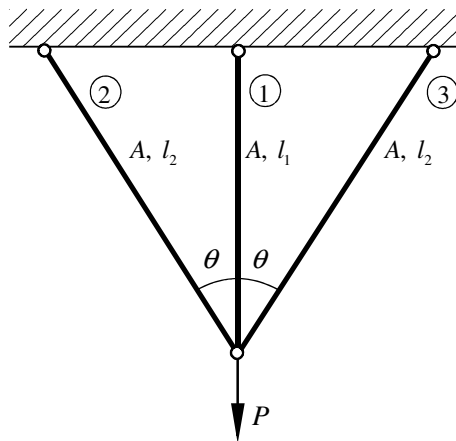

Fig. 5. A three-bar truss subjected to a varying load P.

If $\varepsilon_T = 0$, the solution in Eq. (56) is identical to that of a three-bar truss made of an elastic–perfectly plastic material with the same plastic yield stress σ_y.

The tensile plastic limit load, corresponding to the situation that all the three bars reach the plastic deformation regime (i.e., $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_y$), is

$$p_{PL}^{SMA} = \sigma_y(1 + 2\cos\theta), \tag{57}$$

which is evidently not affected by the phase transformation.

A residual stress field induced by plastic deformation is chosen as

$$\rho_1^p(\varsigma) = -\varsigma\sigma_y \frac{2\sin^2\theta\cos\theta}{1 + 2\cos^3\theta}, \quad \rho_2^p(\varsigma) = \varsigma\sigma_y \frac{\sin^2\theta}{1 + 2\cos^3\theta}, \tag{58}$$

where ς is an adjustable parameter.

Then for a given value of p_{min}, the shakedown limit load p_{SD} of the SMA truss can be obtained from the following mathematical programming problem:

$$p_{SD}^{SMA}(p_{min}) = \max_{\varsigma} p_{max}, \tag{59}$$

subjected to:

$$\begin{aligned}
&-\sigma_y \leqslant \sigma_1^e(p) + \rho_1^p(\varsigma) + \rho_1^{tr}(p,\varsigma) \leqslant \sigma_y, \\
&-\sigma_y \leqslant \sigma_2^e(p) + \rho_2^p(\varsigma) + \rho_2^{tr}(p,\varsigma) \leqslant \sigma_y, \\
&p_{min} \leqslant p \leqslant p_{max},
\end{aligned} \tag{60}$$

where $\rho_\alpha^{tr}(p,\varsigma)$ ($\alpha = 1, 2$) denote the residual stresses induced by the phase transformation due to the cyclic load p on the basis of the assumed $\rho_\alpha^p(\varsigma)$ in Eq. (58). The solution of the problem is

$$p_{SD}^{SMA} = \min\left\{p_{min} + 2p_{EL}^{SMA}, p_{PL}^{SMA}\right\}. \tag{61}$$

It is easy to find that if $\varepsilon_T = 0$, the solution in Eq. (61) reduces to that of a three-bar system made of an elastic–perfectly plastic (EP) material with the same yield stress σ_y, and reads

$$p_{SD}^{EP} = \min\left\{p_{min} + 2p_{EL}^{EP}, p_{PL}^{EP}\right\}, \tag{62}$$

where the elastic and plastic limit loads for the EP truss are

$$p_{EL}^{EP} = \sigma_y(1 + 2\cos^3\theta), \quad p_{PL}^{EP} = \sigma_y(1 + 2\cos\theta), \tag{63}$$

respectively.

When p changes between 0 and p_{max} (i.e., $p_{min} = 0$), the shakedown limit loads in Eqs. (61) and (62) are simplified as

$$p_{SD}^{SMA} = p_{SD}^{EP} = \sigma_y(1 + 2\cos\theta). \tag{64}$$

Evidently, the phase transformation in the system has no impact on the shakedown limit load in this typical case.

In another special case of $p_{min} = -p_{max}$, the solutions in Eqs. (61) and (62) become

$$p_{SD}^{SMA} = p_{EL}^{SMA}, \quad p_{SD}^{EP} = p_{EL}^{EP}, \tag{65}$$

which are compared schematically by the two solid lines in Fig. 6 for representative material parameters. For $\theta = 0$, the shakedown loads are determined as $p_{SD}^{SMA} = p_{SD}^{EP} = 3\sigma_y$ since
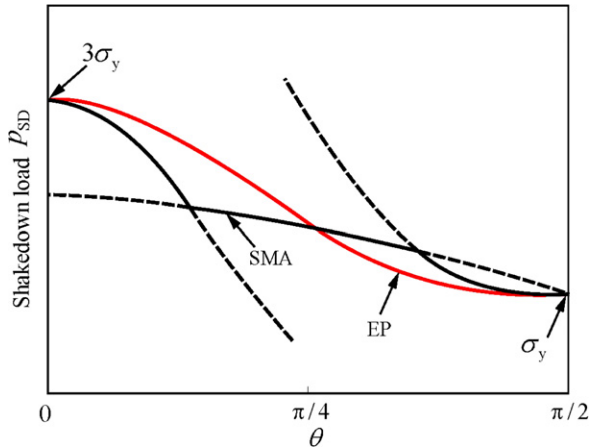
Fig. 6. Shakedown loads of a three-bar truss made of an SMA or an elastic- perfectly plastic (EP) material subjected to a load P varying between $-p_{SD}A$ and $p_{SD}A$.

no residual stress can develop. If $\theta = \pi/2$, then only one bar is active, and the shakedown loads are $p_{SD}^{SMA} = p_{SD}^{EP} = \sigma_y$ since again no residual stress develops. It is interesting to seen that for a larger angle θ, the phase transformation may enhance the shakedown properties of the structure while for a smaller θ, the phase transformation may decrease the shakedown limit load. To show quantitatively the influence of phase transformation, we take the following parameter for a NiTi alloy at room temperature: $\sigma_T = 400$ MPa, $\sigma_y = 1400$ MPa, $\sigma_{RT} = 0$ MPa, $\varepsilon_T = \varepsilon_T = \varepsilon_{RT} = 6\%$, and $E = 50$ GPa. It is easily obtained that due to the effect of phase transformation, the shakedown limit load is increased by 11.1% for $\theta = 72°$ and is decreased by 35.0% for $\theta = 30°$, compared to that of an elasto-plastic structure with the same plastic yield stress and Young's modulus.

This illustrative example clearly demonstrates three possible influences of phase transformation on the load bearing capacity of an SMA structure, namely, strengthening, weakening, and trivial. The phase transformation effect on the mechanical response of a structure depends upon its constitutive parameters (i.e., $\hat{\varepsilon}_T$ and σ_T), structural geometries (i.e., θ in the present example), loading mode and service condition (i.e., temperature). For some structures with high stress concentration (i.e., the three-bar truss with a big angle θ) and subjected to alternating loadings, roughly speaking, phase transformation may lower the magnitude of stress concentration and, therefore, serve as an enhancing mechanism of load bearing. Another well-known example of this case is that phase transformation at a crack tip of such ceramic materials as ZrO_2 may improve significantly the fracture toughness and then is considered as one of the typical toughening mechanisms. For those structures with low stress concentration, however, phase transformation may appear as a softening mechanism of deformation and hence often lower the shakedown load limit.

7. Conclusions and remarks

Analysis of SMA structures under varying loads is complicated, especially when they develop both elastic–plastic deformation and phase transformation. The present paper is focused on static analysis of shakedown of SMA bodies with the assumptions of small

strains and isothermal conditions. We have presented for the first time a theoretical framework of shakedown analysis of SMA bodies based on a simplified phenomenological constitutive model. The Melan or lower bound shakedown theorem has been extended to account for the effect of alternating phase transformation. A lower bound of shakedown limit load can be obtained via a mathematical programming problem without necessity of a step-by-step full analysis following the real loading history. Phase transformation may have different influences on the shakedown behavior and load bearing capacities of SMA structures, depending upon the parameters of constitutive relations, structural geometries and loading modes.

Implementations of the effects of temperature, nonlinear geometry, strain- or work-hardening and dynamics are of interest and can be incorporated in the present theory. In addition, Koiter dynamic shakedown theorem may also be extended to include the effect of phase transformation in SMA structures. This has not been included here since dynamic shakedown analysis is more difficult to be applied in engineering structures and it leads to an upper bound (non-conservative) estimate of the shakedown load domain.

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