Problem Formulation and Solution Methodology of Energy Consumption Optimization for Two-Machine Synchronous Exponential Serial Lines

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***Abstract*—Reducing the energy consumption and improving the energy efficiency of energy-intensive production systems are of significant importance. In the literature, for Bernoulli and geometric lines, the energy consumption optimization problem has been largely studied, while for more practical, e.g., exponential, reliability models, it is barely studied. This paper is intended to investigate the energy consumption optimization for two-machine synchronous exponential lines. Specifically, first, similar for the Bernoulli and geometric lines, this problem is formulated as a constrained nonlinear programming; then, two optimality equations for solving the nonlinear programming are derived; finally, based on the properties of the nonlinear optimality equations, an algorithm is designed to solve the solution of the optimality equations, which is the unique optimal solution of the energy consumption optimization problem. Extensive numerical experiments show that the algorithm is effective and computationally efficient for solving the energy consumption optimization problem.**

***Index Terms*—Production rate, nonlinear programming, optimality equations, monotonicity, sensitivity analysis.**

I. Introduction

PRODUCTION systems consume a huge amount of energy. It is reported by the National Bureau of Statistics of China that, in 2020, the manufacturing sector accounts for 84.07% energy consumption of the industrial production in China and 84.10% of the energy comes from the fossil fuel such as coal, oil, and natural gas [1]. In addition to the intensive energy consumption, most manufacturing systems are energy-inefficient. These energy-intensive and energy-inefficient systems are unfavorable for carbon emission reduction. Hence, reducing the energy consumption and improving the energy efficiency of such systems are of great significance.

In the past several decades, the production systems, usually consisting of unreliable machines and finite buffers, have been extensively studied [2]. The performance metrics interested in most researches are productivity, work-in-process, and production lead time and the energy-related measures have been paid much less attention. Recent years, as more and more concerns are focused on green manufacturing, the energy consumption and energy efficiency have become a new kind of important performance metrics of production systems. By elaborately optimizing efficiencies of the unreliable machines, the energy consumption (or energy efficiency) of the system is reduced (correspondingly, improved) while some traditional metrics such as productivity are ensured.

For two-machine serial lines, in general, the traditional performance metrics of the system can be analytically expressed. In the seminal work [3], the energy consumption optimization problem for two-machine Bernoulli line is formulated as a nonlinear programming. Although qualitative relationships between the optimal solution and system parameters are analyzed based on extensive numerical experiments, no algorithms are developed to solve the optimal solution. For this purpose, in [4], the structural characteristics of the nonlinear programming and mathematical properties of some important functions are analyzed, and based on these derived characteristics and properties, an effective and efficient dichotomy algorithm is designed. This work has been extended to some more practical production systems. First, considering that machine efficiencies are usually confined in a subset of $(0, 1]$ due to the physical limitations, in [5], the energy consumption optimization problem is re-formulated and its optimal solution is constructed based on the results obtained in [4]. Then, considering that the demand-side response is important for reducing energy cost, the energy consumption optimization problem is investigated under the time-of-use electricity pricing in [6]. Finally, in [7], the energy consumption optimization problem for the Bernoulli line is also extended to lines with geometric, which is a more practical, reliability model. In addition, the energy consumption optimization problem for Bernoulli and geometric lines is investigated in [8] and [9], respectively. Different from the previous researches, the energy consumed by a machine in setup and idle time is considered. The energy consumption optimization problem for a Markovian system is formulated and analyzed in [10], where the setup time, as the machine up- and downtime, is modeled as a random variable following the geometric distribution.

As for long lines with multiple machines, the energy consumption optimization problem has also been investigated and only Bernoulli lines have been studied so far. In [11], for small systems, i.e., systems with machines not more than four and buffer capacities not more than two, the optimal solution of the energy consumption optimization problem is solved by an enumeration method; for large systems, the problem is solved by commercial softwares or a heuristic algorithm, which cannot guarantee the optimality of the solution. To solve the optimal solution, a recursive algorithm based on the ones developed in [4] and [5] for two-machine lines is designed in

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[12]. Although the recursive algorithm numerically solves the optimal solution, it is very time-consuming. To alleviate the computational burden, by using the aggregation method in [2], the energy consumption optimization problem for long lines is decomposed into a series of two-machine problems and based on algorithms in [4] and [5], a divide-and-conquer algorithm is developed in [13].

Although the energy consumption optimization problem has been largely studied for Bernoulli and geometric lines, for more practical reliability models, it is barely studied. For this purpose, as a starting point, this paper is devoted to investigate the energy consumption optimization for two-machine synchronous exponential lines as shown in Fig. 1. The methods for analysis and algorithm design for Bernoulli and geometric lines are extended to the exponential lines. Since the problem for the exponential lines has some unique characteristics, the extension is non-trivial.

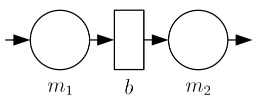


Fig. 1: Two-machine serial line

The rest of the paper is organized as follows. Section II models the two-machine synchronous exponential line and addresses the energy consumption optimization problem investigated in this paper. Section III mathematically formulates the problem as a nonlinear programming and derives its two optimality equations. Section IV analyzes properties of the optimality equations and based on the properties, designs an effective and efficient algorithm to solve the problem. The conclusions and topics for future work are provided in Section V. Proofs of theorems are provided in the Appendix.

II. Production System Modeling and Problem Statement

In this section, the two-machine synchronous exponential serial line is formally modeled and the energy consumption optimization problem is addressed in Subsections II-A and II-B, respectively.

A. System Model

The model of the two-machine serial line in Fig. 1 is assumed as follows:

(i) The system consists of two machines m_1 and m_2, and an intermediate buffer b between the machines.

(ii) The production line is *synchronous*, i.e., these two machines have identical cycle time (namely, processing time), which is denoted by τ (in min).

(iii) Machine m_i, $i = 1, 2$, obeys the exponential reliability model, which is characterized by breakdown rate λ_i and repair rate μ_i (both in 1/min). Specifically, if m_i is up, it will go down during each infinitesimal interval δt with rate λ_i; if it is down, it will go up during δt with rate μ_i. Herein, λ_i is fixed and μ_i can be selected in $(0, +\infty)$. Note that for m_i, $i = 1, 2$, its efficiency is $e_i = \frac{\mu_i}{\lambda_i+\mu_i}$.

(iv) The buffer capacity is N, which is an integer and $0 < N < +\infty$.

(v) The flow model [2] is assumed, i.e., infinitesimal quantity of parts, produced during an infinitesimal time interval, are transferred to and from the buffer. If m_1 breaks down and the buffer is empty, m_2 is starved; if m_2 breaks down and the buffer is full, m_1 is blocked. Machine m_1 is never starved and m_2 never blocked. Machine failures are time-dependent [2], i.e., a machine can be down even if it is starved or blocked.

(vi) When machine m_i, $i = 1, 2$, is up, the power it consumes is P_i; when m_i is down, it doesn't consume any power. Herein, $0 < P_i < +\infty$.

B. Problem Statement

In this subsection, the problem of reducing the total energy consumed by machines of the exponential line defined by model (i)-(vi) will be addressed. To be specific, by elaborately selecting a pair of machine repair rates, (μ_1, μ_2), the total energy consumption is minimized and meanwhile, the system productivity is maintained at a level not less than a required production rate, PR_r. To mathematically formulate the problem, the production rate, PR, of the two-machine synchronous exponential line, is reviewed in the following.

The performance of the two-machine synchronous exponential line has been comprehensively analyzed in [14] and [2]. Specifically, the production rate of the two-machine synchronous exponential line is

$$PR = e_2\big[1 - Q(\tau, \lambda_1, \mu_1, \lambda_2, \mu_2, N)\big]. \tag{1}$$

The Q-function in (1), which is abbreviated as Q, is expressed as

$$Q = \begin{cases} \frac{(1-e_1)(1-\phi)}{1-\phi\exp(-\beta N)}, & \text{if } \frac{\lambda_1}{\mu_1} \neq \frac{\lambda_2}{\mu_2}, \\ \frac{(\lambda_1+\lambda_2)(1-e_1)^2}{N\tau\lambda_1\lambda_2+(\lambda_1+\lambda_2)(1-e_1)}, & \text{if } \frac{\lambda_1}{\mu_1} = \frac{\lambda_2}{\mu_2}, \end{cases} \tag{2}$$

where

$$\begin{aligned} e_i &= \frac{\mu_i}{\lambda_i + \mu_i}, \ i = 1, 2, \\ \phi &= \frac{e_1(1-e_2)}{e_2(1-e_1)}, \\ \beta &= \frac{\tau(\lambda_1 + \lambda_2 + \mu_1 + \mu_2)(\lambda_1\mu_2 - \lambda_2\mu_1)}{(\lambda_1 + \lambda_2)(\mu_1 + \mu_2)} \\ &= \frac{\tau\lambda_1\lambda_2(e_2 - e_1)\big[\lambda_1(1-e_2) + \lambda_2(1-e_1)\big]}{(1-e_1)(1-e_2)(\lambda_1+\lambda_2)\big[\lambda_1 e_1(1-e_2) + \lambda_2 e_2(1-e_1)\big]}. \end{aligned} \tag{3}$$

Let $(\tau, \lambda_1, \mu_1, \lambda_2, \mu_2, N)$ denote the two-machine synchronous exponential line defined by model (i)-(vi). Now we examine the production rate of two lines, $(\tau, \lambda_1, \mu_1, \lambda_2, \mu_2, N)$ and $(1, \lambda_1\tau, \mu_1\tau, \lambda_2\tau, \mu_2\tau, N)$. From (1)-(3), it is easy to check that both production lines have identical e_i, ϕ, and β, and thus, have identical Q and PR. In other words, for any two-machine synchronous exponential line $(\tau, \lambda_1, \mu_1, \lambda_2, \mu_2, N)$, there exists a "production rate equivalent" line, of which the machine cycle time is the unit time. To simplify the analysis, without loss of generality, in the following, we focus on the energy consumption optimization of line $(1, \lambda_1, \mu_1, \lambda_2, \mu_2, N)$, of which the breakdown and repair rates have been adapted accordingly. Meanwhile, $Q(1, \lambda_1, \mu_1, \lambda_2, \mu_2, N)$ is rewritten as $Q(\lambda_1, \mu_1, \lambda_2, \mu_2, N)$.

For $Q(\lambda_1, \mu_1, \lambda_2, \mu_2, N)$ in (2), the expression for $\frac{\lambda_1}{\mu_1} = \frac{\lambda_2}{\mu_2}$ can be derived from the one for $\frac{\lambda_1}{\mu_1} \neq \frac{\lambda_2}{\mu_2}$. In fact, for $\frac{\lambda_1}{\mu_1} \neq \frac{\lambda_2}{\mu_2}$ (i.e., $e_1 \neq e_2$), we have

$$\lim_{e_2 \to e_1} \frac{(1-e_1)(1-\phi)}{1-\phi \exp(-\beta N)} = \frac{(\lambda_1+\lambda_2)(1-e_1)^2}{N\lambda_1\lambda_2 + (\lambda_1+\lambda_2)(1-e_1)}, \quad (4)$$

which implies that the expression of Q-function for $\frac{\lambda_1}{\mu_1} \neq \frac{\lambda_2}{\mu_2}$ could be regarded as a general form and the one for $\frac{\lambda_1}{\mu_1} = \frac{\lambda_2}{\mu_2}$ as its special case. In this case, to facilitate the analysis in the following, the expression of Q-function for $\frac{\lambda_1}{\mu_1} \neq \frac{\lambda_2}{\mu_2}$ is used if not otherwise specified. It should be pointed out that $Q(\lambda_1, \mu_1, \lambda_2, \mu_2, N)$, which represents the probability of part shortage and takes value on $(0, 1)$, is strictly decreasing in μ_1 and strictly increasing in μ_2, and PR is strictly increasing in μ_1 and μ_2 (and thus, in e_1 and e_2), respectively (see [2] for details).

III. Problem Formulation and Analysis

In this section, first, the energy consumption optimization problem for the two-machine synchronous exponential serial line is mathematically formulated in Subsection III-A, and then, two nonlinear optimality equations for solving the energy consumption optimization problem is derived in Subsection III-B.

A. Problem Formulation and Transformation

Considering the expression of the production rate in (1) and the analysis in Subsection II-B, the energy consumption optimization problem for the two-machine synchronous exponential line is mathematically formulated as follows:

$$\text{(P1) min} \quad z = \sum_{i=1}^{2} P_i e_i \quad (5)$$

$$\text{s.t.:} \quad e_2\big[1 - Q(\lambda_1, \mu_1, \lambda_2, \mu_2, N)\big] \geqslant PR_r, \quad (6)$$

$$\mu_i > 0, \; i = 1, 2, \quad (7)$$

where PR_r is the required production rate.

In problem (P1), machine breakdown rates λ_1 and λ_2 are fixed, and machine repair rates μ_1 and μ_2 are decision variables. Noting the expression of machine efficiency in (3), (P1) can be rewritten as

$$\text{(P1') min} \quad z = \sum_{i=1}^{2} P_i e_i \quad (8)$$

$$\text{s.t.:} \quad e_2\big[1 - Q(e_1, e_2, N; \lambda_1, \lambda_2)\big] \geqslant PR_r, \quad (9)$$

$$0 < e_i < 1, \; i = 1, 2, \quad (10)$$

where e_1 and e_2 are decision variables and the Q-function and β have been rewritten as a function of e_i, $i = 1, 2$. Considering that λ_1 and λ_2 are fixed, unless otherwise specified, in the following, the Q-function in (9) will be abbreviated as $Q(e_1, e_2, N)$ or Q. Note that, since e_i, $i = 1, 2$, is strictly increasing in μ_i and $\mu_i \in (0, +\infty)$, it could take any value on $(0, 1)$, which is expressed as constraint (10). Furthermore, considering that PR is a strictly increasing function of μ_i (and thus of e_i), $i = 1, 2$, it is easy to check that the production rate PR in (1) could take any value on $(0, 1)$ as well.

To facilitate to solve (P1'), similar to the Bernoulli and geometric lines, a new problem is introduced as follows:

$$\text{(P2) min} \quad z = \sum_{i=1}^{2} P_i e_i \quad (11)$$

$$\text{s.t.:} \quad e_2\big[1 - Q(e_1, e_2, N)\big] = PR_r, \quad (12)$$

$$0 < e_i < 1, \; i = 1, 2. \quad (13)$$

The only difference between (P2) and (P1') is the production rate constraint. The monotonicity of the optimal objective value of (P2) is analyzed. As a result, we have:

Theorem 3.1: The optimal objective value, z^*, of (P2), is strictly increasing in PR_r.

Proof: See the Appendix.

Theorem 3.1 connects problems (P1') and (P2). Specifically, the relationship between (P1') and (P2) is as follows.

Corollary 3.1: Problem (P1') is essentially equivalent to (P2). In other words, constraint (9) in (P1') can be replaced by (12) in (P2).

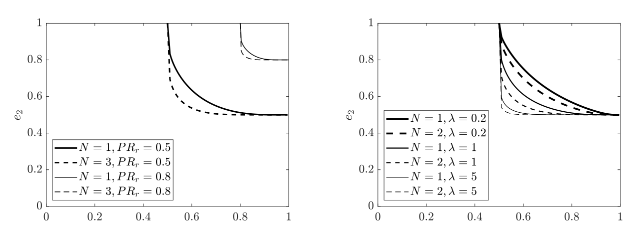
Corollary 3.1 can be proved by contradiction. Due to space limitations, the proof is omitted here.

Corollary 3.1 indicates that (P1) and (P2) are equivalent to each other, which implies that (P1) and (P2) have identical optimal solution. In the following subsections, (P2) is further analyzed and an algorithm to solve the optimal solution is developed.

B. Nonlinear Optimality Equations

Similar to the analysis approach developed for the Bernoulli and geometric lines, for the two-machine synchronous exponential line, two optimality equations that the optimal solution of (P2) satisfies are derived in this section.

Based on the insights gained from the two-machine Bernoulli and geometric lines in [4] and [7], respectively, one of the optimality equations of (P2) is (12). Clearly, (12) can be regarded as a contour of the production rate, on which the relationship between e_1 and e_2 is characterized. Since the production rate is strictly increasing in e_1 and e_2, for a fixed PR_r, e_2 can be regarded as an implicit decreasing function of e_1. The behavior of the implicit function e_2 with respect to e_1 for different PR_r, N, and λ_i's is shown in Fig. 2.



(a) $\lambda_1 = 0.1$, $\lambda_2 = 10$ (b) $\lambda_1 = \lambda_2 = \lambda$ ($PR_r = 0.5$)

Fig. 2: Implicit function e_2 with respect to e_1

Based on the optimality equation (12), the feasible region of e_1 and e_2 can be analyzed. From the results in [2], it follows that $Q \in (0,1)$, which, combining with (12), implies that $e_2 > PR_r$. Taking into account the reversibility of the production line [2], we have $e_1 > PR_r$. Combining the above two inequalities with (13), we have:

$$PR_r < e_i < 1, \ i = 1, 2. \tag{14}$$

As for the other optimality equation, it is derived by the Lagrange multiplier method employed in [4] and [7]. For this purpose, we first solve $\bar{e}$, which is the solution of equation

$$\bar{e}\big[1 - Q(\bar{e}, \bar{e}, N)\big] = PR_r. \tag{15}$$

Clearly, $(\bar{e}, \bar{e})$ is the point that $e_1 = e_2$ on the production rate contour $e_2\big[1 - Q(e_1, e_2, N)\big] = PR_r$. Re-writing (15) and re-arranging the terms, we have

$$\bar{e}^3 - \bar{e}^2 - \Big(\frac{N\lambda_1\lambda_2}{\lambda_1+\lambda_2} + PR_r\Big)\bar{e} + PR_r\Big(\frac{N\lambda_1\lambda_2}{\lambda_1+\lambda_2} + 1\Big) = 0. \tag{16}$$

Theorem 3.2: For Equation (16), it has a unique solution on $(0, 1)$. Specifically, this solution is expressed as

$$\bar{e} = \omega^2 \sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} + \omega \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}} + \frac{1}{3}, \tag{17}$$

where

$$\begin{aligned} \Delta &= \big(\frac{q}{2}\big)^2 + \big(\frac{p}{3}\big)^3, \\ p &= -\frac{N\lambda_1\lambda_2}{\lambda_1+\lambda_2} - PR_r - \frac{1}{3}, \\ q &= \big(PR_r - \frac{1}{3}\big)\frac{N\lambda_1\lambda_2}{\lambda_1+\lambda_2} + \frac{2}{3}PR_r - \frac{2}{27}, \\ \omega &= \frac{-1+\sqrt{3}i}{2}, \end{aligned} \tag{18}$$

and $i = \sqrt{-1}$ is the imaginary unit of complex numbers.

Proof: See the Appendix.

Now we derive the second optimality equation. As a result, we have:

Theorem 3.3: The other optimality function that the optimal solution of (P2) satisfies is

$$f(e_1) = \frac{P_1}{P_2}, \tag{19}$$

where $f(e_1)$ is a positive continuous function expressed as

$$f(e_1) = \begin{cases} \frac{f_n(e_1)}{f_d(e_1)}, & \text{if } e_1 \neq \bar{e}, \\ f(\bar{e}), & \text{if } e_1 = \bar{e}, \end{cases} \tag{20}$$

and

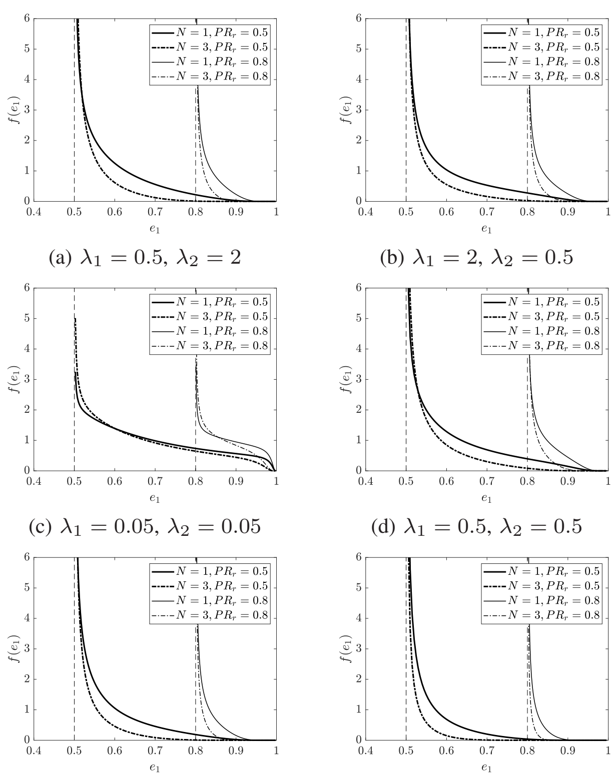
$$\begin{aligned} f_n(e_1) =& e_2(1-e_2)^2(e_1^2 - e_2)(1-e_1)(\lambda_1+\lambda_2) \\ & \cdot \big[\lambda_1 e_1(1-e_2) + \lambda_2 e_2(1-e_1)\big]^2 \\ & + N e_1 e_2 \lambda_1 \lambda_2 (e_1 - e_2)(1-e_2)^2 \\ & \cdot \big[\lambda_1^2(1-e_2)(e_1^2 + e_2 - 2e_1e_2) \\ & + 2\lambda_1\lambda_2(1-e_1)e_2(1-e_2) + \lambda_2^2(1-e_1)^2 e_2\big] \\ & + (1-e_1)^3 e_2^2 (1-e_2)(\lambda_1+\lambda_2) \\ & \cdot \big[\lambda_1 e_1(1-e_2) + \lambda_2 e_2(1-e_1)\big]^2 \exp(\beta N), \end{aligned} \tag{21}$$

$$\begin{aligned} f_d(e_1) =& e_1(1-e_1)^2(e_2^2 - e_1)(1-e_2)(\lambda_1+\lambda_2) \\ & \cdot \big[\lambda_1 e_1(1-e_2) + \lambda_2 e_2(1-e_1)\big]^2 \\ & + N e_1 e_2 \lambda_1 \lambda_2 (1-e_1)^2 (e_2 - e_1) \\ & \cdot \big[\lambda_1^2 e_1(1-e_2)^2 + 2\lambda_1\lambda_2 e_1(1-e_1)(1-e_2) \\ & + \lambda_2^2(1-e_1)(e_1 - 2e_1e_2 + e_2^2)\big] \\ & + e_1^2(1-e_1)(1-e_2)^3(\lambda_1+\lambda_2) \\ & \cdot \big[\lambda_1 e_1(1-e_2) + \lambda_2 e_2(1-e_1)\big]^2 \exp(-\beta N), \end{aligned} \tag{22}$$

$$\begin{aligned} f(\bar{e}) =& \frac{B}{A}, \\ A =& \lambda_1^2 \bar{e}(1-\bar{e})(3PR_r^2 - 5PR_r\bar{e}^2 - PR_r + 4\bar{e}^3 - \bar{e}^2) \\ & + \lambda_1\lambda_2 N(PR_r - \bar{e})^2\big[\lambda_1(3\bar{e}-1) + \lambda_2(\bar{e}-1)\big] \\ & + 2\lambda_1\lambda_2(1-\bar{e})(3PR_r^2\bar{e} - PR_r^2 - 5PR_r\bar{e}^3 + PR_r\bar{e}^2 \\ & + 4\bar{e}^4 - 2\bar{e}^3) + \lambda_2^2(1-\bar{e})(3PR_r^2\bar{e} - 2PR_r^2 \\ & - 5PR_r\bar{e}^3 + 2PR_r\bar{e}^2 + PR_r\bar{e} + 4\bar{e}^4 - 3\bar{e}^3), \\ B =& \lambda_2^2 \bar{e}(1-\bar{e})(3PR_r^2 - 5PR_r\bar{e}^2 - PR_r + 4\bar{e}^3 - \bar{e}^2) \\ & + \lambda_1\lambda_2 N(PR_r - \bar{e})^2\big[\lambda_2(3\bar{e}-1) + \lambda_1(\bar{e}-1)\big] \\ & + 2\lambda_1\lambda_2(1-\bar{e})(3PR_r^2\bar{e} - PR_r^2 - 5PR_r\bar{e}^3 + PR_r\bar{e}^2 \\ & + 4\bar{e}^4 - 2\bar{e}^3) + \lambda_1^2(1-\bar{e})(3PR_r^2\bar{e} - 2PR_r^2 \\ & - 5PR_r\bar{e}^3 + 2PR_r\bar{e}^2 + PR_r\bar{e} + 4\bar{e}^4 - 3\bar{e}^3). \end{aligned} \tag{23}$$

Proof: See the Appendix.

Since function $f(e_1)$ is derived from the Lagrangian function of the energy consumption optimization problem (P2), it is called the energy consumption characteristic function. The energy consumption characteristic function of the two-machine synchronous exponential line for various PR_r, N, and λ_i's is shown in Fig. 3.



(a) $\lambda_1 = 0.5$, $\lambda_2 = 2$ (b) $\lambda_1 = 2$, $\lambda_2 = 0.5$

(c) $\lambda_1 = 0.05$, $\lambda_2 = 0.05$ (d) $\lambda_1 = 0.5$, $\lambda_2 = 0.5$

(e) $\lambda_1 = 1$, $\lambda_2 = 1$ (f) $\lambda_1 = 2$, $\lambda_2 = 2$

Fig. 3: The behavior of function $f(e_1)$

In the next section, an effective and efficient algorithm will be designed to solve the optimality equations (12) and (45).

IV. Solution Methodology

In this section, a method for solving the optimality equations (12) and (45) (and thus, solving the optimal solution of problem (P2)) is proposed. Specifically, in Subsection IV-A, mathematical properties of $f(e_1)$ are further explored; in Subsection IV-B, based on the properties of $f(e_1)$, an effective and efficient algorithm is designed to solve the optimality equations.

A. Properties of the Energy Consumption Characteristic Function

Clearly, to solve problem (P2), it is necessary to solve the optimality equations (12) and (45). Considering that the energy consumption characteristic function $f(e_1)$ is critical for solving the optimality equations, its mathematical properties are explored in this subsection.

First, the domain and range of $f(e_1)$ are analyzed. From (14), it is easy to check that for both problem (P2) and $f(e_1)$, e_1 and e_2 take values on $(PR_r, 1)$, which can be observed in Fig. 3 as well. In addition, considering that

$$\lim_{e_1 \to PR_r} f(e_1) = +\infty, \ \lim_{e_1 \to 1} f(e_1) = 0 \tag{24}$$

and taking into account the continuity of $f(e_1)$, we conclude that $f(e_1)$ takes value on $(0, +\infty)$, which can be observed in Fig. 3.

From Fig. 3, one can also observe that $f(e_1)$ is strictly decreasing in e_1. Although it is very hard to prove the monotonicity, the numerical method is adopted to justify it. To do that, 1000 test cases with parameters randomly and equiprobably selected from the following sets are constructed:

$$\begin{aligned} &\lambda_i \in (0, 10), \ i = 1, 2, \ PR_r \in (0, 1), \\ &N \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}. \end{aligned} \tag{25}$$

For all test cases, e_1 is selected as $PR_r + 0.01k(1 - PR_r)$, $k = 1, 2, \ldots, 99$, and $\Delta f(e_1) = f(e_1 + \Delta e_1) - f(e_1)$ is calculated for each e_1, where $\Delta e_1 = 10^{-4}$. As a result, we have:

Numerical Fact 4.1: For all of the 1000 test cases thus constructed, $\Delta f(e_1)$ is always negative for all values of e_1, which implies that $f(e_1)$ is strictly decreasing in e_1.

Based on Theorem 3.3 and the analysis in the current subsection, we conclude that similar to the Bernoulli and geometric reliability models, the energy consumption characteristic function for the synchronous exponential line is also of good mathematical properties such as positiveness, continuity, and strict monotonicity. On the basis of these properties, an algorithm for solving the optimality equations is designed in the next subsection.

B. Algorithm Design

From the previous sections, it is concluded that $f(e_1)$ is a strictly decreasing function on $(PR_r, 1)$ and takes values on $(0, +\infty)$. Due to the monotonicity of functions $e_2(e_1)$ and $f(e_1)$, for any $\frac{P_1}{P_2}$, the optimality equations (12) and (45) always have a unique solution (e_1^*, e_2^*). It is not difficult to check that this unique solution is the optimal solution of (P2). Based on the value of $\frac{P_1}{P_2}$, the qualitative relationship between e_1^*, e_2^*, and $\bar{e}$ is as follows: if $\frac{P_1}{P_2} = f(\bar{e})$, then $e_1^* = e_2^* = \bar{e}$; if $\frac{P_1}{P_2} < f(\bar{e})$ (or $\frac{P_1}{P_2} > f(\bar{e})$), then $e_2^* < \bar{e} < e_1^*$ (and correspondingly, $e_1^* < \bar{e} < e_2^*$), where $\bar{e}$ and $f(\bar{e})$ are expressed in (17) and (23), respectively. In the following, a method is proposed to effectively and efficiently solve (e_1^*, e_2^*) for $\frac{P_1}{P_2} \neq f(\bar{e})$.

Clearly, it is almost impossible to provide the closed-form expression of the optimal solution (e_1^*, e_2^*) for $\frac{P_1}{P_2} \neq f(\bar{e})$. In this case, a bisection algorithm is proposed to numerically solve (e_1^*, e_2^*). Specifically, if $\frac{P_1}{P_2} < f(\bar{e})$ (or $\frac{P_1}{P_2} > f(\bar{e})$), a bisection search is performed on $(\bar{e}, 1)$ (and correspondingly, $(PR_r, \bar{e})$) to solve e_1^*. Taking the case of $\frac{P_1}{P_2} < f(\bar{e})$ as an example, let $e_1^L = \bar{e}$ and $e_1^U = 1$ denote the initial lower- and upper-endpoint of the bisection interval, respectively. Let $\hat{e}_1 = \frac{e_1^L + e_1^U}{2}$, solve $\hat{e}_2$ from the implicit function $\hat{e}_2\big[1 - Q(\hat{e}_1, \hat{e}_2, N)\big] = PR_r$, and by using $(\hat{e}_1, \hat{e}_2)$, calculate $f(\hat{e}_1)$ based on (20), (21), and (22). If $|f(\hat{e}_1) - \frac{P_1}{P_2}| < \epsilon$ (where ϵ is a predefined small enough positive real number), then $(e_1^*, e_2^*) = (\hat{e}_1, \hat{e}_2)$ and the search ends. Otherwise, if $f(\hat{e}_1) > \frac{P_1}{P_2}$ (or $f(\hat{e}_1) < \frac{P_1}{P_2}$), e_1^L is set to $\hat{e}_1$ and e_1^U remains unchanged (or correspondingly, e_1^U is set to $\hat{e}_1$ and e_1^L remains unchanged). The bisection process is repeated until $|f(\hat{e}_1) - \frac{P_1}{P_2}| < \epsilon$ is satisfied. Similarly, for the case of $\frac{P_1}{P_2} > f(\bar{e})$, initializing $e_1^L = PR_r$ and $e_1^U = \bar{e}$ and performing the above bisection search will obtain (e_1^*, e_2^*). The flowchart of the bisection search algorithm for solving (P2) with $\frac{P_1}{P_2} < f(\bar{e})$ is shown in Fig. 4.

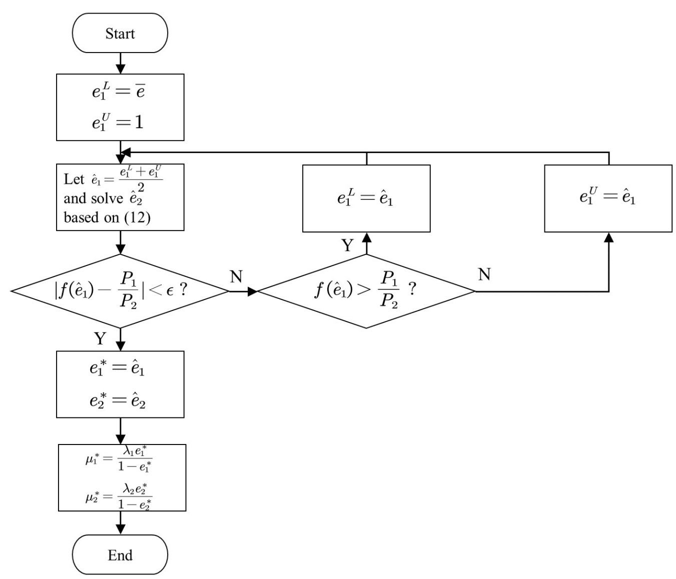


Fig. 4: Flowchart of the algorithm for solving (P2) with $\frac{P_1}{P_2} < f(\bar{e})$

On the basis of the developed bisection search algorithm, extensive numerical experiments for various PR_r, N, $\frac{P_1}{P_2}$, λ_1, and λ_2 have been conducted. Some of the test cases and their optimal solutions are shown in Table I, where, without loss of generality, the optimal objective value z^* is provided for $P_2 = 1$.

TABLE I: Optimal solution of (P2) for various test cases ($P_2 = 1$)

No.	$\frac{P_1}{P_2}$	N	PR_r	λ_1	λ_2	e_1^*	e_2^*	z^*
1	0.5	1	0.5	0.5	0.5	0.7589	0.5498	0.9292
2	0.5	1	0.5	0.5	2	0.7134	0.5419	0.8986
3	0.5	1	0.5	2	0.5	0.7099	0.5522	0.9072
4	0.5	1	0.5	2	2	0.6250	0.5363	0.8488
5	0.5	1	0.9	0.5	0.5	0.9341	0.9071	1.3742
6	0.5	1	0.9	0.5	2	0.9249	0.9060	1.3685
7	0.5	1	0.9	2	0.5	0.9247	0.9062	1.3686
8	0.5	1	0.9	2	2	0.9121	0.9037	1.3597
9	0.5	2	0.5	0.5	0.5	0.6880	0.5465	0.8905
10	0.5	2	0.5	0.5	2	0.6476	0.5347	0.8585
11	0.5	2	0.5	2	0.5	0.6411	0.5442	0.8647
12	0.5	2	0.5	2	2	0.5763	0.5243	0.8124
13	0.5	2	0.9	0.5	0.5	0.9210	0.9056	1.3661
14	0.5	2	0.9	0.5	2	0.9146	0.9042	1.3615
15	0.5	2	0.9	2	0.5	0.9145	0.9043	1.3615
16	0.5	2	0.9	2	2	0.9066	0.9021	1.3554
17	1	1	0.5	0.5	0.5	0.6369	0.6369	1.2739
18	1	1	0.5	0.5	2	0.6270	0.6043	1.2313
19	1	1	0.5	2	0.5	0.6043	0.6270	1.2313
20	1	1	0.5	2	2	0.5732	0.5732	1.1464
21	1	1	0.9	0.5	0.5	0.9184	0.9184	1.8369
22	1	1	0.9	0.5	2	0.9141	0.9138	1.8279
23	1	1	0.9	2	0.5	0.9138	0.9141	1.8279
24	1	1	0.9	2	2	0.9072	0.9072	1.8143
25	1	2	0.5	0.5	0.5	0.6054	0.6054	1.2108
26	1	2	0.5	0.5	2	0.5905	0.5757	1.1662
27	1	2	0.5	2	0.5	0.5757	0.5905	1.1662
28	1	2	0.5	2	2	0.5459	0.5459	1.0917
29	1	2	0.9	0.5	0.5	0.9120	0.9120	1.8240
30	1	2	0.9	0.5	2	0.9086	0.9085	1.8171
31	1	2	0.9	2	0.5	0.9085	0.9086	1.8171
32	1	2	0.9	2	2	0.9040	0.9040	1.8080
33	2	1	0.5	0.5	0.5	0.5498	0.7589	1.8584
34	2	1	0.5	0.5	2	0.5522	0.7099	1.8144
35	2	1	0.5	2	0.5	0.5419	0.7134	1.7972
36	2	1	0.5	2	2	0.5363	0.6250	1.6976
37	2	1	0.9	0.5	0.5	0.9071	0.9341	2.7483
38	2	1	0.9	0.5	2	0.9062	0.9247	2.7372
39	2	1	0.9	2	0.5	0.9060	0.9249	2.7369
40	2	1	0.9	2	2	0.9037	0.9121	2.7194
41	2	2	0.5	0.5	0.5	0.5465	0.6879	1.7809
42	2	2	0.5	0.5	2	0.5442	0.6411	1.7295
43	2	2	0.5	2	0.5	0.5346	0.6476	1.7169
44	2	2	0.5	2	2	0.5243	0.5763	1.6248
45	2	2	0.9	0.5	0.5	0.9056	0.9210	2.7321
46	2	2	0.9	0.5	2	0.9043	0.9145	2.7231
47	2	2	0.9	2	0.5	0.9042	0.9146	2.7230
48	2	2	0.9	2	2	0.9021	0.9066	2.7108

V. Conclusions and Future Work

The energy consumption optimization problem for the two-machine synchronous exponential serial lines has been investigated in this paper. Although this problem has been analyzed and solved for Bernoulli and geometric lines, it is the first time that it is formulated and solved for production lines with continuous reliability models (i.e., continuous probability distributions characterizing the reliability of machines). Similar to the Bernoulli and geometric lines, the energy consumption optimization problem for the two-machine synchronous exponential line is mathematically formulated and analyzed, and an effective and efficient algorithm is designed to solve its optimal solution.

In the future, the research results of this paper will be extended to more complex lines including, but not limited to, two-machine asynchronous exponential serial lines, long exponential serial lines and assembly systems with multiple machines, production lines with non-Markovian, e.g., Weibull, gamma, and log-normal, reliability models, and re-entrant lines for semiconductor manufacturing.

Appendix

Proofs of Theorems

A. Proof of Theorem 3.1

Proof: Re-writing (1) as

$$PR = e_2\big[1 - Q(e_1, e_2, N)\big] \tag{26}$$

and taking into account that the Q-function is continuous, it is concluded that PR is a continuous function of both e_1 and e_2 and takes values on $(0,1)$. Therefore, for any $PR_r \in (0,1)$, (P2) always has at least one feasible solution and thus, has the optimal solution.

To prove the theorem, we choose PR_{r1} and PR_{r2} such that $0 < PR_{r1} < PR_{r2} < 1$, and let (P2') and (P2") denote (P2) with PR_r replaced by PR_{r1} and by PR_{r2}, respectively. In addition, denote the optimal solutions of (P2') and (P2") as $(e^*_{1,r1}, e^*_{2,r1})$ and $(e^*_{1,r2}, e^*_{2,r2})$, respectively, and their corresponding optimal values as z^*_{r1} and z^*_{r2}. Construct a solution $(e^*_{1,r2}, \hat{e}_{2,r1})$ of (P2') which satisfies $PR_{r1} = \hat{e}_{2,r1}\big[1 - Q(e^*_{1,r2}, \hat{e}_{2,r1}, N)\big]$. Considering that the production rate of $(e^*_{1,r2}, e^*_{2,r2})$ is PR_{r2} and taking into account the monotonicity of PR with respect to e_2, we have

$$0 < \hat{e}_{2,r1} < e^*_{2,r2} < 1. \tag{27}$$

Clearly, (27) shows that $0 < \hat{e}_{2,r1} < 1$, which indicates that $(e^*_{1,r2}, \hat{e}_{2,r1})$ is a feasible solution of (P2'). Thus, for the optimal solution $(e^*_{1,r1}, e^*_{2,r1})$ and the feasible solution $(e^*_{1,r2}, \hat{e}_{2,r1})$ of (P2'), and the optimal solution $(e^*_{1,r2}, e^*_{2,r2})$ of (P2"), taking into account (27), we have

$$\begin{aligned} z^*_{r1} &= \sum_{i=1}^{2} P_i e^*_{i,r1} \leqslant P_1 e^*_{1,r2} + P_2 \hat{e}_{2,r1} \\ &< \sum_{i=1}^{2} P_i e^*_{i,r2} = z^*_{r2}, \end{aligned} \tag{28}$$

which completes the proof. ■

B. Proof of Theorem 3.2

Proof: Let

$$g(x) = x^3 - x^2 - \Big(\frac{N\lambda_1\lambda_2}{\lambda_1+\lambda_2} + PR_r\Big)x + PR_r\Big(\frac{N\lambda_1\lambda_2}{\lambda_1+\lambda_2} + 1\Big). \tag{29}$$

Clearly, $\bar{e}$ is one of the solutions of $g(x) = 0$.

To solve $\bar{e}$, the solutions of $g(x) = 0$ should be comprehensively analyzed. Obviously, $g(x) = 0$ is a univariate cubic equation and has three roots in the field of complex numbers. For function $g(x)$, it is easy to obtain

$$\begin{aligned}&\lim_{x\to-\infty} g(x)=-\infty,\ g(0)=PR_r\Big(\frac{N\lambda_1\lambda_2}{\lambda_1+\lambda_2}+1\Big)>0,\\&\lim_{x\to+\infty} g(x)=+\infty,\ g(1)=(PR_r-1)\frac{N\lambda_1\lambda_2}{\lambda_1+\lambda_2}<0,\end{aligned}\tag{30}$$

which, based on the zero-point existence theorem, implies that $g(x)=0$ has at least one root in each of sets $(-\infty,0)$, $(0,1)$, and $(1,+\infty)$. Considering that $g(x)=0$ has three roots in the field of complex numbers, it is concluded that it has three real roots, which are, respectively, located in $(-\infty,0)$, $(0,1)$, and $(1,+\infty)$. Clearly, $\bar{e}$ is the one located in $(0,1)$.

According to Cardano's formula for solving the cubic equations, it is not difficult to derive the expression of $\bar{e}$ in (17). ■

C. Partial Derivatives

Before proving the following theorem, we need to derive the partial derivatives of PR with respect to e_1 and e_2, respectively (see the expression of PR in (1)).

First, we derive the partial derivatives of β, which is expressed in (3), with respect to e_1 and e_2, respectively. Let β_n and β_d denote the numerator and denominator of β, respectively, then

$$\begin{aligned}\beta_n=&\lambda_1\lambda_2(e_2-e_1)\big[\lambda_1(1-e_2)+\lambda_2(1-e_1)\big],\\\beta_d=&(1-e_1)(1-e_2)(\lambda_1+\lambda_2)\\&\cdot\big[\lambda_1e_1(1-e_2)+\lambda_2e_2(1-e_1)\big].\end{aligned}\tag{31}$$

Taking the partial derivative of β with respect to e_1 and e_2, respectively, we have

$$\begin{aligned}\frac{\partial\beta}{\partial e_1}=&\frac{\frac{\partial\beta_n}{\partial e_1}\beta_d-\frac{\partial\beta_d}{\partial e_1}\beta_n}{\beta_d^2},\\\frac{\partial\beta}{\partial e_2}=&\frac{\frac{\partial\beta_n}{\partial e_2}\beta_d-\frac{\partial\beta_d}{\partial e_2}\beta_n}{\beta_d^2},\end{aligned}\tag{32}$$

where

$$\begin{aligned}&\frac{\partial\beta_n}{\partial e_1}\beta_d-\frac{\partial\beta_d}{\partial e_1}\beta_n\\=&\lambda_1\lambda_2(1-e_1)(1-e_2)(\lambda_1+\lambda_2)\\&\cdot\big[\lambda_1(1-e_2)+\lambda_2(1+e_2-2e_1)\big]\\&\cdot\big[\lambda_1e_1(1-e_2)+\lambda_2e_2(1-e_1)\big]\\&-\lambda_1\lambda_2(1-e_2)(e_1-e_2)(\lambda_1+\lambda_2)\\&\cdot\big[\lambda_1(1-2e_1)(1-e_2)-2\lambda_2e_2(1-e_1)\big]\\&\cdot\big[\lambda_1(1-e_2)+\lambda_2(1-e_1)\big]\\=&-\lambda_1\lambda_2(1-e_2)^2(\lambda_1+\lambda_2)\\&\cdot\big[\lambda_1^2(1-e_2)(e_1^2+e_2-2e_1e_2)\\&+2\lambda_1\lambda_2(1-e_1)e_2(1-e_2)+\lambda_2^2(1-e_1)^2e_2\big],\end{aligned}\tag{33}$$

$$\begin{aligned}&\frac{\partial\beta_n}{\partial e_2}\beta_d-\frac{\partial\beta_d}{\partial e_2}\beta_n\\=&\lambda_1\lambda_2(1-e_1)(1-e_2)(\lambda_1+\lambda_2)\\&\cdot\big[\lambda_1(1+e_1-2e_2)+\lambda_2(1-e_1)\big]\\&\cdot\big[\lambda_1e_1(1-e_2)+\lambda_2e_2(1-e_1)\big]\\&-\lambda_1\lambda_2(1-e_1)(e_1-e_2)(\lambda_1+\lambda_2)\\&\cdot\big[\lambda_2(1-e_1)(1-2e_2)-2\lambda_1e_1(1-e_2)\big]\\&\cdot\big[\lambda_1(1-e_2)+\lambda_2(1-e_1)\big]\\=&\lambda_1\lambda_2(1-e_1)^2(\lambda_1+\lambda_2)\\&\cdot\big[\lambda_1^2e_1(1-e_2)^2+2\lambda_1\lambda_2e_1(1-e_1)(1-e_2)\\&+\lambda_2^2(1-e_1)(e_1-2e_1e_2+e_2^2)\big].\end{aligned}\tag{34}$$

From (1)-(3), it is not difficult to obtain the following expression:

$$PR=\frac{e_1e_2\big[1-e_1-(1-e_2)\exp(-\beta N)\big]}{(1-e_1)e_2-(1-e_2)e_1\exp(-\beta N)}.\tag{35}$$

Similarly, let PR_n and PR_d denote the numerator and denominator of PR, respectively, i.e.,

$$\begin{aligned}PR_n=&e_1e_2\big[1-e_1-(1-e_2)\exp(-\beta N)\big],\\PR_d=&(1-e_1)e_2-(1-e_2)e_1\exp(-\beta N).\end{aligned}\tag{36}$$

Then, the partial derivatives of PR with respect to e_1 and e_2 are derived as follows:

$$\begin{aligned}\frac{\partial PR}{\partial e_1}=&\frac{\frac{\partial PR_n}{\partial e_1}PR_d-\frac{\partial PR_d}{\partial e_1}PR_n}{PR_d^2},\\\frac{\partial PR}{\partial e_2}=&\frac{\frac{\partial PR_n}{\partial e_2}PR_d-\frac{\partial PR_d}{\partial e_2}PR_n}{PR_d^2},\end{aligned}\tag{37}$$

where

$$\begin{aligned}&\frac{\partial PR_n}{\partial e_1}PR_d-\frac{\partial PR_d}{\partial e_1}PR_n\\=&\Big[e_2(1-2e_1)+e_2(1-e_2)\Big(-1+e_1\frac{\partial\beta}{\partial e_1}N\Big)\exp(-\beta N)\Big]\\&\cdot\big[(1-e_1)e_2-(1-e_2)e_1\exp(-\beta N)\big]\\&-\Big[-e_2+(1-e_2)\Big(-1+e_1\frac{\partial\beta}{\partial e_1}N\Big)\exp(-\beta N)\Big]\\&\cdot e_1e_2\big[1-e_1-(1-e_2)\exp(-\beta N)\big]\\=&(1-e_1)^2e_2^2+\big[e_2(1-e_2)(e_1^2-e_2)\\&+e_1e_2(1-e_1)(1-e_2)(e_2-e_1)\frac{\partial\beta}{\partial e_1}N\big]\exp(-\beta N),\\&\frac{\partial PR_n}{\partial e_2}PR_d-\frac{\partial PR_d}{\partial e_2}PR_n\\=&\Big\{\Big[-e_1(1-2e_2)+e_1e_2(1-e_2)\frac{\partial\beta}{\partial e_2}N\Big]\exp(-\beta N)\\&+e_1(1-e_1)\Big\}\big[(1-e_1)e_2-(1-e_2)e_1\exp(\beta N)\big]\\&-\Big\{1-e_1+e_1\Big[1+(1-e_2)\frac{\partial\beta}{\partial e_2}N\Big]\exp(-\beta N)\Big\}\\&\cdot e_1e_2\big[1-e_1-(1-e_2)\exp(-\beta N)\big]\\=&\exp(-\beta N)\Big[e_1e_2(1-e_1)(1-e_2)(e_2-e_1)\frac{\partial\beta}{\partial e_2}N\\&+e_1(1-e_1)(e_2^2-e_1)\Big]+e_1^2(1-e_2)^2\exp(-2\beta N).\end{aligned}\tag{38}$$

D. Proof of Theorem 3.3

Proof: To prove the theorem, construct the Lagrangian function of (P2) as follows:

$$z=\sum_{i=1}^{2}P_ie_i+\eta\big[e_2(1-Q)-PR_r\big], \tag{39}$$

where η is the Lagrangian multiplier. Taking the partial derivative of z with respect to e_1 and e_2, respectively, and noting that Q is continuously differentiable, we have

$$\begin{aligned} &P_1-\eta e_2\frac{\partial Q}{\partial e_1}=0,\\ &P_2+\eta\left[(1-Q)-e_2\frac{\partial Q}{\partial e_2}\right]=0, \end{aligned} \tag{40}$$

which imply

$$\frac{P_1}{P_2}=\frac{e_2\frac{\partial Q}{\partial e_1}}{e_2\frac{\partial Q}{\partial e_2}-(1-Q)}. \tag{41}$$

From (1), it follows

$$\frac{\frac{\partial PR}{\partial e_1}}{\frac{\partial PR}{\partial e_2}}=\frac{e_2\frac{\partial Q}{\partial e_1}}{e_2\frac{\partial Q}{\partial e_2}-(1-Q)}. \tag{42}$$

Thus, combining the above two equations, we have

$$\frac{P_1}{P_2}=\frac{\frac{\partial PR}{\partial e_1}}{\frac{\partial PR}{\partial e_2}}. \tag{43}$$

Let

$$\begin{aligned} f_n(e_1):=&\frac{\frac{\partial PR}{\partial e_1}PR_d^2\exp(\beta N)\beta_d^2}{(1-e_1)(1-e_2)(\lambda_1+\lambda_2)}\\ =&\frac{\left(\frac{\partial PR_n}{\partial e_1}PR_d-\frac{\partial PR_d}{\partial e_1}PR_n\right)\exp(\beta N)\beta_d^2}{(1-e_1)(1-e_2)(\lambda_1+\lambda_2)},\\ f_d(e_1):=&\frac{\frac{\partial PR}{\partial e_2}PR_d^2\exp(\beta N)\beta_d^2}{(1-e_1)(1-e_2)(\lambda_1+\lambda_2)}\\ =&\frac{\left(\frac{\partial PR_n}{\partial e_2}PR_d-\frac{\partial PR_d}{\partial e_2}PR_n\right)\exp(\beta N)\beta_d^2}{(1-e_1)(1-e_2)(\lambda_1+\lambda_2)},\\ f(e_1):=&\frac{f_n(e_1)}{f_d(e_1)}, \end{aligned} \tag{44}$$

then we have the optimality equation

$$f(e_1)=\frac{P_1}{P_2} \tag{45}$$

and

$$\begin{aligned} f_n(e_1)=&e_2(1-e_2)^2(e_1^2-e_2)(1-e_1)(\lambda_1+\lambda_2)\\ &\cdot\big[\lambda_1e_1(1-e_2)+\lambda_2e_2(1-e_1)\big]^2\\ &+Ne_1e_2\lambda_1\lambda_2(e_1-e_2)(1-e_2)^2\\ &\cdot\big[\lambda_1^2(1-e_2)(e_1^2+e_2-2e_1e_2)\\ &+2\lambda_1\lambda_2(1-e_1)e_2(1-e_2)+\lambda_2^2(1-e_1)^2e_2\big]\\ &+(1-e_1)^3e_2^2(1-e_2)(\lambda_1+\lambda_2)\\ &\cdot\big[\lambda_1e_1(1-e_2)+\lambda_2e_2(1-e_1)\big]^2\exp(\beta N), \end{aligned} \tag{46}$$

$$\begin{aligned} f_d(e_1)=&e_1(1-e_1)^2(e_2^2-e_1)(1-e_2)(\lambda_1+\lambda_2)\\ &\cdot\big[\lambda_1e_1(1-e_2)+\lambda_2e_2(1-e_1)\big]^2\\ &+Ne_1e_2\lambda_1\lambda_2(1-e_1)^2(e_2-e_1)\\ &\cdot\big[\lambda_1^2e_1(1-e_2)^2+2\lambda_1\lambda_2e_1(1-e_1)(1-e_2)\\ &+\lambda_2^2(1-e_1)(e_1-2e_1e_2+e_2^2)\big]\\ &+e_1^2(1-e_1)(1-e_2)^3(\lambda_1+\lambda_2)\\ &\cdot\big[\lambda_1e_1(1-e_2)+\lambda_2e_2(1-e_1)\big]^2\exp(-\beta N). \end{aligned} \tag{47}$$

Since PR is increasing in e_1 and e_2, respectively, it is clear that $f_n(e_1)\geqslant 0$ and $f_d(e_1)\geqslant 0$. Note that if and only if $e_1=\bar{e}$, $f_n(e_1)=f_d(e_1)=0$. In other words, $f(e_1)$ is a positive continuous function when $e_1\neq\bar{e}$. Although expressions in (44) define $f(e_1)$ only for $e_1\neq\bar{e}$, they could be regarded as the general definition of $f(e_1)$ and $f(\bar{e})$ be derived by investigating $\lim_{e_1\to\bar{e}}f(e_1)$. Adopting this concept, the expression of $f(\bar{e})$ is derived and the positiveness and continuity of $f(e_1)$ at $e_1=\bar{e}$ are proved in the following.

First, we derive the expression of $f(\bar{e})$ from expressions in (44). Based on the optimality equation (12) and production rate expression(1)-(3), we have

$$\exp(\beta N)=\frac{e_1(1-e_2)(e_2-PR_r)}{e_2(1-e_1)(e_1-PR_r)}. \tag{48}$$

To facilitate the derivation, re-denote $f_n(e_1)$ and $f_d(e_1)$ as $G_n(e_1,e_2)$ and $G_d(e_1,e_2)$, respectively, and take into account (48), then we have

$$\begin{aligned} G_n(e_1,e_2)=&e_2(1-e_2)^2(e_1^2-e_2)(1-e_1)(\lambda_1+\lambda_2)\\ &\cdot\big[\lambda_1e_1(1-e_2)+\lambda_2e_2(1-e_1)\big]^2\\ &+Ne_1e_2\lambda_1\lambda_2(e_1-e_2)(1-e_2)^2\\ &\cdot\big[\lambda_1^2(1-e_2)(e_1^2+e_2-2e_1e_2)\\ &+2\lambda_1\lambda_2(1-e_1)e_2(1-e_2)+\lambda_2^2(1-e_1)^2e_2\big]\\ &+(1-e_1)^2e_1e_2(1-e_2)^2(\lambda_1+\lambda_2)\\ &\cdot\big[\lambda_1e_1(1-e_2)+\lambda_2e_2(1-e_1)\big]^2\frac{e_2-PR_r}{e_1-PR_r}, \end{aligned} \tag{49}$$

$$\begin{aligned} G_d(e_1,e_2)=&e_1(1-e_1)^2(e_2^2-e_1)(1-e_2)(\lambda_1+\lambda_2)\\ &\cdot\big[\lambda_1e_1(1-e_2)+\lambda_2e_2(1-e_1)\big]^2\\ &+Ne_1e_2\lambda_1\lambda_2(1-e_1)^2(e_2-e_1)\\ &\cdot\big[\lambda_1^2e_1(1-e_2)^2+2\lambda_1\lambda_2e_1(1-e_1)(1-e_2)\\ &+\lambda_2^2(1-e_1)(e_1-2e_1e_2+e_2^2)\big]\\ &+e_1e_2(1-e_1)^2(1-e_2)^2(\lambda_1+\lambda_2)\\ &\cdot\big[\lambda_1e_1(1-e_2)+\lambda_2e_2(1-e_1)\big]^2\frac{e_1-PR_r}{e_2-PR_r}. \end{aligned} \tag{50}$$

Thus,

$$f(\bar{e})=\lim_{e_1\to\bar{e}}f(e_1)=\lim_{e_1\to\bar{e}}\frac{G_n(e_1,e_2)}{G_d(e_1,e_2)}. \tag{51}$$

Furthermore, it is easy to check that

$$G_n(\bar{e},\bar{e})=0,\ G_d(\bar{e},\bar{e})=0. \tag{52}$$

To derive $f(\bar{e})$ using the Taylor expansion, partial derivatives of bivariate functions $G_n(e_1,e_2)$ and $G_d(e_1,e_2)$ with respect to e_1 and e_2 are obtained as follows:

$$\begin{aligned}A_{n1} =&\frac{\partial G_n(e_1,e_2)}{\partial e_1}\Big|_{e_1=e_2=\bar{e}}\\ =&\bar{e}^3(1-\bar{e})^4(\lambda_1+\lambda_2)^2\\ &\cdot\left[\frac{(\bar{e}^2-PR_r)(1-\bar{e})(\lambda_1+\lambda_2)}{\bar{e}-PR_r}+N\lambda_1\lambda_2\right],\end{aligned} \quad (53)$$

$$\begin{aligned}A_{n2} =&\frac{\partial G_n(e_1,e_2)}{\partial e_2}\Big|_{e_1=e_2=\bar{e}}\\ =&-\bar{e}^3(1-\bar{e})^4(\lambda_1+\lambda_2)^2\\ &\cdot\left[\frac{(\bar{e}^2-PR_r)(1-\bar{e})(\lambda_1+\lambda_2)}{\bar{e}-PR_r}+N\lambda_1\lambda_2\right],\end{aligned} \quad (54)$$

$$\begin{aligned}A_{d1} =&\frac{\partial G_d(e_1,e_2)}{\partial e_1}\Big|_{e_1=e_2=\bar{e}}\\ =&-\bar{e}^3(1-\bar{e})^4(\lambda_1+\lambda_2)^2\\ &\cdot\left[\frac{(\bar{e}^2-PR_r)(1-\bar{e})(\lambda_1+\lambda_2)}{\bar{e}-PR_r}+N\lambda_1\lambda_2\right],\end{aligned} \quad (55)$$

$$\begin{aligned}A_{d2} =&\frac{\partial G_d(e_1,e_2)}{\partial e_2}\Big|_{e_1=e_2=\bar{e}}\\ =&\bar{e}^3(1-\bar{e})^4(\lambda_1+\lambda_2)^2\\ &\cdot\left[\frac{(\bar{e}^2-PR_r)(1-\bar{e})(\lambda_1+\lambda_2)}{\bar{e}-PR_r}+N\lambda_1\lambda_2\right].\end{aligned} \quad (56)$$

Clearly,

$$\begin{aligned}&A_{n1}=-A_{n2}=-A_{d1}=A_{d2}\\ =&\bar{e}^3(1-\bar{e})^4(\lambda_1+\lambda_2)^2\\ &\cdot\left[\frac{(\bar{e}^2-PR_r)(1-\bar{e})(\lambda_1+\lambda_2)}{\bar{e}-PR_r}+N\lambda_1\lambda_2\right]\\ =&\bar{e}^3(1-\bar{e})^4(\lambda_1+\lambda_2)^3\\ &\cdot\left[\frac{\bar{e}^3-\bar{e}^2-(\frac{N\lambda_1\lambda_2}{\lambda_1+\lambda_2}+PR_r)\bar{e}+PR_r(\frac{N\lambda_1\lambda_2}{\lambda_1+\lambda_2}+1)}{PR_r-\bar{e}}\right].\end{aligned} \quad (57)$$

Taking into account (16), we have

$$A_{n1}=A_{n2}=A_{d1}=A_{d2}=0. \quad (58)$$

Since all first-order partial derivatives are 0, the second-order partial derivatives of $G_n(e_1,e_2)$ and $G_d(e_1,e_2)$ are derived. As a result, we have

$$\begin{aligned}&B_{n1}=\frac{\partial^2 G_n(e_1,e_2)}{\partial e_1^2}\Big|_{e_1=e_2=\bar{e}}\\ =&\big[\lambda_1^2\lambda_2 N(\bar{e}-1)(PR_r-\bar{e})^2+\lambda_1\lambda_2^2 N(3\bar{e}-1)(PR_r-\bar{e})^2\\ &+2\lambda_1\lambda_2(1-\bar{e})(3PR_r^2\bar{e}-PR_r^2-5PR_r\bar{e}^3+PR_r\bar{e}^2+4\bar{e}^4-2\bar{e}^3)\\ &+\lambda_2^2\bar{e}(1-\bar{e})(3PR_r^2-5PR_r\bar{e}^2-PR_r+4\bar{e}^3-\bar{e}^2)\\ &+\lambda_1^2(3PR_r^2\bar{e}-2PR_r^2-5PR_r\bar{e}^3+2PR_r\bar{e}^2+PR_r\bar{e}+4\bar{e}^4-3\bar{e}^3)\\ &\cdot(1-\bar{e})\big]\frac{2\bar{e}^2(\bar{e}-1)^3(\lambda_1+\lambda_2)}{(\bar{e}-PR_r)^2},\end{aligned} \quad (59)$$

$$\begin{aligned}&B_{n2}=2\frac{\partial^2 G_n(e_1,e_2)}{\partial e_1\partial e_2}\Big|_{e_1=e_2=\bar{e}}\\ =&\big[\lambda_1^2\lambda_2 N(5\bar{e}-1)(PR_r-\bar{e})^2+\lambda_1\lambda_2^2 N(\bar{e}-1)(PR_r-\bar{e})^2\\ &+2\lambda_1\lambda_2(1-\bar{e})(2PR_r^2\bar{e}-PR_r^2-4PR_r\bar{e}^2+2PR_r\bar{e}+\bar{e}^4)\\ &+\lambda_2^2(1-\bar{e})(2PR_r^2\bar{e}-3PR_r^2-2PR_r\bar{e}^2+4PR_r\bar{e}+\bar{e}^4-2\bar{e}^3)\\ &+\lambda_1^2(1-\bar{e})(2PR_r^2\bar{e}+PR_r^2-6PR_r\bar{e}^2+\bar{e}^4+2\bar{e}^3)\big]\\ &\cdot\frac{2\bar{e}^2(\bar{e}-1)^3(\lambda_1+\lambda_2)}{(\bar{e}-PR_r)^2},\end{aligned} \quad (60)$$

$$\begin{aligned}&B_{n3}=\frac{\partial^2 G_n(e_1,e_2)}{\partial e_2^2}\Big|_{e_1=e_2=\bar{e}}\\ =&\big[2\lambda_1^2\lambda_2 N(3\bar{e}-1)(PR_r-\bar{e})+2\lambda_1\lambda_2^2 N(2\bar{e}-1)(PR_r-\bar{e})\\ &+\lambda_1^2(5\bar{e}-1)(\bar{e}-1)(\bar{e}^2-PR_r)+\lambda_2^2(5\bar{e}-3)(\bar{e}-1)(\bar{e}^2-PR_r)\\ &-2\lambda_1\lambda_2(5\bar{e}-2)(1-\bar{e})(\bar{e}^2-PR_r)\big]\frac{2\bar{e}^2(1-\bar{e})^3(\lambda_1+\lambda_2)}{PR_r-\bar{e}},\end{aligned} \quad (61)$$

$$\begin{aligned}&B_{d1}=\frac{\partial^2 G_d(e_1,e_2)}{\partial e_1^2}\Big|_{e_1=e_2=\bar{e}}\\ =&\big[2\lambda_1^2\lambda_2 N(2\bar{e}-1)(PR_r-\bar{e})+2\lambda_1\lambda_2^2 N(3\bar{e}-1)(PR_r-\bar{e})\\ &+\lambda_1^2(5\bar{e}-3)(\bar{e}-1)(\bar{e}^2-PR_r)+\lambda_2^2(5\bar{e}-1)(\bar{e}-1)(\bar{e}^2-PR_r)\\ &-2\lambda_1\lambda_2(5\bar{e}-2)(\bar{e}-1)(PR_r-\bar{e}^2)\big]\frac{2\bar{e}^2(1-\bar{e})^3(\lambda_1+\lambda_2)}{PR_r-\bar{e}},\end{aligned} \quad (62)$$

$$\begin{aligned}&B_{d2}=2\frac{\partial^2 G_d(e_1,e_2)}{\partial e_1\partial e_2}\Big|_{e_1=e_2=\bar{e}}\\ =&\big[\lambda_1^2\lambda_2 N(\bar{e}-1)(PR_r-\bar{e})^2+\lambda_1\lambda_2^2 N(5\bar{e}-1)(PR_r-\bar{e})^2\\ &+2\lambda_1\lambda_2(1-\bar{e})(2PR_r^2\bar{e}-PR_r^2-4PR_r\bar{e}^2+2PR_r\bar{e}+\bar{e}^4)\\ &+\lambda_1^2(1-\bar{e})(2PR_r^2\bar{e}-3PR_r^2-2PR_r\bar{e}^2+4PR_r\bar{e}+\bar{e}^4-2\bar{e}^3)\\ &+\lambda_2^2(1-\bar{e})(2PR_r^2\bar{e}+PR_r^2-6PR_r\bar{e}^2+\bar{e}^4+2\bar{e}^3)\big]\\ &\cdot\frac{2\bar{e}^2(\bar{e}-1)^3(\lambda_1+\lambda_2)}{(\bar{e}-PR_r)^2},\end{aligned} \quad (63)$$

$$\begin{aligned}&B_{d3}=\frac{\partial^2 G_d(e_1,e_2)}{\partial e_2^2}\Big|_{e_1=e_2=\bar{e}}\\ =&\big[\lambda_1^2\lambda_2 N(3\bar{e}-1)(PR_r-\bar{e})^2+\lambda_1\lambda_2^2 N(\bar{e}-1)(PR_r-\bar{e})^2\\ &+2\lambda_1\lambda_2(1-\bar{e})(3PR_r^2\bar{e}-PR_r^2-5PR_r\bar{e}^3+PR_r\bar{e}^2+4\bar{e}^4-2\bar{e}^3)\\ &+\lambda_1^2\bar{e}(1-\bar{e})(3PR_r^2-5PR_r\bar{e}^2-PR_r+4\bar{e}^3-\bar{e}^2)\\ &+\lambda_2^2(3PR_r^2\bar{e}-2PR_r^2-5PR_r\bar{e}^3+2PR_r\bar{e}^2+PR_r\bar{e}+4\bar{e}^4-3\bar{e}^3)\\ &\cdot(1-\bar{e})\big]\frac{2\bar{e}^2(\bar{e}-1)^3(\lambda_1+\lambda_2)}{(\bar{e}-PR_r)^2},\end{aligned} \quad (64)$$

which indicate

$$\begin{aligned}&B_{n1}+B_{n2}+B_{n3}=0,\\ &B_{d1}+B_{d2}+B_{d3}=0,\\ &B_{n1}+B_{d1}-B_{n3}-B_{d3}=0.\end{aligned} \quad (65)$$

Let $e_1=\bar{e}+\Delta e_1$ and $e_2=\bar{e}+\Delta e_2$, the second-order Taylor expansions of $G_n(e_1,e_2)$ and $G_d(e_1,e_2)$ at $e_1=e_2=\bar{e}$ are, respectively,

$$
\begin{aligned}
G_n(e_1,e_2) =& G_n(\bar e,\bar e) + \frac{\partial G_n(e_1,e_2)}{\partial e_1}\Big|_{e_1=e_2=\bar e}\cdot \Delta e_1 \\
&+ \frac{\partial G_n(e_1,e_2)}{\partial e_2}\Big|_{e_1=e_2=\bar e}\cdot \Delta e_2 \\
&+ \frac{1}{2}\Bigg[\frac{\partial^2 G_n(e_1,e_2)}{\partial e_1^2}\Big|_{e_1=e_2=\bar e}\cdot \Delta e_1^2 \\
&+ 2\frac{\partial^2 G_n(e_1,e_2)}{\partial e_1 \partial e_2}\Big|_{e_1=e_2=\bar e}\cdot \Delta e_1\Delta e_2 \\
&+ \frac{\partial^2 G_n(e_1,e_2)}{\partial e_2^2}\Big|_{e_1=e_2=\bar e}\cdot \Delta e_2^2\Bigg] \\
&+ o(\Delta e_1^2) + o(\Delta e_1\Delta e_2) + o(\Delta e_2^2) \\
=& \frac{1}{2}\left[B_{n1}\Delta e_1^2 + B_{n2}\Delta e_1\Delta e_2 + B_{n3}\Delta e_2^2\right] \\
&+ o(\Delta e_1^2) + o(\Delta e_1\Delta e_2) + o(\Delta e_2^2),
\end{aligned} \tag{66}
$$

$$
\begin{aligned}
G_d(e_1,e_2) =& G_d(\bar e,\bar e) + \frac{\partial G_d(e_1,e_2)}{\partial e_2}\Big|_{e_1=e_2=\bar e}\cdot \Delta e_1 \\
&+ \frac{\partial G_d(e_1,e_2)}{\partial e_2}\Big|_{e_1=e_2=\bar e}\cdot \Delta e_2 \\
&+ \frac{1}{2}\Bigg[\frac{\partial^2 G_d(e_1,e_2)}{\partial e_1^2}\Big|_{e_1=e_2=\bar e}\cdot \Delta e_1^2 \\
&+ 2\frac{\partial^2 G_d(e_1,e_2)}{\partial e_1 \partial e_2}\Big|_{e_1=e_2=\bar e}\cdot \Delta e_1\Delta e_2 \\
&+ \frac{\partial^2 G_d(e_1,e_2)}{\partial e_2^2}\Big|_{e_1=e_2=\bar e}\cdot \Delta e_2^2\Bigg] \\
&+ o(\Delta e_1^2) + o(\Delta e_1\Delta e_2) + o(\Delta e_2^2) \\
=& \frac{1}{2}\left[B_{d1}\Delta e_1^2 + B_{d2}\Delta e_1\Delta e_2 + B_{d3}\Delta e_2^2\right] \\
&+ o(\Delta e_1^2) + o(\Delta e_1\Delta e_2) + o(\Delta e_2^2).
\end{aligned} \tag{67}
$$

Since e_1 and e_2 satisfy (12), taking into account (15), it is clear that when $\Delta e_1 \to 0$, $\Delta e_2 \to 0$. Thus, from (51), it follows

$$
\begin{aligned}
f(\bar e) &= \lim_{e_1\to\bar e} f(e_1) \\
&= \lim_{\Delta e_1\to 0}\frac{B_{n1}\Delta e_1^2 + B_{n2}\Delta e_1\Delta e_2 + B_{n3}\Delta e_2^2}{B_{d1}\Delta e_1^2 + B_{d2}\Delta e_1\Delta e_2 + B_{d3}\Delta e_2^2} \\
&= \lim_{\Delta e_1\to 0}\frac{B_{n1} + B_{n2}\frac{\Delta e_2}{\Delta e_1} + B_{n3}(\frac{\Delta e_2}{\Delta e_1})^2}{B_{d1} + B_{d2}\frac{\Delta e_2}{\Delta e_1} + B_{d3}(\frac{\Delta e_2}{\Delta e_1})^2} \\
&= \lim_{\Delta e_1\to 0}\frac{B_{n1} + B_{n2}e_2' + B_{n3}(e_2')^2}{B_{d1} + B_{d2}e_2' + B_{d3}(e_2')^2}.
\end{aligned} \tag{68}
$$

To obtain $f(\bar e)$, e_2' is derived. Specifically, taking the total derivative of both sides of (12), we have

$$
(1-Q)de_2 - e_2\left[\frac{\partial Q}{\partial e_1}de_1 + \frac{\partial Q}{\partial e_2}de_2\right] = 0, \tag{69}
$$

which implies

$$
e_2' = \frac{de_2}{de_1} = -\frac{e_2\frac{\partial Q}{\partial e_1}}{e_2\frac{\partial Q}{\partial e_2} - (1-Q)}. \tag{70}
$$

Taking into account (42) and (44), we have

$$
e_2' = -f(e_1). \tag{71}
$$

Thus, we obtain the equation of $f(\bar e)$ as follows:

$$
f(\bar e) = \frac{B_{n1} - B_{n2}f(\bar e) + B_{n3}f^2(\bar e)}{B_{d1} - B_{d2}f(\bar e) + B_{d3}f^2(\bar e)}, \tag{72}
$$

which can be rewritten as

$$
\begin{aligned}
&B_{d3}f^3(\bar e) - (B_{d2} + B_{n3})f^2(\bar e) \\
&+ (B_{d1} + B_{n2})f(\bar e) - B_{n1} = 0.
\end{aligned} \tag{73}
$$

Taking into account (65), we have

$$
(f(\bar e)+1)^2(B_{d3}f(\bar e) - B_{n1}) = 0. \tag{74}
$$

Considering that $f(e_1)$ is positive, we have

$$
f(\bar e) = \frac{B_{n1}}{B_{d3}}, \tag{75}
$$

which, by re-arranging terms, can be rewritten as (23). Clearly, $f(e_1)$ is positive and continuous at $e_1 = \bar e$ as well, which implies that $f(e_1)$ is a positive and continuous function and completes the proof. ■

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