Energy Consumption Optimization for Two-Machine Bernoulli Serial Lines Processing Perishable Products

Sheng Zhang, *Student Member, IEEE,* Lingchen Liu, *Student Member, IEEE,* Chao-Bo Yan, *Senior Member, IEEE*

***Abstract*—Reducing energy consumption and raw material waste in manufacturing systems processing perishable products are of significant importance. While there has been extensive research on yield analysis, control, and energy consumpiton optimization in serial lines, studies considering both yield assurance and energy consumption optimization are relatively scarce. This paper aims to investigate energy consumption optimization in two-machine Bernoulli lines with constraints on production rate and yield. Specifically, a chance constrained programming model for the problem is first formulated. Next, we analyze the properties of opportunity constraints and simplify the problem. Finally, based on the structural characteristics of the problem, we propose optimality conditions and design a numerical algorithm to obtain the unique optimal solution. Extensive numerical experiments demonstrate the effectiveness of the algorithm in solving the energy consumption optimization problem.**

***Index Terms*—Production rate, yield, lead time, chance constrained programming, monotonicity.**

I. Introduction

Production systems consume a significant amount of energy during operation. Taking semiconductor production lines as an example, statistical data from 27 semiconductor corporations worldwide indicates that the total energy consumption of these semiconductor companies in 2021 was 1.49×10^{11} kWh[1]. In addition to energy consumption, many production systems that porcess perishable products impose high requirements on the lead time (i.e. residence time, waiting time, flow time or sojourn time) of parts in the buffer, and incur significant costs to handle parts with excessively long lead time. These high-energy-consumption and material-wasting production systems are not conducive to reducing carbon emissions and conserving resources. Therefore, reducing the energy consumption of these systems while minimizing waste is of great practical significance.

For some production lines, the lead time of products, which defined as the time taken for a part to enter and leave a buffer, is limited, and the quality of products deteriorates with increasing lead time. These types of production systems are referred to as systems prosessing perishable products and are common in industries such as semiconductors, food, chemicals, and steel. Next, we will review research on production systems for perishable products from two perspectives: performance analysis and control strategies.

In terms of performance analysis, some scholars have studied the part lead time in production systems composed of unreliable machines and limited buffers. The average lead time in two-Markovian-machine (i.e., machines following Bernoulli, geometric, or exponential reliability models) serial lines is provided in [2]. The part lead time distribution in two-machine geometric serial lines has been investigated and derived in [3] and [4]. The latter also proposed a numerical algorithm for computing the part lead time distribution of two-machine serial lines with general Markovian machines. Other scholars have analyzed the performence of Bernoulli and geometric serial production lines with lead time constraints in [5] and [6], respectively. Furthermore, some studies focus on reducing product wastage by designing buffer capacities [7] or control strategies [8] in production systems.

The above studies primarily focus on part lead time, production rate, and yield, with much less attention given to energy-related indicators. In terms of reducing the energy consumption of manufacturing systems, Yan et al. established a mathematical model for optimizing energy consumption in two-machine Bernoulli serial lines in [9] and designed an effective numerical algorithm to obtain theoretical optimal solutions. Considering the practical application where machine efficiency ranges from $(0, 1)$, [10] addressed the two-machine Bernoulli optimization problem with machine efficiency constraints. More complex energy consumption optimization problems for two-machine geometric serial lines were investigated in [11] and [12].

For the energy consumption optimization in longer serial lines, in [13], the method is applicable to longer production lines but can only obtain optimal solutions for small-scale problems (i.e., systems with no more than 4 machines and 2 buffer capacities). To obtain the optimal solution for the energy consumption optimization problem in long Bernoulli serial lines, recursive and divide-and-conquer methods were proposed in [14] and [15], respectively. Compared to the former, the latter significantly improves the efficiency of computing solution.

While there is extensive research on the analysis, evaluation, and control of the lead time, as well as energy optimization for unreliable serial lines, there is relatively less research on energy optimization under the premise of ensuring yield, which is a function of part lead time. Although [16] considered the average lead time constraint, the yield constraint holds

S. Zhang, L. Liu and C.-B. Yan are with the State Key Laboratory for Manufacturing Systems Engineering, and also with the School of Automation Science and Engineering, Faculty of Electronic and Information Engineering, Xi'an Jiaotong University, Xi'an, Shaanxi 710049, China (e-mail: chaoboyan@mail.xjtu.edu.cn).

more practical significance. To address this gap, we investigate the energy optimization problem for two-machine Bernoulli serial lines processing perishable products, as illustrated in Fig. 1.

The following is the outline of the subsequent content. In section II, a model for two-machine Bernoulli serial lines processing perishable products is established and the energy optimization problem is addressed. In Section III, by deriving the expression and properties of the part lead time distribution, the problem is mathematically formulated as a nonlinear programming. Then we deduce the optimality conditions and develop an effective algorithm to solve the problem in Section IV. The proof of lemmas and theorems are provided in the appendix.

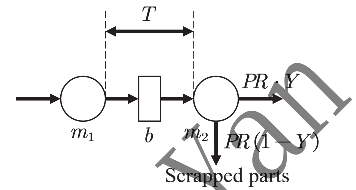


Fig. 1: Two-machine Bernoulli serial lines processing perishable products

II. Production System Modeling and Problem Statement

In this section, the two-machine Bernoulli serial line is formally modeled and the energy consumption optimization problem is addressed in Subsections II-A and II-B, respectively.

A. System Model

The model of the two-machine Bernoulli serial line in Fig. 1 is assumed as follows:

(i) The system consists of two machines m_1 and m_2, and an intermediate buffer b between the machines.

(ii) Machine m_i, $i = 1, 2$, obeys the Bernoulli reliability model, which is characterized by its efficiency p_i. Specifically, during a cycle time, m_i is up with probability p_i and down with $1-p_i$. Herein, p_i can be selected in $(0, 1]$. Both machines have identical cycle time (namely, processing time), which is denoted by τ.

(iii) The time is divided in to equal time slots of τ. The status of a machine (i.e., up or down) is determined at the beginning of each time slot, and the state of the buffer (i.e., the occupancy) is determined at the end of each time slot. Transportation time from m_1 to b and that from b to m_2 are ignored.

(iv) The buffer capacity is N, which is an integer and $0 < N < +\infty$.

(v) Blocking before service is assumed. Specifically, if the buffer is empty at the beginning of a time slot, then m_2 is starved; if buffer b is full of parts while m_2 fails to take a part from it, then m_1 is blocked. Machine m_1 is never starved, and m_2 is never blocked.

(vi) When machine m_i, $i = 1, 2$, is up, the power it consumes is P_i; when m_i is down, it doesn't consume any power. Herein, $0 < P_i < +\infty$.

(vii) The yield is defined as a fraction of effective parts delivered by the line. The effective rate of parts within the lead time interval $[LT_{j-1}, LT_j)$, denoted as γ_j for $j = 1, 2, \cdots, S$, decreases as the lead time increases. Herein, $0 < \gamma_S < \gamma_{S-1} < \cdots < \gamma_1 = 1$.

To avoid confusion, performance metrics related to the two-machine Bernoulli line in Fig.1 are defined. These metrics are all measured when the system is in a steady state. Specifically, they are:

- Yield Y: The ratio of effective parts produced by m_2 to the total number of parts produced during a cycle.
- Production rate PR: The average number of parts produced by m_2 during a cycle.
- Scrap rate SR: The average number of scrapped parts at m_2 during a cycle. Herein, $SR = PR(1-Y)$.

B. Problem Statement

In this subsection, the problem of reducing energy consumption in two-machine Bernoulli serial lines defined by models (i)-(vii) is addressed. Specifically, by selecting appropriate machine parameters (i.e. p_1 and p_2), this article aims to minimize system energy consumption while maintaining system production rate and yield above given thresholds PR_r and Y_r, respectively. In the following, with the intention of mathematically formulating the problem, the preformance metrics (i.e. production rate and yield) are reviewed separately.

The steady-state performance of the two-machine Bernoulli lines has been comprehensively analyzed in [2]. Specifically, the production rate of the two-machine Bernoulli line is

$$\begin{aligned} PR &= p_1 \left[1 - Q\left(p_2, p_1, N\right)\right] \\ &= p_2 \left[1 - Q\left(p_1, p_2, N\right)\right], \end{aligned} \tag{1}$$

where

$$Q(x, y, N) = \begin{cases} \frac{1-x}{N+1-x}, & \text{if } x = y, \\ \frac{(1-x)(1-\alpha)}{1-\frac{x}{y}\alpha^N}, & \text{if } x \neq y, \end{cases} \tag{2}$$

$$\alpha = \frac{x(1-y)}{y(1-x)}. \tag{3}$$

The domain, range and continuity of the Q-function are explored in [9]. Based on [9], for the sake of simplicity, we denote $Q(p_1, p_2, N)$ as Q. In the following, Q and α are expressed as follows, if not otherwise specified:

$$Q = Q\left(p_1, p_2, N\right) = \frac{p_2 - p_1}{p_2 - p_1 \alpha^N}, \tag{4}$$

$$\alpha = \frac{p_1\left(1 - p_2\right)}{p_2\left(1 - p_1\right)}. \tag{5}$$

Apart from production rate, the yield of production lines processing perishable products has also been investigated intensively. Based on [4] and [5], we further extend the definition of yield. Specifically, the yield is defined as the weighted sum

Copyrights@Chao-Bo Yan

of the effective rate of products with different lead time, which is expressed as:

$$Y = \sum_{i=1}^{S} \gamma_i \mathcal{P}\left\{L_{i-1} \leqslant LT < L_i\right\}. \tag{6}$$

where LT is part lead time in two-machine Bernoulli serial lines.

It is worth noting that when $S = 1$ and $\gamma_1 = 1$, $Y = \mathcal{P}\{LT \leqslant L_1\}$, consistent with the definition in the [4]. In the following section, based on previous research [3] and [5], an analytical expression for Y is provided, and its properties are proposed and analyzed.

III. Problem Formulation and Analysis

In this section, the energy consumption optimization problem with production rate and yield constraints mentioned in Section II is formulated, analyzed, and transformed. Specifically, in III-A, the probability mass function and cumulative distribution function of part lead time distribution in two-machine Bernoulli serial lines are derived, and the properties of the functions are analyzed. In III-B, the energy consumption optimization problem is formulated and transformed into an equivalent problem based on its structural characteristics.

A. Derivation and Analysis of Part Lead Time Distribution

For the purpose of deriving the steady-state probability mass function, we adopt a similar approach to that presented in [3]. Some random variables and events are defined to accurately describe the state of the two-machine system. It is widely adopted that $s_i(t) = 0$ or 1, for $i = 1, 2$, indicates machine m_i is down or up at the beginning of time slot t and $H(t)$ is the occupancy of the buffer at the end of time slot t. The event that the reference part is added to the buffer at the end of time slot t is defined as $A(t)$. The time slots a part spends in the buffer under the condition that it arrives at time t is denoted as $T(t)$.

A conditional probability model is formulated to calculate the conditional probability of the time spent by a part from arriving the buffer to leaving it, given the system state. Based on assumption(ii), the states of m_2 at time slot t and $t+1$ are independent. Consequently, the definition of $P\{T(t) = k\}$ in [3] is rewritten as:

$$\begin{aligned}\mathcal{P}\{T(t) = k\} = \sum_{h=1}^{\min\{k,N\}} \Big[&\mathcal{P}\{T(t) = k | H(t) = h, A(t)\} \\ &\cdot \mathcal{P}\{H(t) = h | A(t)\}\Big].\end{aligned} \tag{7}$$

To avoid recursion, the event $\{T(t) = k | H(t) = h, A(t)\}$ is construed as two simultaneous independent events. One of them is processing $h-1$ parts in $k-1$ time slots (from t to $t+k-1$). The other is that the reference part is processed by m_2 at time slot $t+k$. Consistent with the findings in [5], $\{T(t) = k | H(t) = h, A(t)\}$ follows the negative binomial distribution, $NB(h, p_2)$. Then, we obtain the following expression:

$$\begin{aligned}&\mathcal{P}\{T(t) = k | H(t) = h, A(t)\} \\ &= \begin{cases} \binom{k-1}{h-1} p_2^h (1-p_2)^{k-h}, & \text{if } k \geqslant h, \\ 0, & \text{if } k < h, \end{cases}\end{aligned} \tag{8}$$

Subsequently, we turn our attention to the event that the reference part arrives at the buffer. When the system is in steady state, the probability mass function of buffer occupancy do not change with t. The probablity of $H(t) = h$ occuring when the reference part arrives at the buffer in steady state has been proposed in [5]. It is rewritten as

$$\mathcal{P}\{H(t) = h | A(t)\} = \frac{\alpha^{h-1} Q}{(1-p_1)(1-\alpha^N Q)}, 1 \leqslant h \leqslant N, \tag{9}$$

where Q and α are defined in (4) and (5), respectively.

Based on (7), (8), (9) and the law of total probability, we draw the following conclusion.

Lemma 3.1: The steady-state probability mass function of part lead time is formulated as

$$\begin{aligned}&\mathcal{P}\{T = k\} \\ &= \begin{cases} \frac{p_2(1-p_2)^{k-1} Q}{(1-p_1)^k (1-\alpha^N Q)}, & \text{if } 1 \leqslant k \leqslant N, \\ \frac{p_2(1-p_2)^{k-1} Q}{(1-p_1)(1-\alpha^N Q)} \sum_{i=0}^{N-1} \frac{\binom{k-1}{i} p_1^i}{(1-p_1)^i}, & \text{if } k \geqslant N+1, \end{cases}\end{aligned} \tag{10}$$

and the cumulative distribution function is formulated as

$$\begin{aligned}&\mathcal{P}\{T \leqslant k\} \\ &= \begin{cases} \frac{p_2 Q}{(1-p_1)(1-\alpha^N Q)} \sum_{i=0}^{k-1} \frac{(1-p_2)^i}{(1-p_1)^i}, \text{if } 1 \leqslant k \leqslant N, \\ \frac{p_2 Q}{(1-p_1)(1-\alpha^N Q)} \Big[\sum_{i=0}^{N-1} \frac{(1-p_2)^i}{(1-p_1)^i} \\ \quad + \sum_{j=N}^{k-1} (1-p_2)^j \sum_{i=0}^{N-1} \frac{\binom{j}{i} p_1^i}{(1-p_1)^i} \Big], \text{if } k > N, \end{cases}\end{aligned} \tag{11}$$

which is denoted as $F_{CDF}(p_1, p_2, k, N)$.

Proof: See the Appendix.

Additonally, the properties of the cumulative distribution function are analyzed. The conclusions are as the following.

Lemma 3.2: For $N = 1$, $F_{CDF}(p_1, p_2, k, N)$ is a constant function of p_1 and an strictly increasing function of p_2.

Proof: See the Appendix.

Lemma 3.3: For $N > 1$ and $k \leqslant N$, $F_{CDF}(p_1, p_2, k, N)$ is a strictly decreasing function of p_1 and an strictly increasing function of p_2.

Proof: See the Appendix.

Due to the complexity of $F_{CDF}(p_1, p_2, k, N)$ for $N > 1$ and $k > N$, we adopted a numerical method to explore the monotonicity of $F_{CDF}(p_1, p_2, k, N)$ with respect to p_1 and p_2, respectively. To do that, we constructed 1000 test cases with parameters selected randomly and equiprobably from the following sets,

$$N \in \{2, 3, \cdots, 10\}, k \in \{N+1, N+2, N+3, \cdots, N+10\}. \tag{12}$$

For all test cases, (p_1, p_2) are set as $(0.01v_1, 0.01v_2), v_1 = 1, 2, 3, \cdots, 99, v_2 = 1, 2, 3, \cdots, 99$, $\delta_1 = F_{CDF}(p_1 + 10^{-4}, p_2, k, N) - F_{CDF}(p_1, p_2, k, N)$ and $\delta_2 = F_{CDF}(p_1, p_2 + 10^{-4}, k, N) - F_{CDF}(p_1, p_2, k, N)$ are calculated for each (p_1, p_2). As a result, we have:

Numerical Fact 3.1: For all of 1000 cases constructed above, $F_{CDF}(p_1, p_2, k, N)$ is strictly decreasing in p_1 and strictly increasing in p_2.

Theorem 3.1: For any k and N, provided $k \geqslant 1$ and $N \geqslant 1$, $F_{CDF}(p_1, p_2, k, N)$ is a diffenertiable function defined on $0 < p_i < 1, i = 1, 2$. Specifically, $F_{CDF}(p_1, p_2, k, N)$ is a strictly decreasing function (or a constant function) of p_1 for $N > 1$ (or $N = 1$, correspondingly) and strictly increasing in p_2.

Proof: Theorem 3.1 is the summary of Lemma 3.2, Lemma 3.3 and Numerical Fact 3.1.

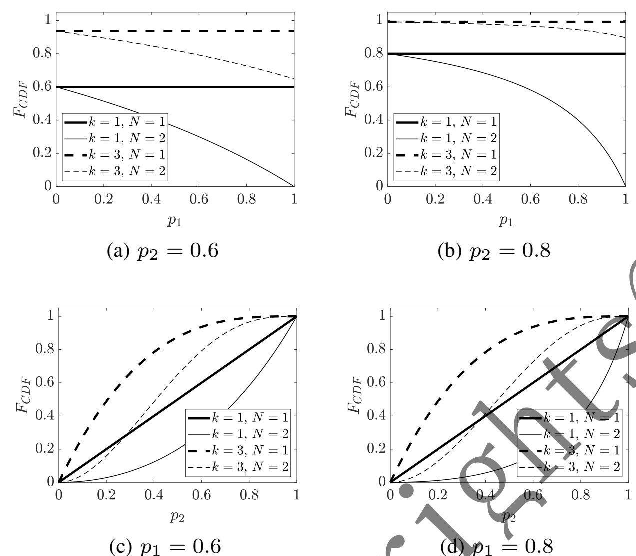


Fig. 2: The behavior of function F_{CDF}

B. Problem Formulation and Transformation

Considering the expression of the production rate in (1) and the expression of the yield in (6), the energy consumption optimization problem for the two-machine Bernoulli line is mathematically formulated as follows:

$$
\begin{aligned}
\text{(P1) min} \quad & z = \sum_{i=1}^{2} P_i p_i && (13)\\
\text{s.t.:} \quad & p_2\big[1 - Q(p_1, p_2, N)\big] \geqslant PR_r, && (14)\\
& Y = \sum_{i=1}^{S} \gamma_i \mathcal{P}\left\{L_{i-1} \leqslant LT < L_i\right\} \geqslant Y_r, && (15)\\
& 0 < p_i \leqslant 1, \; i = 1, 2, && (16)
\end{aligned}
$$

where PR_r and Y_r are the required production rate and required yield, respectively.

From problem (P1), it can be observed that the objective function is a weighted sum of machine efficiencies which indicates that the goal of (P1) is to minimize the total energy used by machines during a production cycle. It should be noted that machines m_1 and m_2 have the same cycle time, τ, which has been omitted in the objective function.

In (P1), constraint (14) is nonlinear and constraint (15) is a complicated chance constraint. For the purpose of simplifing the constraint, the expression of yield in (15) is rewirtten as

$$Y = \sum_{i=1}^{S} (\gamma_i - \gamma_{i+1})\, \mathcal{P}\{LT < L_i\}, \tag{17}$$

where $\gamma_{S+1} = 0$. Note that $L_i, i = 1, 2, \cdots, S$ is a positive real number, while the part lead time in two-machine Bernoulli lines is an integer multiple of τ, we define $n_i, i = 1, 2, \cdots, S$ as

$$n_i = \begin{cases} \frac{L_i}{\tau} - 1, & \text{if } \frac{L_i}{\tau} \text{ is an integer,} \\ \lfloor \frac{L_i}{\tau} \rfloor, & \text{if } \frac{L_i}{\tau} \text{ is not an integer,} \end{cases} \tag{18}$$

where $\lfloor \frac{L_i}{\tau} \rfloor$ is the maximum integer not larger than $\frac{L_i}{\tau}$, which suggests $n_i\tau < L_i \leqslant (n_i + 1)\tau$. That is, the event $\{LT < L_i\}$ is equivalent to $\{T \leqslant n_i\}$. Consequently, we rewritten the expression of yield as

$$
\begin{aligned}
Y &= \sum_{i=1}^{S} (\gamma_i - \gamma_{i+1})\, \mathcal{P}\{T \leqslant n_i\} \\
&= \sum_{i=1}^{S} (\gamma_i - \gamma_{i+1})\, F_{CDF}(p_1, p_2, n_i, N).
\end{aligned} \tag{19}
$$

Based on the assumption (vii), $\gamma_i - \gamma_{i+1}$ is a positive real number on $(0, 1)$. Considering Theorem 3.1, we have the following.

Corollary 3.1: Y is a differentiable function defined on $0 < p_i < 1, i = 1, 2$. Specifically, Y is a strictly decreasing function (or a constant function) of p_1 for $N > 1$ (or $N = 1$, correspondingly) and strictly increasing in p_2.

In problem (P1), p_1 and p_2 are decision variables. In the following, the function of production rate and yield with respect to p_1 and p_2 will be abbreviated as $F_{PR}(p_1, p_2)$ and $F_Y(p_1, p_2)$, respectively. Therefore, problem (P1) is rewritten as

$$
\begin{aligned}
\text{(P1') min} \quad & z = \sum_{i=1}^{2} P_i p_i && (20)\\
\text{s.t.:} \quad & F_{PR}(p_1, p_2) \geqslant PR_r, && (21)\\
& F_Y(p_1, p_2) \geqslant Y_r, && (22)\\
& 0 < p_i \leqslant 1, \; i = 1, 2. && (23)
\end{aligned}
$$

To facilitate to solve (P1'), similar to the energy consumption optimization problem with sole prodution rate constraint in Markovian lines (namely, Bernoulli and geometric lines) [9] and [11], a new theorem is introduced as follows:

Theorem 3.2: The optimal objective value, z^*, of (P1'), is non-decreasing in PR_r and Y_r, respectively. For $0 < PR_{r1} < PR_{r2} < 1$ and $0 < Y_{r1} < Y_{r2} < 1$, $z^*(PR_{r2}, Y_{r2})$ is larger than $z^*(PR_{r1}, Y_{r1})$.

Proof: See the Appendix.

Corollary 3.2: Let (p_1^*, p_2^*) denote the optimal solution of (P1'), at least one of the two equations,

$$F_{PR}(p_1^*, p_2^*) = PR_r \tag{24}$$

and

$$F_Y(p_1^*, p_2^*) = Y_r \tag{25}$$

is true.

Corollary 3.2 can be proved by contradiction. The proof is omitted because of space limitation.

Equation (24) and (25) in Corollary 3.2 determine two implicit functions, which are denoted as $F_{I,PR}(p_1; PR_r)$ and

$F_{I,Y}(p_1;Y_r)$, respectively. The efficiency of m_2, given the efficiency of m_1 and required production rate, PR_r, (or required yield, Y_r) can be computed using $F_{I,PR}(p_1;PR_r)$ (or $F_{I,Y}(p_1;Y_r)$, correspondingly). In the following, the properties of $F_{I,PR}(p_1;PR_r)$ and $F_{I,Y}(p_1;Y_r)$ will be investigated.

The continuity of the function $F_{I,PR}(p_1;PR_r)$ can be inferred from F_{PR}. From the results in [9], it follows that $PR_r < p_i < 1$ when $F_{PR}(p_1,p_2) = PR_r$ is true. Specifically, when p_1 approaches PR_r (or 1), p_2 approaches 1 (or PR_r, correspondingly). Since the function F_{PR} is strictly increasing in p_1 and p_2, function $F_{I,PR}$ is a decreasing function of p_1.

However, the properties of $F_{I,Y}(p_1;Y_r)$ are quite different. Based on (11) and (19), $F_{I,Y}$ is a positive continuous function. Additionally, according to Corollary 3.1, the function $F_{I,Y}$ is an increasing function (or a constant function) of p_1 for $N > 1$ (or for $N = 1$, correspondingly). Based on the supremum and infimum principle and the monotonicity of $F_{I,Y}(p_1;Y_r)$, the supremum and infimum of $F_{I,Y}(p_1;Y_r)$ exist, which are denoted as $p_{2,Ymax} = \lim_{p_1\to1^-} F_{I,Y}(p_1;Y_r)$ and $p_{2,Ymin} = \lim_{p_1\to0^+} F_{I,Y}(p_1;Y_r)$, respectively.

In the following, we will further investigate the relationship between $F_{I,PR}(p_1;PR_r)$ and $F_{I,Y}(p_1;Y_r)$. Let $F_{dif}(p_1)$ denote $F_{I,PR}(p_1;PR_r) - F_{I,Y}(p_1;Y_r)$, $F_{dif}(p_1)$ is a continuous function on $(PR_r,1)$. It is clear that $\lim_{p_1\to PR_r^+} F_{dif}(p_1) > 0$. Considering the monotonicity of $F_{I,PR}(p_1;PR_r)$ and $F_{I,Y}(p_1;Y_r)$, $F_{dif}(p_1)$ is a decreasing function. If $\lim_{p_1\to1^-} F_{dif}(p_1) \geqslant 0$, (i.e. $PR_r > p_{2,Ymax}$), $F_{dif}(p_1)$ is a positive function (i.e. $F_{I,PR}(p_1;PR_r) > F_{I,Y}(p_1;Y_r)$). Addtionally, we have $F_Y(p_1,F_{I,PR}(p_1;PR_r)) > F_Y(p_1,F_{I,Y}(p_1;Y_r)) = Y_r$, which suggests that the points of curve $F_{PR}(p_1,p_2) = PR_r$ satisfy the constraint (22). If $\lim_{p_1\to1^-} F_{dif}(p_1) < 0$, based on the zero point theorem and the monotonicity of $F_{dif}(p_1)$, the only zero of $F_{dif}(p_1)$, denoted as $\hat{p}_1$, is on $(PR_r,1)$. When p_1 is on $(PR_r,\hat{p}_1)$, $F_{dif}(p_1)$ is positive (i.e. $F_{I,PR}(p_1;PR_r) > F_{I,Y}(p_1;Y_r)$). The points of curve $F_{PR}(p_1,p_2) = PR_r, PR_r < p_1 < \hat{p}_1$ satisfy the constraint (22). When p_1 is on $(\hat{p}_1,1)$, $F_{dif}(p_1)$ is negative (i.e. $F_{I,PR}(p_1;PR_r) < F_{I,Y}(p_1;Y_r)$). We have $F_{PR}(p_1,F_{I,Y}(p_1;Y_r)) > F_{PR}(p_1,F_{I,PR}(p_1;PR_r)) = PR_r$, which suggests that the points of curve $F_Y(p_1,p_2) = Y_r$ satisfy the constraint (21). Consequently, we define a new function,

$$F(p_1,p_2) = \begin{cases} F_Y(p_1,p_2) - Y_r, & \text{if } PR_r < p_{2,Ymax} \text{ and } \hat{p}_1 < p_1 < 1, \\ F_{PR}(p_1,p_2) - PR_r, & \text{otherwise.} \end{cases} \quad (26)$$

As depicted in Fig. 3, the points of curve $F(p_1,p_2) = 0$ satisfy both (21) and (22).

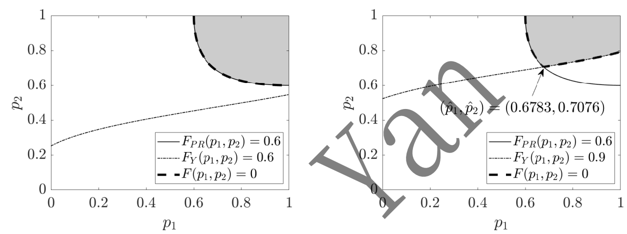
Herein, we introduce a new problem:

$$\text{(P2) min} \quad z = \sum_{i=1}^{2} P_i p_i \quad (27)$$

$$\text{s.t.:} \quad F(p_1,p_2) = 0, \quad (28)$$

$$0 < p_i \leqslant 1, \; i = 1,2. \quad (29)$$

The only difference between (P2) and (P1') is constraint. Based on Corollary 3.2, (P2) is equivalent to (P1), which suggests that (P1') and (P2) have identical optimal solution. In the next subsection, the optimality conditions of (P2) is presented and analyzed.



(a) $PR_r = 0.6, Y_r = 0.6$ (b) $PR_r = 0.6, Y_r = 0.9$

Fig. 3: The behavior of function $F(p_1,p_2)$ with $N = 2$, $\gamma_1 = 1$, $\gamma_2 = 0.1$, $n_1 = 3$ and $n_2 = 5$

IV. Solution Methodology

In this section, a method for solving the optimal solution of problem (P2) is proposed. Specifically, in Subsection IV-A, optimality conditions of (P2) are further explored; in Subsection IV-B, based on the properties of $F_{I,PR}(p_1;PR_r)$ and $F_{I,Y}(p_1;Y_r)$, an effective and efficient algorithm is designed to solve the optimal solution.

A. Optimality Conditions

For the purpose of solving (P2) effectively, the optimality condition that the optimal solution of (P2) satisfies are derived in this section.

From the insights gained from solving the energy consumption optimization problem in two-machine Bernoulli serial lines, $F(p_1,p_2) = 0$ is a optimality function of (P2). Based on the analysis in Subsection III-B, p_2 can be regard as an implicit function of p_1, which is derived from $F(p_1,p_2) = 0$. The implicit function is defined as

$$F_I(p_1;PR_r,Y_r) = \begin{cases} F_{I,Y}(p_1;Y_r), & \text{if } PR_r < p_{2,Ymax} \text{ and } \hat{p}_1 < p_1 < 1 \\ F_{I,PR}(p_1;PR_r), & \text{otherwise.} \end{cases}, \quad (30)$$

The domain of the differentiable function, $F_I(p_1,PR_r,Y_r)$, is $(PR_r,1)$. Additionally, we define the function $f(p_1)$ as:

$$f_I(p_1) = -\frac{\mathrm{d}F_I}{\mathrm{d}p_1} \quad (31)$$

If $PR_r < p_{2,Ymax}$ and $\hat{p}_1 < p_1 < 1$, we have $f_I(p_1) = -\frac{\mathrm{d}F_{I,Y}}{\mathrm{d}p_1}$. Otherwise, we have $f_I(p_1) = f(p_1) = -\frac{\mathrm{d}F_{I,PR}}{\mathrm{d}p_1}$ which has been derived in [9]. Let

$$\bar{p} = \frac{(N+1)PR_r}{N+PR_r}, \quad (32)$$

$f(p_1)$ is expressed as

$$f(p_1) = \frac{p_2^2}{p_1^2}, \quad (33)$$

for $N = 1$ and

$$f(p_1) = \begin{cases} 1, & \text{if } p_1 = \bar{p}, \\ \frac{p_2(1-p_2)(p_2-PR_r)(p_1-\bar{p})}{p_1(1-p_1)(p_1-PR_r)(\bar{p}-p_2)}, & \text{if } p_1 \neq \bar{p}, \end{cases} \quad (34)$$

for $N > 1$. Let

$$\tilde{p}_1 = \begin{cases} 1, \text{if } \sqrt{\frac{P_1}{P_2}} \leqslant PR_r, \\ \frac{\left(1+\sqrt{P_2/P_1}\right)PR_r}{1+PR_r}, \text{if } \min\left(\sqrt{\frac{P_1}{P_2}}, \sqrt{\frac{P_2}{P_1}}\right) > PR_r, \\ PR_r, \text{if } \sqrt{\frac{P_2}{P_1}} \leqslant PR_r, \end{cases} \quad (35)$$

for $N = 1$ and $\tilde{p}_1$ equals the solution of $f(p_1) = \frac{P_1}{P_2}$ for $N > 1$, we have the following.

Theorem 4.1: The optimal solution of (P2) (i.e. (p_1^*, p_2^*)) satisfies

$$p_1^* = \min(\hat{p}_1, \tilde{p}_1), \quad (36)$$
$$p_2^* = F_{I,PR_r}(p_1^*, PR_r), \quad (37)$$

for $PR_r < p_{2,Ymax}$ and

$$p_1^* = \tilde{p}_1, \quad (38)$$
$$p_2^* = F_{I,PR_r}(p_1^*, PR_r), \quad (39)$$

for $PR_r \geqslant p_{2,Ymax}$

Proof: See the Appendix.

In the next subsection, an effective and efficient algorithm will be designed to solve p_1^* and p_2^* in Theorem 4.1.

B. Algorithm Design

Based on Theorem 4.1, the solution of (P2) can be divided into three steps. The first step involves determining the relationship between PR_r and $p_{2,Ymax}$. In the second step, we solve for $\tilde{p}_1$. The algorithm for solving $f(p_1) = \frac{P_1}{P_2}$ was designed in [9]. If $PR_r < p_{2,Ymax}$, in the third step, we solve for $\hat{p}_1$. Otherwise, the third step can be skipped. In the following, we will design an algorithm for solving $\hat{p}_1$.

From the previous sections, when $PR_r < p_{2,Ymax}$, $\hat{p}_1$ is the zero of $F_{dif}(p_1)$ which is a strictly decreasing function. Although we could directly solve $F_{dif}(p_1) = 0$ using existing mathematical software (such as MATLAB) or numerical methods (such as the bisection method), given the complexity of the expression for function F_{dif}, we devised an indirect approach, which is on the basis of the bisection method, to solve this equation. Specifically, the bisection method is performed on $(PR_r, 1)$. Let p_1^L and p_2^L denote the lower and upper endpoints for the bisection method during each iteration, respectively. Clearly, we have $p_1^L = PR_r$ and $p_2^L = 1$ as the initial endpoints. Let $p_1^M = \frac{p_1^L + p_1^U}{2}$, solve p_2^M from $F_{I,PR}(p_1^M; PR_r)$. To avoid computing $F_{I,Y}(p_1; Y_r)$, on the basis of the monotonicity of $F_Y(p_1, p_2)$ with respect to p_2, we have the following:

$$\begin{aligned} \text{sgn}\big(F_{dif}(p_1)\big) =& \text{sgn}\big(F_{I,PR}(p_1; PR_r) - F_{I,Y}(p_1; Y_r)\big) \\ =& \text{sgn}\Big(F_Y\big(p_1, F_{I,PR}(p_1; PR_r)\big) \\ & - F_Y\big(p_1, F_{I,Y}(p_1; Y_r)\big)\Big) \\ =& \text{sgn}\Big(F_Y\big(p_1, F_{I,PR}(p_1; PR_r)\big) - Y_r\Big), \end{aligned} \quad (40)$$

where $\text{sgn}(x)$ is defined as

$$\text{sgn}(x) = \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x > 0. \end{cases} \quad (41)$$

If $F_Y(p_1^M, p_2^M) - Y_r = 0$ or $p_1^U - p_1^L < \epsilon$, where ϵ (namely, precision) is a predefined small positive real number, then $\hat{p}_1 = p_1^M$ and the algorithm ends. Otherwise, if $F_Y(p_1^M, p_2^M) - Y_r > 0$ (or $F_Y(p_1^M, p_2^M) - Y_r < 0$), p_1^L is set to p_1^M and p_1^U remains unchanged (or correspondingly, p_1^L remains unchanged and p_1^U is set to p_1^M) and then $p_1^M = \frac{p_1^L + p_1^U}{2}$. Bisecting the interval, (p_1^L, p_1^U) and selecting the subinterval is repeated until $F_Y(p_1^M, p_2^M) - Y_r = 0$ or $p_1^U - p_1^L < \epsilon$ is satisfied.

The flowchart of the bisection algorithm for solving $F_{dif}(p_1) = 0$ is shown in Fig. 4.

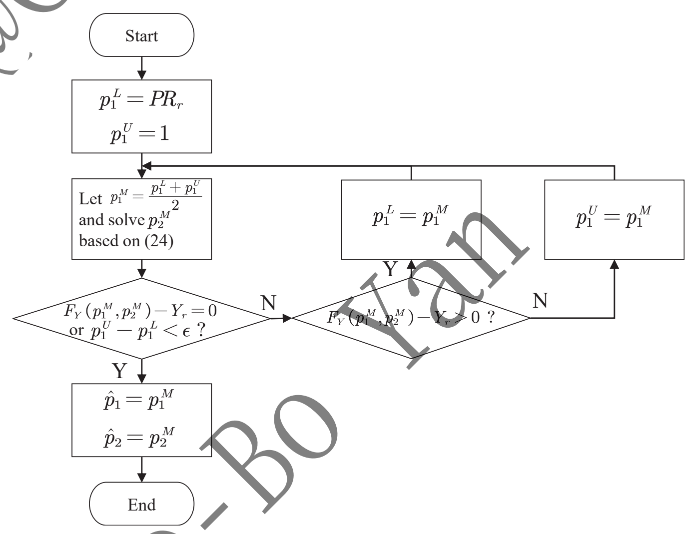


Fig. 4: Flowchart of the algorithm for solving $\hat{p}_1$

With the developed algorithm, extensive numerical experiments for various $\frac{P_1}{P_2}$, N, PR_r, and Y_r have been conducted. Some of the test cases and their optimal solutions are shown in Table I, where the optimal objective value z^* is provided for $P_2 = 1$, $\gamma_1 = 1$, $\gamma_2 = 0.1$, $L_1 = 2$ and $L_2 = 4$.

V. Conclusions and Future Work

The energy consumption optimization problem for two-machine Bernoulli serial lines processing perishable products is proposed and sovled in this paper. Specifically, we derive and analyze analytical expressions and mathematical properties of yield and establish a mathematical model. To solve this problem, we analyze the structural characteristics of the model and derive the optimality conditions, based on which we design an effective and efficient numerical algorithm to find the optimal solution.

In the realm of energy optimization for systems processing perishable products, there are still many valuable research topics. In the future, we will extend the findings of this paper to long Bernoulli serial lines and to more practical production systems, for example, re-entrant lines and assembly

TABLE I: Optimal solution of (P2) for various test cases ($P_2 = 1, \gamma_1 = 1, \gamma_2 = 0.1, L_1 = 2, L_2 = 4$)

No.	$\frac{P_1}{P_2}$	N	PR_r	Y_r	p_1^*	p_2^*	z^*
1	0.5	1	0.6	0.6	0.9053	0.6402	1.0928
2	0.5	1	0.6	0.9	0.8538	0.6687	1.0956
3	0.5	1	0.9	0.6	1.0000	0.9000	1.4000
4	0.5	1	0.9	0.9	1.0000	0.9000	1.4000
5	0.5	3	0.6	0.6	0.6369	0.7075	1.0259
6	0.5	3	0.6	0.9	0.6039	0.9900	1.2919
7	0.5	3	0.9	0.6	0.9292	0.9177	1.3823
8	0.5	3	0.9	0.9	0.9030	0.9616	1.4131
9	2	1	0.6	0.6	0.6402	0.9053	2.1857
10	2	1	0.6	0.9	0.6402	0.9053	2.1857
11	2	1	0.9	0.6	0.9000	1.0000	2.8000
12	2	1	0.9	0.9	0.9000	1.0000	2.8000
13	2	3	0.6	0.6	0.6350	0.7112	1.9812
14	2	3	0.6	0.9	0.6039	0.9900	2.1977
15	2	3	0.9	0.6	0.9100	0.9414	2.7613
16	2	3	0.9	0.9	0.9030	0.9616	2.7675

systems, with more practical reliability models, including but not limited to geometric models, exponential models, and non-Markovian models.

APPENDIX

PROOFS OF THEOREMS

A. Proof of Lemma 3.1

Proof:

Based on (7), (8) and (9), we have

$$\begin{aligned}
&\mathcal{P}\{T=k\}\\
&=\sum_{h=1}^{\min\{k,N\}}\binom{k-1}{h-1}p_2^h(1-p_2)^{k-h}\frac{\alpha^{h-1}Q}{(1-p_1)\,(1-\alpha^N Q)}\\
&=\frac{(1-p_2)^k Q}{(1-p_1)\,(1-\alpha^N Q)}\sum_{h=1}^{\min\{k,N\}}\binom{k-1}{h-1}\frac{p_2^h\alpha^{h-1}}{(1-p_2)^h}\\
&=\frac{p_2(1-p_2)^{k-1}Q}{(1-p_1)\,(1-\alpha^N Q)}\sum_{h=1}^{\min\{k,N\}}\binom{k-1}{h-1}\frac{p_1^{h-1}}{(1-p_1)^{h-1}}.
\end{aligned} \tag{42}$$

For $k \leqslant N$, on the basis of Binomial theorem, we have:

$$\sum_{h=1}^{k}\binom{k-1}{h-1}\frac{p_1^{h-1}}{(1-p_1)^{h-1}}=\frac{1}{(1-p_1)^k}. \tag{43}$$

Considering (42) and (43), the expression of the probability mass function is (10).

From (10)), for $1 \leqslant k \leqslant N$, we have:

$$\begin{aligned}
\mathcal{P}\{T\leqslant k\}&=\sum_{i=1}^{k}\mathcal{P}\{T=i\}\\
&=\frac{p_2 Q}{(1-p_1)\,(1-\alpha^N Q)}\sum_{i=0}^{k-1}\frac{(1-p_2)^i}{(1-p_1)^i},
\end{aligned} \tag{44}$$

and for $k > N$, we have:

$$\begin{aligned}
\mathcal{P}\{T\leqslant k\}=&\mathcal{P}\{T\leqslant N\}+\sum_{i=N+1}^{k}\mathcal{P}\{T=i\}\\
=&\frac{p_2 Q}{(1-p_1)\,(1-\alpha^N Q)}\Bigg[\sum_{i=0}^{N-1}\frac{(1-p_2)^i}{(1-p_1)^i}\\
&+\sum_{j=N}^{k-1}(1-p_2)^j\sum_{i=0}^{N-1}\frac{\binom{j}{i}{p_1}^i}{(1-p_1)^i}\Bigg],
\end{aligned} \tag{45}$$

which completes the proof. ■

B. Proof of Lemma 3.2

Proof: From (2), we have:

$$Q(p_1,p_2,1)=\frac{(1-p_1)\,p_2}{p_2+p_1-p_1p_2}. \tag{46}$$

Based on (46), the expression of $F_{CDF}(p_1,p_2,k,1)$ is rewritten as follows:

$$F_{CDF}(p_1,p_2,k,1)=1-(1-p_2)^k. \tag{47}$$

Clearly, it can be observed that $F_{CDF}(p_1,p_2,k,1)$ is a constant function of p_1. Taking the partial derivative of F_{CDF} with respect to p_2, we have

$$\frac{\partial F_{CDF}}{\partial p_2}=k(1-p_2)^{k-1}\geqslant 0, \tag{48}$$

indicating that $F_{CDF}(p_1,p_2,k,1)$ is an increasing function of p_2. ■

C. Partial derivatives of $F_{CDF}(p_1,p_2,k,N)$ for $N > 1$ and $1 \leqslant k \leqslant N$

Before the proof of the following lemma, the partial derivatives of $F_{CDF}(p_1,p_2,k,N)$ for $1 \leqslant k \leqslant N$ (see in (11)) with respect to p_1 and p_2 are derived.

Since Q is a continous function, $F_{CDF}(p_1,p_2,k,N)$ is continous on $0 < p_i < 1, i = 1, 2$. Based on (2) and (11), the expression of $F_{CDF}(p_1,p_2,k,N)$ for $1 \leqslant k \leqslant N$ is rewritten as follows

$$F_{CDF}(p_1,p_2,k,N)=\begin{cases}\frac{p_1 k}{N}, \text{if } p_1=p_2,\\ \frac{1-\left(\frac{p_2}{p_1}\alpha\right)^k}{1-\alpha^N}, \text{if } p_1\neq p_2.\end{cases} \tag{49}$$

To facilitate the derivation of derivatives, we adopt the expression of $F_{CDF}(p_1,p_2,k,N)$ for $p_1 \neq p_2$ as a general formula. Let $F_{CDF,n}$ and $F_{CDF,d}$ denote the numerator and denominator of F_{CDF}, respectively, then

$$F_{CDF,n}=1-(\frac{p_2}{p_1}\alpha)^k, \tag{50}$$

$$F_{CDF,d}=1-\alpha^N. \tag{51}$$

Taking the partial derivatives of $F_{CDF,n}$ and $F_{CDF,d}$ with respect to p_1 and p_2, respectively, the results are as follows

$$\frac{\partial F_{CDF,n}}{\partial p_1}=-\frac{k(1-p_2)^k}{(1-p_1)^{k+1}}, \tag{52}$$

Copyrights@Chao-Bo Yan

$$\frac{\partial F_{CDF,n}}{\partial p_2} = \frac{k(1-p_2)^{k-1}}{(1-p_1)^k}, \tag{53}$$

$$\frac{\partial F_{CDF,d}}{\partial p_1} = -\frac{N\alpha^N}{p_1(1-p_1)}, \tag{54}$$

$$\frac{\partial F_{CDF,d}}{\partial p_2} = \frac{N\alpha^N}{p_2(1-p_2)}. \tag{55}$$

Then the derivatives of F_{CDF} with respect to p_1 and p_2 are obtained as

$$\begin{aligned}\frac{\partial F_{CDF}}{\partial p_1} =& \frac{\frac{F_{CDF,n}}{\partial p_1}F_{CDF,d} - \frac{\partial F_{CDF,d}}{\partial p_1}F_{CDF,n}}{F^2_{CDF,d}} \\ =& \Big[Np_1^{N-1}\left(1-p_2\right)^{N-k}\left(1-p_1\right)^k - kp_2^N\left(1-p_1\right)^N \\ & - Np_1^{N-1}\left(1-p_2\right)^N + kp_1^N\left(1-p_2\right)^N\Big] \\ & \cdot \frac{\left(1-p_2\right)^k}{p_2^N\left(1-p_1\right)^{N+k+1}F^2_{CDF,d}}\end{aligned} \tag{56}$$

and

$$\begin{aligned}\frac{\partial F_{CDF}}{\partial p_2} =& \frac{\frac{F_{CDF,n}}{\partial p_2}F_{CDF,d} - \frac{\partial F_{CDF,d}}{\partial p_2}F_{CDF,n}}{F^2_{CDF,d}} \\ =& \Big[kp_2^{N+1}\left(1-p_1\right)^N - N\left(1-p_1\right)^k p_1^N\left(1-p_2\right)^{N-k} \\ & - kp_2p_1^N\left(1-p_2\right)^N + Np_1^N\left(1-p_2\right)^N\Big] \\ & \cdot \frac{\left(1-p_2\right)^{k-1}}{p_2^{N+1}\left(1-p_1\right)^{N+k}F^2_{CDF,d}}\end{aligned} \tag{57}$$

D. Proof of Lemma 3.3

Proof:

To prove this lemma, we construct an auxiliary function, $G(x)$, as follows

$$G(x) = \frac{1-x^k}{1-x^N}, \tag{58}$$

where x is on $(0,+\infty)$ and $1 \leqslant k < N$. Noting that the signs of $1-x^k$ and $1-x^N$ are identical for $x>0$, $G(x)$ is positive. Taking derivative of $G(x)$ with respect to x, we obtain

$$\frac{\mathrm{d}G}{\mathrm{d}x} = \frac{x^{k-1}}{\left(1-x^N\right)^2}\left[\left(k-N\right)x^N + Nx^{N-k} - k\right]. \tag{59}$$

For the purpose of investigating the sign of $\frac{\mathrm{d}G}{\mathrm{d}x}$, we define $G_{deri,part}(x)$ as

$$G_{deri,part}(x) = \left(k-N\right)x^N + Nx^{N-k} - k. \tag{60}$$

Taking derivatives of $G_{deri,part}(x)$, we have

$$\frac{\mathrm{d}G_{deri,part}(x)}{\mathrm{d}x} = N\left(k-N\right)x^{N-k-1}\left(x^k-1\right). \tag{61}$$

For $1 \leqslant k < N$, the sign of $\frac{\mathrm{d}G_{deri,part}(x)}{\mathrm{d}x}$ can be inferred, which is

$$\operatorname{sgn}\left(\frac{\mathrm{d}G_{deri,part}(x)}{\mathrm{d}x}\right) = \begin{cases}1, & \text{if } 0<x<1,\\ 0, & \text{if } x=1,\\ -1, & \text{if } x>1.\end{cases} \tag{62}$$

Considering (62) and $\frac{\mathrm{d}G_{deri,part}(x)}{\mathrm{d}x}|_{x=1} = 0$, $G_{deri,part}(x)$ is strictly increasing (or strictly decreasing) on $(0,1)$ (or correspondingly, $(1,+\infty)$) and it is negative on $(0,1)\cup(1,\infty)$. Additionally, noting

$$\frac{\mathrm{d}G}{\mathrm{d}x} = \frac{x^{k-1}}{\left(1-x^N\right)^2}G_{deri,part}(x). \tag{63}$$

we obtain the sign of $\frac{\mathrm{d}G}{\mathrm{d}x}$ as follows

$$\operatorname{sgn}\left(\frac{\mathrm{d}G(x)}{\mathrm{d}x}\right) = \begin{cases}0, & \text{if } x=1,\\ -1, & \text{if } x\in(0,1)\cup(1,\infty).\end{cases} \tag{64}$$

With (58), the expression of $F_{CDF}(p_1,p_2,k,N)$ can be rewritten as

$$\begin{aligned}& F_{CDF}(p_1,p_2,k,N) \\ =& \begin{cases}G(\frac{1-p_2}{1-p_1})F_{CDF}(p_1,p_2,N,N), & \text{if } 1\leqslant k<N,\\ F_{CDF}(p_1,p_2,N,N), & \text{if } k=N.\end{cases}\end{aligned} \tag{65}$$

To obtain the monotonicity of $F_{CDF}(p_1,p_2,k,N)$, the signs of $\frac{\partial G(\frac{1-p_2}{1-p_1})}{\partial p_1}$, $\frac{\partial G(\frac{1-p_2}{1-p_1})}{\partial p_2}$, $\frac{\partial F_{CDF}(p_1,p_2,N,N)}{\partial p_1}$ and $\frac{\partial F_{CDF}(p_1,p_2,N,N)}{\partial p_2}$ are explored as follows.

First, taking partial derivatives of $G(\frac{1-p_2}{1-p_1})$ with respect to p_1 and p_2, respectively, the results are

$$\frac{\partial G(\frac{1-p_2}{1-p_1})}{\partial p_1} = \frac{1-p_2}{(1-p_1)^2}\cdot\left.\frac{\mathrm{d}G(x)}{\mathrm{d}x}\right|_{x=\frac{1-p_2}{1-p_1}}, \tag{66}$$

and

$$\frac{\partial G(\frac{1-p_2}{1-p_1})}{\partial p_2} = -\frac{1}{1-p_1}\cdot\left.\frac{\mathrm{d}G(x)}{\mathrm{d}x}\right|_{x=\frac{1-p_2}{1-p_1}}. \tag{67}$$

on the basis of (64), we have $\frac{\partial G(\frac{1-p_2}{1-p_1})}{\partial p_1} < 0$ and $\frac{\partial G(\frac{1-p_2}{1-p_1})}{\partial p_2} > 0$ for $p_1 \neq p_2$.

Then, according to (56), the partial derivative of $F_{CDF}(p_1,p_2,N,N)$ with respect to p_1 and p_2 are rewritten as follows

$$\begin{aligned}& \frac{\partial F_{CDF}(p_1,p_2,N,N)}{\partial p_1} \\ =& -\frac{N\left(1-p_2\right)^N\left[\frac{p_2^N}{p_1^{N-1}} + \frac{(1-p_2)^N}{(1-p_1)^{N-1}} - 1\right]}{p_1^{N-1}p_2^N\left(1-p_1\right)^{3N}F^2_{CDF,d}},\end{aligned} \tag{68}$$

$$\begin{aligned}& \frac{\partial F_{CDF}(p_1,p_2,N,N)}{\partial p_2} \\ =& \frac{N\left(1-p_2\right)^{N-1}\left[\frac{p_2^{N+1}}{p_1^N} + \frac{(1-p_2)^{N+1}}{(1-p_1)^N} - 1\right]}{p_1^Np_2^{N+1}\left(1-p_1\right)^{3N}F^2_d},\end{aligned} \tag{69}$$

Construct an auxiliary function $F_{auxi}(p_2;W)$, which is defined as

$$F_{auxi}(p_2;W) = \frac{p_2^{W+1}}{p_1^W} + \frac{\left(1-p_2\right)^{W+1}}{\left(1-p_1\right)^W} - 1 \tag{70}$$

where W is a positive integer. Taking derivative of $F_{auxi}(p_2;W)$, we have

$$\frac{\mathrm{d}F_{auxi}}{\mathrm{d}p_2} = (W+1)\frac{p_2^W}{p_1^W} - (W+1)\frac{\left(1-p_2\right)^W}{\left(1-p_1\right)^W}, \tag{71}$$

Copyrights@Chao-Bo Yan

and the sign of (71) is

$$\mathrm{sgn}(F_{auxi}) = \begin{cases} -1, & \text{if } 0 < p_2 < p_1, \\ 1, & \text{if } p_1 < p_2 < 1. \end{cases} \tag{72}$$

which suggests that $F_{auxi}(p_2; W)$ is strictly decreasing on $(0, p_1)$ and strictly increasing on $(p_1, 1)$. Considering $F_{auxi}(p_1; W) = 0$, $F_{auxi}(p_2; W)$ is positive on $(0, p_1) \cup (p_1, 1)$. Therefore, $\frac{\partial F_{CDF}(p_1, p_2, N, N)}{\partial p_1}$ is negative and $\frac{\partial F_{CDF}(p_1, p_2, N, N)}{\partial p_1}$ is positive which proves the monotonicity of $F_{CDF}(p_1, p_2, N, N)$ with respect to p_1 and p_2 for $k = N$.

For $1 \leqslant k < N$, based on the sign of $\frac{\partial G(\frac{1-p_2}{1-p_1})}{\partial p_1}$, $\frac{\partial G(\frac{1-p_2}{1-p_1})}{\partial p_2}$, $\frac{\partial F_{CDF}(p_1, p_2, N, N)}{\partial p_1}$ and $\frac{\partial F_{CDF}(p_1, p_2, N, N)}{\partial p_2}$, one can derive that $F_{CDF}(p_1, p_2, k, N)$ is strictly decreasing in p_1 and strictly increasing in p_2, which completes the proof.

■

E. Proof of Theorem 3.2

Proof:

For a two-machine Bernoulli serial line where both machines are reliable, the prodution rate and yield are both 1. Therefore, for any $PR_r \in (0, 1)$ and $Y_r \in (0, 1)$, (P1') always has at least one feasible solution and thus, it has optimal solution.

To prove the theorem, we choose PR_{r1}, PR_{r2}, Y_{r1} and Y_{r2} such that $0 < PR_{r1} < PR_{r2} < 1$ and $0 < Y_{r1} < Y_{r2} < 1$. Let (P1'-1) and (P1'-2) denote (P1') with PR_r (and Y_r) replaced by PR_{r1} (and correspondingly, Y_{r1}) and by PR_{r2} (and correspondingly, Y_{r2}), respectively. In addition, denote the optimal solutions of (P1'-1) and (P1'-2) as $(p^*_{1,r1}, p^*_{2,r1})$ and $(p^*_{1,r2}, p^*_{2,r2})$, respectively, and their corresponding optimal values as z^*_{r1} and z^*_{r2}. Construct two solution $(p^*_{1,r1}, p_{2,PR_{r1}})$ and $(p^*_{1,r1}, p_{2,Y_{r1}})$, which satisfy $F_{PR}(p^*_{1,r1}, p_{2,PR_{r1}}) = PR_{r1}$ and $F_Y(p^*_{1,r1}, p_{2,Y_{r1}}) = Y_{r1}$, respectively. Considering $F_{PR}(p^*_{1,r2}, p^*_{2,r2}) \geqslant PR_{r2}$ and $F_Y(p^*_{1,r2}, p^*_{2,r2}) \geqslant Y_{r2}$, and taking into account the monotonicity of $F_{PR}(p_1, p_2)$ with respect to p_2 and that of $F_Y(p_1, p_2)$ with respect to p_2, we have

$$\begin{aligned} &0 < p_{2,PR_{r1}} < p^*_{2,r2} \leqslant 1, \\ &0 < p_{2,Y_{r1}} < p^*_{2,r2} \leqslant 1. \end{aligned} \tag{73}$$

Let $\check{p}_{2,r1}$ denote $\max(p_{2,PR_{r1}}, p_{2,Y_{r1}})$, we have

$$\begin{aligned} &F_{PR}(p^*_{1,r2}, \check{p}_{2,r1}) \geqslant F_{PR}(p^*_{1,r2}, p_{2,PR_{r1}}) = PR_{r1} \\ &F_Y(p^*_{1,r2}, \check{p}_{2,r1}) \geqslant F_Y(p^*_{1,r2}, p_{2,Y_{r1}}) = Y_{r1} \end{aligned} \tag{74}$$

Clearly, (74) indicates that $(p^*_{1,r2}, \check{p}_{2,r1})$ is a feasible solution of (P1'-1). Thus, for the optimal solution $(p^*_{1,r1}, p^*_{2,r1})$ and the feasible solution $(p^*_{1,r2}, \check{p}_{2,r1})$ of (P1'-1), and the optimal solution $(p^*_{1,r2}, p^*_{2,r2})$ of (P1'-2), taking into account (73), we have

$$\begin{aligned} z^*_{r1} &= \sum_{i=1}^{2} P_i p^*_{i,r1} \leqslant P_1 p^*_{1,r2} + P_2 \check{p}_{2,r1} \\ &< \sum_{i=1}^{2} P_i p^*_{i,r2} = z^*_{r2}, \end{aligned} \tag{75}$$

For the cases of $0 < PR_{r1} = PR_{r2} < 1$, $0 < Y_{r1} < Y_{r2} < 1$ or $0 < PR_{r1} < PR_{r2} < 1$, $0 < Y_{r1} = Y_{r2} < 1$, employing the same derivation process as above leads to the conclusion that $z^*_{r1} \leqslant z^*_{r2}$, which indicates that z^* is non-decreasing in PR_r and Y_r, respectively.

■

F. Proof of Theorem 4.1

Proof:

Considering (27), (28), and (30), the objective function of (P2), denoted as $z(p_1)$, is rewritten as the following,

$$z(p_1) = P_1 p_1 + P_2 F_I(p_1, PR_r, Y_r), \tag{76}$$

which indicates that z is a function of p_1. Based on (31), the derivative of z with respect to p_1 is expressed as

$$\frac{\mathrm{d}z(p_1)}{\mathrm{d}p_1} = P_1 - P_2 f_I(p_1). \tag{77}$$

For $PR_r < p_{2,Ymax}$, we investigated the optimal solution of (P2) under the condition where $\hat{p}_1 < \tilde{p}_1$ or $\hat{p}_1 \geqslant \tilde{p}_1$ holds, respectively. On the basis of the results in [9], we have

$$\mathrm{sgn}\big(P_1 - P_2 f(p_1)\big) = \begin{cases} -1, & \text{if } PR_r < p_1 < \tilde{p}_1, \\ 0, & \text{if } p_1 = \tilde{p}_1, \\ 1, & \text{if } \tilde{p}_1 < p_1 < 1. \end{cases} \tag{78}$$

Since $F_{I,Y}(p_1; Y_r)$ is strictly increasing in p_1, we have

$$P_1 - P_2\left(-\frac{\mathrm{d}F_{I,Y}}{\mathrm{d}p_1}\right) > 0. \tag{79}$$

for $PR_r < p_1 < 1$. Taking into account (78) and (79), for $\hat{p}_1 < \tilde{p}_1$, we have

$$\mathrm{sgn}\left(\frac{\mathrm{d}z(p_1)}{\mathrm{d}p_1}\right) = \begin{cases} -1, & \text{if } PR_r < p_1 < \hat{p}_1, \\ 1, & \text{if } \hat{p}_1 < p_1 < 1, \end{cases} \tag{80}$$

which indicates that $z(p_1)$ is strictly decreasing on $(PR_r, \hat{p}_1)$ and strictly increasing on $(\hat{p}_1, 1)$. Therefore, the minimum of z is $z(\hat{p}_1)$ and $p^*_1 = \hat{p}_1$. Considering the definition of $\hat{p}_1$, we have $p^*_2 = F_{I,PR_r}(p^*_1, PR_r)$. For $\hat{p}_1 \geqslant \tilde{p}_1$, we have

$$\mathrm{sgn}\left(\frac{\mathrm{d}z(p_1)}{\mathrm{d}p_1}\right) = \begin{cases} -1, & \text{if } PR_r < p_1 < \tilde{p}_1, \\ 0, & \text{if } p_1 = \tilde{p}_1, \\ 1, & \text{if } \tilde{p}_1 < p_1 < 1, \end{cases} \tag{81}$$

which indicates $p^*_1 = \hat{p}_1$ and $p^*_2 = F_{I,PR_r}(p^*_1, PR_r)$. Consequently, we have (36) and (37) for $PR_r < p_1 < 1$.

For $PR_r \geqslant p_{2,Ymax}$, from the previous analysis, the derivative of z with respect to p_1 is expressed as

$$\frac{\mathrm{d}z(p_1)}{\mathrm{d}p_1} = P_1 - P_2 f(p_1). \tag{82}$$

Considering (78), $\tilde{p}_1$ is the global minimum point of $z(p_1)$. Therefore, the optimal solution of (P2) for $PR_r \geqslant p_{2,Ymax}$ satisties (38) and (39).

■

REFERENCES

[1] Q. Wang, N. Huang, Z. Chen, X. Chen, H. Cai, and Y. Wu, "Environmental data and facts in the semiconductor manufacturing industry: An unexpected high water and energy consumption situation," *Water Cycle*, vol. 4, pp. 47–54, 2023.

[2] J. Li and S. M. Meerkov, *Production Systems Engineering*. New York: Springer, 2009.

[3] C. Shi and S. B. Gershwin, "Part sojourn time distribution in a two-machine line," *European Journal of Operational Research*, vol. 248, no. 1, pp. 146–158, Jan. 2016.

[4] M. Colledani, A. Angius, and A. Horvath, "Lead time distribution in unreliable production lines processing perishable products," in *Proceedings of the 2014 IEEE Emerging Technology and Factory Automation (ETFA)*. Barcelona, Spain: IEEE, Sep. 2014, pp. 1–8.

[5] J.-H. Lee, J. Li, and J. A. Horst, "Serial production lines with waiting time limits: Bernoulli reliability model," *IEEE Transactions on Engineering Management*, vol. 65, no. 2, pp. 316–329, May 2018.

[6] F. Wang and F. Ju, "Transient and steady-state analysis of multistage production lines with residence time limits," *IEEE Transactions on Automation Science and Engineering*, vol. 18, no. 1, pp. 122–134, Jan. 2021.

[7] J.-H. Lee, C. Zhao, J. Li, and C. T. Papadopoulos, "Analysis, design, and control of bernoulli production lines with waiting time constraints," *Journal of Manufacturing Systems*, vol. 46, pp. 208–220, Jan. 2018.

[8] F. Wang and F. Ju, "Simulation-based real-time production control with different classes of residence time constraints," in *2022 IEEE 18th International Conference on Automation Science and Engineering (CASE)*. Mexico City, Mexico: IEEE, Aug. 2022, pp. 1860–1865.

[9] C.-B. Yan, X. Cheng, F. Gao, and X. Guan, "Formulation and solution methodology for reducing energy consumption in two-machine Bernoulli serial lines," *IEEE Transactions on Automation Science and Engineering*, vol. 19, no. 1, pp. 522–530, 2022.

[10] C.-B. Yan, "Analysis and optimization of energy consumption in two-machine Bernoulli lines with general bounds on machine efficiency," *IEEE Transactions on Automation Science and Engineering*, vol. 18, no. 1, pp. 151–163, 2021.

[11] C.-B. Yan and L. Liu, "Problem formulation and solution methodology of energy consumption optimization for two-machine geometric serial lines," *IEEE Transactions on Automation Science and Engineering*, pp. 1–14, 2023.

[12] P. Yang and Z. Pei, "Energy-saving manufacturing system design with two geometric machines," *Sustainability*, vol. 14, no. 18, p. 11448, 2022.

[13] W. Su, X. Xie, J. Li, L. Zheng, and S. C. Feng, "Reducing energy consumption in serial production lines with Bernoulli reliability machines," *International Journal of Production Research*, vol. 55, no. 24, pp. 7356–7379, 2017.

[14] C.-B. Yan and Z. Zheng, "Problem formulation and solution methodology for energy consumption optimization in Bernoulli serial lines," *IEEE Transactions on Automation Science and Engineering*, vol. 18, no. 2, pp. 776–790, 2021.

[15] ——, "An effective and efficient divide-and-conquer algorithm for energy consumption optimisation problem in long Bernoulli serial lines," *International Journal of Production Research*, vol. 59, no. 23, pp. 7018–7036, 2021.

[16] X. Ma, S. Zhang, L. Liu, and C.-B. Yan, "Energy consumption optimization for two-machine bernoulli serial lines with time-sensitive constraint," in *2023 IEEE 19th International Conference on Automation Science and Engineering (CASE)*. IEEE, pp. 1–6. [Online]. Available: https://ieeexplore.ieee.org/document/10260668/