Problem Formulation and Solution Methodology of Energy Consumption Optimization for Two-Machine Geometric Serial Lines

Chao-Bo Yan, *Senior Member, IEEE,* Lingchen Liu, *Graduate Student Member, IEEE*

***Abstract*—Manufacturing systems consume a tremendous amount of energy and contribute about a quarter of greenhouse gas emissions. To achieve the sustainable production, it is vital to reduce the total energy consumption and improve the energy efficiency of manufacturing systems, especially energy-intensive manufacturing systems. In this paper, the energy consumption optimization problem for a two-machine geometric line is investigated. Specifically, it is formulated as a nonlinear programming which minimizes the energy consumption of the system while maintaining a required production rate. For this nonlinear programming with complex constraints, two optimality equations are explored and their mathematical properties are analyzed. Based on these properties, an effective and computationally efficient algorithm is developed to solve the optimal solution of the energy consumption optimization problem. In addition, the sensitivity of the optimal solution with respect to system parameters is analyzed. Finally, several extensions of the problem with an alternative objective and more practical considerations, are addressed as well.**

***Note to Practitioners*—For energy-intensive manufacturing systems, reducing the energy consumption and improving the energy efficiency are of both economic and ecological significance. In the literature, almost all existing related researches assume that machines obey the Bernoulli reliability model, which is not applicable for modeling some production lines. In this paper, we extend the problem formulation and solution methodology for the two-machine line with Bernoulli reliability model to that with geometric model, which has wider applications in practical systems. However, the extension is non-trivial since it is much more complicated to solve the energy consumption optimization problem for geometric lines and more insights on the results are gained. Although the line studied is short, as a step stone, this research will be extended to long geometric lines and production lines with more practical, e.g., exponential and non-exponential, reliability models.**

***Index Terms*—Productivity, nonlinear programming, optimality equations, monotonicity, sensitivity analysis.**

I. Introduction

MANUFACTURING systems consume a tremendous amount of energy every year with massive greenhouse gas emissions. According to the U.S. Department of Energy, the industry sector accounts for 24% of greenhouse gas emissions in 2020, nearly 1.5 billion metric tons [1]. With worldwide concerns on climate change and urgent needs of environmental protection, *green manufacturing* (or interchangeably, *sustainable manufacturing*), which advocates to optimize, control, and redesign the manufacturing systems to reduce emissions and minimize energy use, is proposed [2].

In the last several decades, abundant efforts have been devoted to performance analysis and optimization of manufacturing systems, most of which focus on performance metrics such as productivity, work-in-process, production lead time, and makespan, etc. (see books [3]–[5] and reviews [6]–[8]). These studies have encompassed diverse types of production systems with various machine reliability models, including Bernoulli [9], [10], geometric [11], [12], exponential [13], and non-exponential [14], models. Recent years, as the sustainable manufacturing is paid more and more attention, the energy consumption and energy efficiency have become important and indispensable measures and been studied extensively for different types of manufacturing systems, e.g., for single equipments [15], workshops [16], and flexible production lines [17], among others.

Depending on whether system structures or production processes are changed, researches on energy savings and emission reduction in manufacturing systems can be divided into two main categories: one is redesign of the production systems or innovation of processing technologies, the other is system optimization and control.

System redesign and process improvement for energy saving and emission reduction, have been studied extensively. It is reported in [18] that an energy recovery system with energy efficiency being a key measure is installed in the early layout of production systems. In [19], the life cycle assessment to evaluate environmental impacts of automobile paint shops is applied and suggestions at the process and plant level are provided. Apart from upgrading the whole system, buffer design is a viable alternative for sustainable manufacturing. In [20], by redesigning the repair capacity of automotive paint process, the number of repainted jobs is reduced so that the energy use is saved. In [21], by appropriately conducting the buffer allocation, the energy efficiency of the system is improved. Besides, upgrading production processes can also achieve low-carbon manufacturing [22], [23].

In addition to system redesign and process upgrades, system optimization and control, which reduce energy consumption without changing system structures and manufacturing technologies, are major alternatives to achieve energy-efficient production. For most manufacturing enterprises, redesigning

This paper has been partially presented at the *16th IEEE Conference on Automation Science and Engineering*, Hong Kong, China, 2020.

C.-B. Yan and L. Liu are with the State Key Laboratory for Manufacturing Systems Engineering, and also with the School of Automation Science and Engineering, Faculty of Electronic and Information Engineering, Xi'an Jiaotong University, Xi'an, Shaanxi 710049, China (e-mail: chaoboyan@mail.xjtu.edu.cn).

the system or upgrading the production process is costly in both capital and technologies. Thus, system optimization and control methods are preferable and have been extensively studied.

Control and scheduling are two typical methods to reduce energy consumption of manufacturing systems. The main idea of control methods is cutting back the energy consumption by switching machines ON/OFF according to the state of production systems (i.e., machine status and/or buffer occupancy) [24]–[28]. Different from the control methods, the main concept of scheduling methods is saving energy consumption by adjusting machine operation schedules or task assignments. In [29], the energy consumption reduction problem of Bernoulli serial lines is formulated and effectively solved by scheduling the on-off time of machines. In [30] and [31], the total energy consumption in flexible manufacturing systems is reduced through effectively allocating production tasks.

From the optimization perspective, recently, a novel idea for energy savings in unreliable production systems has emerged. It minimizes the energy consumption of the production systems by elaborately optimizing efficiencies of all unreliable machines. Specifically, in [32], the energy consumption optimization problem for a two-machine Bernoulli serial line is formulated. Based on a large number of numerical experiments, the qualitative relationship between the optimal solution of the problem and system parameters are investigated, but no algorithms are developed to solve the optimal solution. On this basis, an effective and computationally efficient algorithm, which numerically solves the optimal solution of the energy consumption optimization problem, is designed in [33]. Moreover, for the two-machine Bernoulli serial line, this problem is extended to broader scenarios, including that machine efficiencies are limited in a given range [34], energy cost minimization under time-of-use electricity pricing [35], and energy consumption reduction with machine's setup and idleness considered [36].

Meanwhile, the energy consumption optimization for long production lines with more than two machines has also been studied. Specifically, in [37], the energy consumption optimization problem for long Bernoulli serial lines is mathematically formulated and solved by commercial softwares or heuristic algorithms. Due to the lack of effective and computationally efficient algorithms, in [38], this problem is comprehensively analyzed and a solution method is proposed to numerically solve the optimal solution. The solution method, which is time-consuming for long lines with lots of machines, is improved in [39], where the algorithm developed is much more computationally efficient.

Although the energy consumption optimization for Bernoulli serial lines has been comprehensively studied and optimally solved, for lines with more practical reliability, e.g., geometric, exponential, and non-exponential, models, this problem is far from being well studied. Among the limited related researches, article [40] focuses on the energy savings of a two-machine Markovian serial line with setups, and formulates an integrated problem which minimizes the system energy consumption while maximizing the productivity. Some qualitative properties of the problem are analyzed and the Pareto frontier is obtained, but no algorithms are provided. The energy consumption optimization problem for a two-machine geometric line with machine's setup and idleness is investigated in [41], where the formulation and solution method in [36] is extended to the geometric lines.

To comprehensively study the energy consumption optimization for geometric serial lines, as a step stone, this paper focuses on the two-machine lines show in Fig. 1. The solution methodology, which is based on rigorous mathematical derivations and first developed in [33], is extended to solve the problem in the current paper. Specifically, for the energy consumption optimization problem, i.e., a nonlinear programming with complex constraints, first, two optimality equations are explored and their mathematical properties are elaborately analyzed. Then, based on the properties, an effective and efficient algorithm is designed to solve the optimal solution of the problem. Finally, the sensitivity of the optimal solution with respect to system parameters is analyzed. On the basis of the solution, some extended problems with more practical considerations are discussed and solved as well. It should be pointed out that although the Bernoulli reliability model could be regarded as a special case of the geometric model, the extension of the solution method is non-trivial since it is much more complicated to solve the problem for geometric lines and more insights on the results are gained. It should also be pointed out that although the problem formulation and solution approach have been preliminarily presented in [42], the current paper extensively analyzes properties of the energy consumption characteristic function, carries out the sensitivity analysis of the optimal solution, and extends the results of the fundamental energy consumption optimization model to more general and practical models.

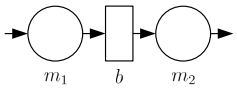


Fig. 1: Two-machine serial line

The rest of this paper is organized as follows. In Section II, the two-machine geometric serial line in Fig. 1 is formalized and the energy consumption optimization problem is formulated. In Section III, two optimality equations of the energy consumption optimization problem is explored. In Section IV, an effective and efficient algorithm is proposed to solve the optimality equations and thus, solve the optimal solution of the energy consumption optimization problem. In Section V, sensitivity analysis of the optimal solution is conducted. In Section VI, some extended problems are discussed. The conclusions and topics for future work are presented in Section VII. All proofs are provided in the Appendix.

II. Production System Modeling and Problem Formulation

In this section, the two-machine geometric line shown in Fig. 1 is modeled in Subsection II-A, and the energy consumption optimization problem investigated in this paper is formulated and transformed in Subsection II-B.

A. System Model

To formulate the energy consumption problem, the model of the two-machine production line in Fig. 1 is formalized as follows:

(i) The system consists of two machines and an intermediate buffer, which are denoted as m_1, m_2, and b, respectively.

(ii) Both machines have identical cycle time (i.e., processing time), which is denoted by τ. The time is slotted with slot duration τ. The status of a machine (i.e., up or down) is determined at the beginning of each time slot, and the state of the buffer (i.e., the occupancy) is determined at the end of each time slot.

(iii) Machine m_i, $i = 1, 2$, obeys the geometric reliability model, which is characterized by breakdown probability p_i and repair probability r_i. Specifically, as shown in Fig. 2, if m_i is up, in the next production cycle, it will be down with probability p_i and up with $1 - p_i$, where $p_i \in (0, 1)$; if it is down, it will be up with probability r_i and down with $1 - r_i$. Herein, p_i is fixed and r_i can be selected in $(0, 1]$. Note that for m_i, $i = 1, 2$, its efficiency is $e_i = \frac{r_i}{p_i + r_i}$.

(iv) The repair resources are sufficient so that any machine can be repaired in one cycle time (correspondingly, the repair probability is 1). In other words, the cost of repair resources is ignored and by elaborately adjusting the repair resources, repair probability r_i, $i = 1, 2$, can take any value in $(0, 1]$.

(v) The capacity of buffer b is N, which is an integer and $0 < N < +\infty$.

(vi) The blocking mechanism is assumed to be blocking before service. Specifically, at the beginning of a time slot, if the buffer is empty, m_2 is then starved; if the buffer is full and m_2 fails to take a part from it for processing, m_1 is then blocked. Machine m_1 is never starved and m_2 never blocked.

(vii) When machine m_i, $i = 1, 2$, is up, the power it consumes is P_i; when m_i is down, it doesn't consume any power. Herein, $0 < P_i < +\infty$.

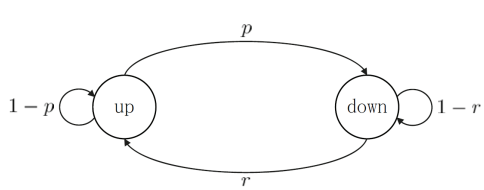


Fig. 2: State transition diagram of a machine with geometric reliability model

Remark 2.1: In Assumption (iii), machines are assumed to obey the geometric reliability model, which is commonly used in lines of machining, heat treatment, washing, etc., where the downtime of the machine is typically much longer than its cycle time [5].

Remark 2.2: It should be pointed out that, if $p_i + r_i = 1$, $i = 1, 2$, the geometric reliability model degrades to the Bernoulli model. In this case, the machine efficiency e_i is equal to r_i.

Remark 2.3: In Assumption (iii), the breakdown probability is assumed to be a constant. It makes sense because malfunctions, the intrinsic attribute of a machine, are governed by physical and statistical laws, which implies that, in a relatively long time, the breakdown probability is usually invariant.

Remark 2.4: To implement the optimal repair probability r^* in a practical production system, let $T_{down,i}$, $i = 1, 2, \ldots$, be the ith required downtime randomly generated from a geometric distribution with parameter r^*. Clearly, $T_{down,i} \in \{\tau, 2\tau, \ldots\}$. For example, $T_{down,1} = 2\tau$, $T_{down,2} = 3\tau$, $T_{down,3} = \tau$, etc., as shown in Fig. 3, where the uptimes are omitted and represented by ellipsis. Since the repair resources are sufficient (see Assumption (iv)), for i^{th} breakdown, by elaborately adjusting the repair resources, the practical downtime will be $T_{down,i}$ as desired.

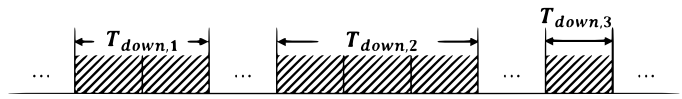


Fig. 3: Repair times randomly generated according to the optimal repair probability for a practical production system

B. Problem Formulation and Transformation

Based on model (i)-(vii), the problem investigated in this paper is addressed in this subsection. Specifically, by elaborately choosing the pair of machine repair probability, (r_1, r_2), this paper intends to minimize the energy consumed by the machines during a production cycle, while ensuring that the productivity of the line is not less than a required production rate, PR_r. Considering that the production rate of the two-machine geometric line is expressed as

$$PR = e_2\big[1 - Q(p_1, r_1, p_2, r_2, N)\big] \tag{1}$$

and is increasing in r_1 and r_2, respectively (see [43] for details), the energy consumption optimization problem is formulated as follows:

$$\text{(P1) min} \quad z = \sum_{i=1}^{2} P_i e_i \tag{2}$$

$$\text{s.t.:} \quad e_2\big[1 - Q(p_1, r_1, p_2, r_2, N)\big] \geqslant PR_r, \tag{3}$$

$$0 < r_i \leqslant 1, \ i = 1, 2, \tag{4}$$

where

$$e_i = \frac{r_i}{p_i + r_i}, \tag{5}$$

$$Q(p_1, r_1, p_2, r_2, N) = \begin{cases} \frac{p_1 \beta_2}{(p_1 + r_1)(r_1 + r_2 - r_1 r_2)}, & \text{if } N = 1, \\ \frac{p_1 \alpha_1 \alpha_2 \beta_2^2 (p_2 + r_2)}{A + B + C + D}, & \text{if } N \neq 1, \end{cases} \tag{6}$$

and

$$\begin{aligned}
\alpha_1 =& p_1 + p_2 - p_1p_2 - p_2r_1,\\
\alpha_2 =& p_1 + p_2 - p_1p_2 - p_1r_2,\\
\beta_1 =& r_1 + r_2 - r_1r_2 - p_1r_2,\\
\beta_2 =& r_1 + r_2 - r_1r_2 - p_2r_1,\\
\sigma =& \frac{\alpha_2\beta_1}{\alpha_1\beta_2},\\
A =& p_1r_2\alpha_1\alpha_2\beta_2(p_2+\beta_2),\\
B =& p_1r_1r_2\alpha_2\big[\beta_2^2 + p_2(\alpha_1+\beta_1)(\alpha_2+2\beta_2)\big],\\
C =& \sum_{k=2}^{N-1} p_1p_2r_1r_2(\alpha_2+\beta_2)^3\sigma^{k-1},\\
D =& p_2r_1\alpha_1\beta_2\big[r_2(\alpha_1+\beta_1) + \alpha_2(p_1+r_1)\big]\sigma^{N-1}.
\end{aligned} \tag{7}$$

Note that α_1, α_2, β_1, β_2, σ, A, B, C, and D are always positive. Specifically, for $N = 1$, we have

$$\begin{aligned}
Q =& \frac{(1-e_1)\big[p_1e_1(1-e_2) + p_2e_2(1-e_1) - p_1p_2e_1\big]}{p_1e_1(1-e_2) + p_2e_2(1-e_1) - p_1p_2e_1e_2}\\
PR =& e_1e_2 + \frac{p_1p_2e_1e_2(1-e_1)(1-e_2)}{p_1e_1(1-e_2) + p_2e_2(1-e_1) - p_1p_2e_1e_2}.
\end{aligned} \tag{8}$$

In problem (P1), p_1 and p_2 are fixed, and r_1 and r_2 are decision variables. Taking into account the expression of machine efficiency in (5), (P1) can be transformed into the following problem with e_1 and e_2 being decision variables:

$$\text{(P1')} \min \quad z = \sum_{i=1}^{2} P_ie_i \tag{9}$$

$$\text{s.t.:} \quad e_2\big[1 - Q(e_1, e_2, N; p_1, p_2)\big] \geqslant PR_r, \tag{10}$$

$$0 < e_i \leqslant \frac{1}{1+p_i},\ i = 1, 2. \tag{11}$$

Clearly, (P1') is equivalent to (P1). It should be pointed out that, with a slight abuse of notations, the Q-function in (P1') has been re-written as a function of e_i, $i = 1, 2$. Since p_1 and p_2 are fixed, in the following, the Q-function in (10) will be abbreviated as $Q(e_1, e_2, N)$ or Q if not otherwise specified. It should also be pointed out that, since e_i, $i = 1, 2$, is strictly increasing in r_i, it is easy to obtain the lower- and upper-bound of e_i as shown in (11). Furthermore, considering that PR is a strictly increasing function of r_i (and thus, of e_i), $i = 1, 2$, the maximum attainable production rate of the two-machine geometric line is

$$PR_{max} = \frac{1}{1+p_2}\left[1 - Q\Big(\frac{1}{1+p_1}, \frac{1}{1+p_2}, N\Big)\right]. \tag{12}$$

Note that when $N = 1$, the maximum attainable production rate can be re-written as

$$PR_{max} = \frac{1+p_1p_2}{(1+p_1)(1+p_2)}. \tag{13}$$

Thus, to ensure that (P1) and (P1') have feasible solutions, PR_r should not be greater than PR_{max}, i.e.,

$$0 < PR_r \leqslant PR_{max}. \tag{14}$$

Similar to the analysis approach for the energy consumption optimization problem in the two-machine Bernoulli line [33], we present the following problem

$$\text{(P2)} \min \quad z = \sum_{i=1}^{2} P_ie_i \tag{15}$$

$$\text{s.t.:} \quad e_2\big[1 - Q(e_1, e_2, N)\big] = PR_r, \tag{16}$$

$$0 < e_i \leqslant \frac{1}{1+p_i},\ i = 1, 2, \tag{17}$$

and analyze the monotonicity of its optimal objective value with respect to PR_r. As a result, we have:

Theorem 2.1: The optimal objective value, z^*, of (P2), is strictly increasing in PR_r.

Proof: See the Appendix.

Based on this theorem, the relationship between problems (P1') and (P2) is analyzed and concluded in the following.

Corollary 2.1: Problem (P1') is essentially equivalent to (P2). In other words, constraint (10) in (P1') can be replaced by (16) in (P2).

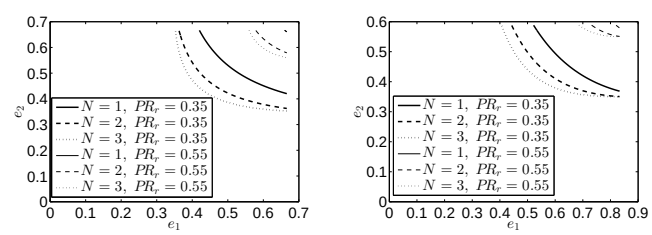
The proof is omitted since from Theorem 2.1, it is easy to draw the above conclusion.

Based on Corollary 2.1 and taking into account that (P1) and (P1') are equivalent to each other, one can conclude that solving (P2) leads to solving (P1). Since no direct properties of (P2) can be used for algorithm design, in Section III, two optimality equations will be derived and analyzed. Based on properties of these optimality equations, an effective algorithm for solving the optimal solution of (P2) will be developed in Section IV.

III. Optimality Equations

Similar to the analysis approach in [33], in this section, two optimality equations for solving (P2) are explored.

From the insights on the energy consumption optimization for the two-machine Bernoulli line [33], one of the optimality equations of (P2) is (16), which, being regarded as a contour of the production rate, characterizes the relationship between e_1 and e_2 on the contour. Specifically, since the production rate is strictly increasing in e_1 and e_2, respectively [43], for a fixed PR_r, e_2 can be regarded as an implicit decreasing function of e_1. The behavior of the implicit function e_2 with respect to e_1 for different N and p_i's is shown in Fig. 4.



(a) $p_1 = p_2 = 0.5$ (b) $p_1 = 0.2$, $p_2 = 0.7$

Fig. 4: Implicit function e_2 with respect to e_1

Based on the optimality equation (16), the feasible region of e_1 and e_2 can be analyzed. From the results in [43], it follows that $Q \in (0, 1)$, which, combining with (16), implies

that $e_2 > PR_r$. Taking into account the reversibility of the production line [5], we have $e_1 > PR_r$. Combining the above two inequalities with (17), we have:

$$PR_r < e_i \leqslant \frac{1}{1+p_i},\ i=1,2. \tag{18}$$

Based on the relationship between e_2 and e_1 analyzed above, the feasible region of (P2) can be further specified. Specifically, since e_2 is a strictly decreasing function of e_1 on the production rate contour, it takes its minimum attainable value when e_1 takes its maximum, $\frac{1}{1+p_1}$. Let $e_{2,min}$ denote the minimum attainable value of e_2, then it satisfies

$$e_{2,min}\left[1-Q\Big(\frac{1}{1+p_1}, e_{2,min}, N\Big)\right] = PR_r. \tag{19}$$

It should be pointed out that $e_{2,min} > PR_r$. The argument is as follows: if $p_1 = 0$, then $e_1 = 1$ and $Q(e_1, e_2, N) = 0$, thus we have

$$PR_r\big[1-Q(1, PR_r, N)\big] = PR_r. \tag{20}$$

Taking into account that e_2 is strictly decreasing in e_1 and e_1 is in turn strictly decreasing in p_1 and noting that $p_1 > 0$, we conclude $e_{2,min} > PR_r$.

Similarly, the minimum attainable value of e_1, namely, $e_{1,min}$, satisfies

$$\frac{1}{1+p_2}\left[1-Q\Big(e_{1,min}, \frac{1}{1+p_2}, N\Big)\right] = PR_r. \tag{21}$$

Note that $e_{1,min} > PR_r$, which can be validated by the similar justification above.

Thus, the feasible region of (P2) is the intersection of the contour and the rectangle area characterized by

$$e_{i,min} \leqslant e_i \leqslant e_{i,max},\ i=1,2, \tag{22}$$

where

$$e_{i,max} = \frac{1}{1+p_i},\ i=1,2, \tag{23}$$

and $e_{i,min}$ can be uniquely solved from (19) or (21) in terms of the strict monotonicity of the production rate with respect to e_1 and e_2, respectively. It should be pointed out that although $e_{1,max}$ and $e_{2,max}$ in (23) are constant, clearly, both $e_{1,min}$ and $e_{2,min}$ are strictly increasing in PR_r. Also note that when $N = 1$, $e_{1,min}$ and $e_{2,min}$ have closed-form expressions, i.e.,

$$e_{1,min} = \frac{PR_r(1+p_2)}{1+p_1p_2},\ e_{2,min} = \frac{PR_r(1+p_1)}{1+p_1p_2}. \tag{24}$$

As for the other optimality equation, it is derived in the following. Construct the Lagrangian function of (P2) as follows:

$$z = \sum_{i=1}^{2} P_i e_i + \mu\big[e_2(1-Q) - PR_r\big], \tag{25}$$

where μ is the Lagrangian multiplier. Taking the partial derivative of z with respect to e_1 and e_2, respectively, and noting that Q is continuously differentiable, we have

$$\begin{aligned} &P_1 - \mu e_2 \frac{\partial Q}{\partial e_1} = 0, \\ &P_2 + \mu\Big[(1-Q) - e_2\frac{\partial Q}{\partial e_2}\Big] = 0, \end{aligned} \tag{26}$$

which imply

$$\frac{P_1}{P_2} = \frac{e_2\frac{\partial Q}{\partial e_1}}{e_2\frac{\partial Q}{\partial e_2} - (1-Q)}. \tag{27}$$

Since e_2 is an implicit function of e_1 (see Fig. 4), the right-hand side of (27) can be regarded as a function of e_1 only. Define it as a function of e_1, i.e.,

$$\begin{aligned} f(e_1) :=& \left.\frac{e_2\frac{\partial Q}{\partial e_1}}{e_2\frac{\partial Q}{\partial e_2} - (1-Q)}\right|_{PR=PR_r} \\ =& \left.\frac{e_2^2\frac{\partial Q}{\partial e_1}}{e_2^2\frac{\partial Q}{\partial e_2} - PR_r}\right|_{PR=PR_r}, \end{aligned} \tag{28}$$

which is called the energy consumption characteristic function. Based on the definition in (28), the expression of $f(e_1)$ can be obtained as follows:

$$f(e_1) = \begin{cases} \frac{p_2e_2^2(PR_r-e_1e_2)^2+p_1p_2e_1^2e_2^2(1-e_2)(e_2-PR_r)}{p_1e_1^2(PR_r-e_1e_2)^2+p_1p_2e_1^2e_2^2(1-e_1)(e_1-PR_r)}, & \text{if } N=1, \\ \frac{e_2^2\frac{\partial Q}{\partial e_1}}{e_2^2\frac{\partial Q}{\partial e_2}-PR_r}, & \text{if } N>1, \end{cases} \tag{29}$$

where the expressions of $\frac{\partial Q}{\partial e_1}$ and $\frac{\partial Q}{\partial e_2}$, which are omitted here because of their complicated forms, can be found in [44]. From [44], it follows

$$\frac{\partial PR}{\partial e_1} = -e_2\frac{\partial Q}{\partial e_1},\ \frac{\partial PR}{\partial e_2} = (1-Q) - e_2\frac{\partial Q}{\partial e_2}, \tag{30}$$

which implies

$$f(e_1) = \left.\left(\frac{\partial PR}{\partial e_1}\Big/\frac{\partial PR}{\partial e_2}\right)\right|_{PR=PR_r}. \tag{31}$$

It is worth noting that the above equation holds for the Bernoulli reliability model as well. Thus, it is hypothesized that Eq. (31) holds for general reliability models and can be used as an alternative definition of the energy consumption characteristic function f.

It should be pointed out that both $e_2\frac{\partial Q}{\partial e_1}$ and $e_2\frac{\partial Q}{\partial e_2} - (1-Q)$ (correspondingly, $e_2^2\frac{\partial Q}{\partial e_1}$ and $e_2^2\frac{\partial Q}{\partial e_2} - PR_r$) in (28) are negative [44], which implies that $f(e_1)$ is always positive. It should also be pointed out that taking the derivative of both sides of (16) with respect to e_1 and re-arranging the terms result in

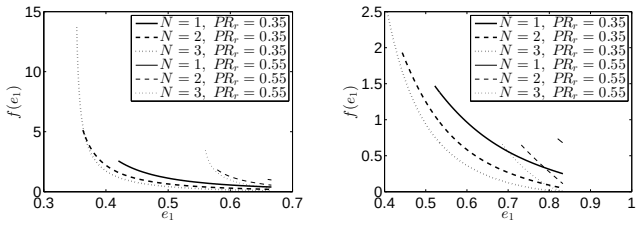
$$e_2' = \frac{de_2}{de_1} = -\left.\frac{e_2\frac{\partial Q}{\partial e_1}}{e_2\frac{\partial Q}{\partial e_2} - (1-Q)}\right|_{PR=PR_r} = -f(e_1). \tag{32}$$

The behavior of the energy consumption characteristic function $f(e_1)$ for various N and p_i's is shown in Fig. 5.

By using $f(e_1)$ defined in (28), Eq. (27) can be re-written as

$$f(e_1) = \frac{P_1}{P_2}. \tag{33}$$

Clearly, to solve (P2), it is necessary to solve optimality equations (16) and (33). In the subsequent section, properties of the optimality equations will be analyzed and based on these properties, an effective method will be proposed to solve the optimality equations and thus, to solve problem (P2).



(a) $p_1 = p_2 = 0.5$ (b) $p_1 = 0.2$, $p_2 = 0.7$

Fig. 5: The behavior of the energy consumption characteristic function $f(e_1)$

IV. Solution Methodology

In this section, the methodology proposed in [33] for energy consumption optimization in two-machine Bernoulli lines is employed to solve problem (P2). Specifically, in Subsection IV-A, some useful properties of $f(e_1)$ are analyzed; in Subsection IV-B, based on the properties of $f(e_1)$, an efficient and effective algorithm is developed to numerically solve the optimal solution of (P2); in Subsection IV-C, numerical experiments are presented to reveal the effectiveness of the proposed algorithm.

A. Properties of $f(e_1)$

To solve the optimality equations derived in Section III, in this subsection, more properties of function $f(e_1)$ in (28) are analyzed. To be specific, the continuity, strict monotonicity, and range of $f(e_1)$ are analyzed.

First, the continuity of $f(e_1)$ is analyzed based on the definition in (28). From [44], it follows that both $e_2^2\frac{\partial Q}{\partial e_1}$ and $e_2^2\frac{\partial Q}{\partial e_2}-PR_r$ in (28) are continuous and always negative, which implies that $f(e_1)$ is continuous.

Then, the monotonicity of $f(e_1)$ is analyzed. To do that, the derivative of $f(e_1)$ is deduced. As a result, we have:

Proposition 4.1: For the energy consumption characteristic function $f(e_1)$ in (29), its derivative is

$$f'(e_1)=\begin{cases}\frac{1}{e_1e_2v^3}\big[2uvp_2e_1^2e_2^3(PR_r-e_1e_2)\\ \quad -2uvp_1p_2e_1^3e_2^3(1+PR_r-e_1-e_2)\\ \quad -2v^2p_2e_2^3PR_r(PR_r-e_1e_2)\\ \quad +2uvp_1e_1^3e_2^2(PR_r-e_1e_2)\\ \quad -2u^2p_1e_1^3PR_r(PR_r-e_1e_2)\big], & \text{if } N=1,\\ \frac{e_2^2\frac{\partial^2 Q}{\partial e_1^2}}{e_2^2\frac{\partial Q}{\partial e_2}-PR_r}-\frac{e_2^3\frac{\partial Q}{\partial e_1}}{(e_2^2\frac{\partial Q}{\partial e_2}-PR_r)^3}\Big[2e_2^3\frac{\partial Q}{\partial e_2}\frac{\partial^2 Q}{\partial e_1\partial e_2}\\ \quad -2PR_r\big(\frac{\partial Q}{\partial e_1}+e_2\frac{\partial^2 Q}{\partial e_1\partial e_2}\big)-e_2^3\frac{\partial Q}{\partial e_1}\frac{\partial^2 Q}{\partial e_2^2}\Big], & \text{if } N>1,\end{cases} \tag{34}$$

where

$$\begin{aligned}u=&p_2e_2^2(PR_r-e_1e_2)^2+p_1p_2e_1^2e_2^2(1-e_2)(e_2-PR_r),\\ v=&p_1e_1^2(PR_r-e_1e_2)^2+p_1p_2e_1^2e_2^2(1-e_1)(e_1-PR_r),\end{aligned} \tag{35}$$

and the expressions of $\frac{\partial Q}{\partial e_1}$, $\frac{\partial Q}{\partial e_2}$, $\frac{\partial^2 Q}{\partial e_1^2}$, $\frac{\partial^2 Q}{\partial e_2^2}$, and $\frac{\partial^2 Q}{\partial e_1\partial e_2}$ are omitted and can be found in [44]. Moreover, $f'(e_1)$ is continuous, which implies that $f(e_1)$ is continuously differentiable.

Proof: See the Appendix.

Although when $p_i + r_i = 1$, $i = 1, 2$, the geometric reliability model degrades to the Bernoulli model, comparing the expressions of $f'(e_1)$ in (34) with those for Bernoulli model in [33], one can see that $f'(e_1)$ for the geometric model does not degrade to the Bernoulli model anymore, which implies that the extension of the solution method from Bernoulli to geometric model is non-trivial.

In the following, we validate $f(e_1)$ is strictly decreasing in e_1 for $N = 1$ and $N > 1$ separately.

For $N = 1$, let $g(e_1) = \frac{\partial PR}{\partial e_1}\big|_{PR=PR_r}$ and $h(e_1) = \frac{\partial PR}{\partial e_2}\big|_{PR=PR_r}$, then we have $f(e_1) = \frac{g(e_1)}{h(e_1)}$. From [44], it follows

$$\begin{aligned}g(e_1)=&\frac{e_2-PR_r}{1-e_1}+\frac{(PR_r-e_1e_2)^2}{p_1e_1^2(1-e_1)(1-e_2)},\\ h(e_1)=&\frac{e_1-PR_r}{1-e_2}+\frac{(PR_r-e_1e_2)^2}{p_2e_2^2(1-e_1)(1-e_2)}.\end{aligned} \tag{36}$$

As a result, we have:

Lemma 4.1: For $N = 1$, $g(e_1)$ is strictly decreasing in e_1.

Proof: See the Appendix.

Lemma 4.2: For $N = 1$, $h(e_1)$ is strictly increasing in e_1.

Proof: See the Appendix.

Theorem 4.1: For $N = 1$, $f(e_1)$ is strictly decreasing in e_1.

Proof: It is easy to draw this conclusion based on Lemmas 4.1 and 4.2. ■

As for $N > 1$, it is very difficult to prove the monotonicity of $f(e_1)$. Thus, the numerical justification is adopted. For this purpose, 5000 test cases have been constructed, with parameters randomly and equiprobably selected from the following sets:

$$\begin{aligned}&p_1,p_2\in(0,1),\ PR_r\in(0,PR_{max}],\\ &N\in\{2,3,4,5,6,7,8,9,10,15,20\}.\end{aligned} \tag{37}$$

As a result, we have:

Numerical Fact 4.1: For all of the 5000 test cases constructed for $N > 1$, $f'(e_1)$ in (34) is always negative.

Based on Theorem 4.1 and Numerical Fact 4.1, we conclude that $f(e_1)$ is strictly decreasing in e_1. The strict monotonicity of $f(e_1)$ can also be observed in Fig. 5.

Finally, by taking advantage of the monotonicity results of $f(e_1)$, its range is analyzed. Denote the minimum and maximum values of $f(e_1)$ as f_{min} and f_{max}, respectively. Noting that the range of e_i in (22) and taking into account the monotonicity of $f(e_1)$, we have

$$f_{min}=f(e_{1,max}),\ f_{max}=f(e_{1,min}), \tag{38}$$

where $f(e_1)$ is expressed in (29) and $e_{i,min}$ and $e_{i,max}$, $i = 1, 2$, are, respectively, defined in (19), (21), and (23).

To explore more properties of $f(e_1)$, the monotonicity of f_{min} and f_{max} is investigated. For this purpose, 6000 test cases are constructed and the system parameters are selected randomly and equiprobably from the following sets:

$$p_1,p_2\in(0,1),\ N\in\{1,2,\ldots,10\}. \tag{39}$$

For all test cases, f_{min} and f_{max} are evaluated with all PR_r in the following set:

$$PR_r\in\{0.1PR_{max},0.2PR_{max},0.3PR_{max},\ldots,PR_{max}\}, \tag{40}$$

where PR_{max} is calculated in terms of (12). As a result, we have:

Numerical Fact 4.2: For all 6000 constructed test cases, f_{min} and f_{max} are, respectively, strictly increasing and decreasing in PR_r.

It is clear that, similar to the Bernoulli reliability model, the f-function of the two-machine line with geometric reliability model is also of good mathematical properties, i.e., positiveness, continuity, strict monotonicity, and continuous differentiability.

B. Algorithm Design

Based on the properties explored in Subsection IV-A, an efficient and effective method will be developed to solve optimality equations (16) and (33) so that problem (P2) is solved.

To do that, we first assume that $\frac{P_1}{P_2} \in [f_{min}, f_{max}]$ and develop a method to solve the optimality equations (16) and (33), and then extend the method to the general case $\frac{P_1}{P_2} \in (0, +\infty)$.

For $\frac{P_1}{P_2} \in [f_{min}, f_{max}]$, clearly, optimality equations (16) and (33) always have a solution. Considering that both $f(e_1)$ and e_2 are strictly monotonic functions of e_1, the optimality equations have a unique solution. Furthermore, based on the monotonicity of $f(e_1)$ and e_2, this unique solution can be solved by a binary search method (i.e., the dichotomy method) in the following. Let $e_1^L = e_{1,min}$ and $e_1^U = e_{1,max}$ denote, respectively, the initial lower- and upper-endpoint for the binary search. Let $\hat{e}_1 = \frac{e_1^L + e_1^U}{2}$, solve $\hat{e}_2$ based on (16) with e_1 replaced by $\hat{e}_1$, and calculate $f(\hat{e}_1)$ with $(\hat{e}_1, \hat{e}_2)$. If $|f(\hat{e}_1) - \frac{P_1}{P_2}| < \varepsilon$ (where ε is a predetermined small enough positive real number), then $(e_1^*, e_2^*) = (\hat{e}_1, \hat{e}_2)$ and the search ends; otherwise, if $f(\hat{e}_1) < \frac{P_1}{P_2}$ (or $f(\hat{e}_1) > \frac{P_1}{P_2}$), let $e_1^U = \hat{e}_1$ (correspondingly, let $e_1^L = \hat{e}_1$) and continue the binary search using the updated lower- and upper-endpoints. The flowchart for solving (e_1^*, e_2^*) using the binary search method is shown in Fig. 6.

As for the general case, i.e., $\frac{P_1}{P_2} \in (0, +\infty)$, we have

$$\frac{dz}{de_1} = P_1 + P_2 \frac{de_2}{de_1} = P_2 \Big[\frac{P_1}{P_2} - f(e_1) \Big], \tag{41}$$

where $e_1 \in [e_{1,min}, e_{1,max}]$. If $\frac{P_1}{P_2} < f_{min}$, i.e., $\frac{P_1}{P_2} < f_{min} = f(e_{1,max}) \leqslant f(e_1)$, then $\frac{dz}{de_1} < 0$, which implies that z is strictly decreasing in e_1 and achieves its minimum at $e_{1,max}$. In other words, $(e_1^*, e_2^*) = (e_{1,max}, e_{2,min})$. Similarly, if $\frac{P_1}{P_2} > f_{max}$, then $\frac{dz}{de_1} > 0$, which implies that z is strictly increasing in e_1 and achieves its minimum at $e_{1,min}$. In this case, $(e_1^*, e_2^*) = (e_{1,min}, e_{2,max})$.

C. Numerical Experiments

Based on the method proposed in Subsection IV-B, extensive test cases with various PR_r, N, $\frac{P_1}{P_2}$, p_1, and p_2 are constructed and solved. Some of these test cases and their solutions are provided in Table I. Note that the optimal solution (e_1^*, e_2^*), which depends on the value of $\frac{P_1}{P_2}$, is independent of the absolute values of P_1 and P_2. Considering that the optimal objective value, z^*, depends on the absolute value of P_1 and P_2, without loss of generality, in Table I, z^* is provided for $P_2 = 1$.

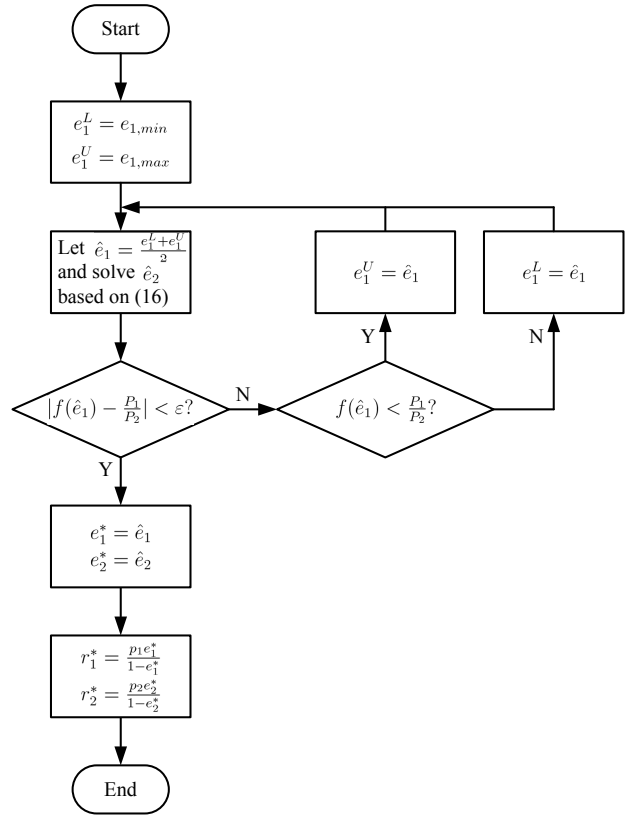


Fig. 6: Flowchart of solving (P2) when $\frac{P_1}{P_2} \in [f_{min}, f_{max}]$

From Table I, one can observe:

- e_1^*, e_2^*, and z^* are increasing in PR_r while decreasing in N;
- $e_{1,min}$ and f_{min} are also increasing in PR_r and decreasing in N;
- f_{max} is decreasing in PR_r and increasing in N.

The quantitative impact of system parameters on the optimal solution and on the optimal objective value will be analyzed in the next section.

V. Sensitivity Analysis

From Table I, we observe that the values of all system parameters have impacts on the optimal solution and the optimal objective value of (P2). In this section, we analyze the sensitivity of the optimal solution and the optimal objective value with respect to these system parameters. Specifically, the impact of PR_r is analyzed in Subsection V-A and impacts of other parameters are investigated in Subsection V-B.

A. Impact of PR_r

In this subsection, the impact of PR_r on the optimal solution and on the optimal objective value is analyzed. To do that, the expression of $\frac{de_i^*}{dPR_r}$, $i = 1, 2$, is derived. First, with a slight

TABLE I: Optimal solution of (P2) for various test cases

Case No.	$\frac{P_1}{P_2}$	N	p_1	p_2	PR_r	$e_{1,min}$	$e_{1,max}$	f_{min}	f_{max}	e_1^*	e_2^*	r_1^*	r_2^*	z^* $(P_2=1)$
1	0.5	1	0.1	0.2	0.1	0.118	0.909	0.094	8.014	0.401	0.213	0.067	0.054	0.414
2	0.5	1	0.1	0.2	0.4	0.471	0.909	0.388	1.983	0.823	0.469	0.464	0.177	0.881
3	0.5	1	0.1	0.2	0.75	0.882	0.909	0.860	0.973	0.909	0.809	1	0.846	1.263
4	0.5	1	0.9	0.8	0.1	0.105	0.526	0.043	33.766	0.211	0.156	0.241	0.148	0.261
5	0.5	1	0.9	0.8	0.3	0.314	0.526	0.294	3.967	0.460	0.357	0.767	0.444	0.587
6	0.5	1	0.9	0.8	0.5	0.523	0.526	1.037	1.074	0.526	0.552	1	0.987	0.815
7	0.5	2	0.1	0.2	0.1	0.110	0.909	0.071	9.565	0.358	0.197	0.056	0.049	0.376
8	0.5	2	0.1	0.2	0.4	0.439	0.909	0.296	2.333	0.761	0.464	0.318	0.173	0.844
9	0.5	2	0.1	0.2	0.75	0.827	0.909	0.640	1.064	0.909	0.764	1	0.647	1.219
10	0.5	2	0.9	0.8	0.1	0.100	0.526	0.009	424.368	0.164	0.130	0.177	0.119	0.212
11	0.5	2	0.9	0.8	0.3	0.300	0.526	0.063	59.905	0.390	0.331	0.576	0.396	0.526
12	0.5	2	0.9	0.8	0.5	0.502	0.526	0.754	5.009	0.526	0.515	1	0.851	0.779
13	0.5	3	0.1	0.2	0.1	0.105	0.909	0.028	12.707	0.326	0.185	0.048	0.045	0.348
14	0.5	3	0.1	0.2	0.4	0.422	0.909	0.120	3.017	0.714	0.463	0.249	0.172	0.819
15	0.5	3	0.1	0.2	0.75	0.802	0.909	0.306	1.191	0.887	0.763	0.782	0.645	1.206
16	0.5	3	0.9	0.8	0.1	0.100	0.526	0.001	23208.128	0.146	0.120	0.153	0.109	0.193
17	0.5	3	0.9	0.8	0.3	0.300	0.526	0.008	1942.750	0.363	0.322	0.514	0.380	0.504
18	0.5	3	0.9	0.8	0.5	0.500	0.526	0.434	28.541	0.524	0.507	0.991	0.822	0.769
19	2	1	0.1	0.2	0.1	0.118	0.909	0.094	8.014	0.226	0.392	0.029	0.129	0.844
20	2	1	0.1	0.2	0.4	0.471	0.909	0.388	1.983	0.471	0.833	0.089	1	1.775
21	2	1	0.1	0.2	0.75	0.882	0.909	0.860	0.973	0.882	0.833	0.750	1	2.598
22	2	1	0.9	0.8	0.1	0.105	0.526	0.043	33.766	0.149	0.218	0.157	0.223	0.515
23	2	1	0.9	0.8	0.3	0.314	0.526	0.294	3.967	0.344	0.473	0.471	0.717	1.160
24	2	1	0.9	0.8	0.5	0.523	0.526	1.037	1.074	0.523	0.556	0.988	1	1.602
25	2	2	0.1	0.2	0.1	0.110	0.909	0.071	9.565	0.217	0.342	0.028	0.104	0.776
26	2	2	0.1	0.2	0.4	0.439	0.909	0.296	2.333	0.467	0.771	0.088	0.674	1.706
27	2	2	0.1	0.2	0.75	0.827	0.909	0.640	1.064	0.827	0.833	0.479	1	2.488
28	2	2	0.9	0.8	0.1	0.100	0.526	0.009	424.368	0.126	0.167	0.130	0.161	0.420
29	2	2	0.9	0.8	0.3	0.300	0.526	0.063	59.905	0.328	0.393	0.438	0.518	1.048
30	2	2	0.9	0.8	0.5	0.502	0.526	0.754	5.009	0.508	0.538	0.929	0.930	1.553
31	2	3	0.1	0.2	0.1	0.105	0.909	0.028	12.707	0.207	0.307	0.026	0.089	0.722
32	2	3	0.1	0.2	0.4	0.422	0.909	0.120	3.017	0.471	0.716	0.089	0.504	1.657
33	2	3	0.1	0.2	0.75	0.802	0.909	0.306	1.191	0.802	0.833	0.404	1	2.437
34	2	3	0.9	0.8	0.1	0.100	0.526	0.001	23208.128	0.118	0.147	0.121	0.138	0.384
35	2	3	0.9	0.8	0.3	0.300	0.526	0.008	1942.750	0.320	0.365	0.424	0.460	1.005
36	2	3	0.9	0.8	0.5	0.500	0.526	0.434	28.541	0.506	0.524	0.924	0.882	1.537

abuse of notations, let $\frac{\partial Q}{\partial e_i^*}$ denote $\frac{\partial Q}{\partial e_i}\big|_{e_i=e_i^*}$, $i=1,2$, and let

$$
\begin{aligned}
E &= -\frac{1}{e_2^*\frac{\partial Q}{\partial e_1^*}},\\
F &= \begin{cases} \frac{2}{e_1^*}P_2p_2e_2^{*2}(PR_r-e_1^*e_2^*)PR_r - P_1p_1e_1^{*2}e_2^*\big[2(PR_r \\ \quad -e_1^*e_2^*)+p_2e_2^*(e_1^*-PR_r)-p_2e_2^*(1-e_1^*)\big], & \text{if } N=1,\\ e_2^{*2}\frac{\partial^2 Q}{\partial e_1^*\partial e_2^*}-\frac{P_2}{P_1}e_2^{*2}\frac{\partial^2 Q}{\partial e_1^{*2}}, & \text{if } N>1,\end{cases}\\
G &= \begin{cases} -\frac{2}{e_2^*}P_1p_1e_1^{*2}(PR_r-e_1^*e_2^*)PR_r + P_2p_2e_1^*e_2^{*2}\big[2(PR_r \\ \quad -e_1^*e_2^*)+p_1e_1^*(e_2^*-PR_r)-p_1e_1^*(1-e_2^*)\big], & \text{if } N=1,\\ 2e_2^*\frac{\partial Q}{\partial e_2^*}+e_2^{*2}\frac{\partial^2 Q}{\partial e_2^{*2}}-2\frac{P_2}{P_1}e_2^*\frac{\partial Q}{\partial e_1^*}-\frac{P_2}{P_1}e_2^{*2}\frac{\partial^2 Q}{\partial e_1^*\partial e_2^*}, & \text{if } N>1,\end{cases}\\
H &= \begin{cases} -P_1p_1e_1^{*2}\big[2(PR_r-e_1^*e_2^*)-p_2e_2^{*2}(1-e_1^*)\big] \\ \quad +P_2p_2e_2^{*2}\big[2(PR_r-e_1^*e_2^*)-p_1e_1^{*2}(1-e_2^*)\big], & \text{if } N=1,\\ 1, & \text{if } N>1.\end{cases}
\end{aligned}
\tag{42}
$$

Then, we have:

Proposition 5.1: For the system defined by model (i)-(vii), the derivative of the optimal solution of (P2) with respect to PR_r is

$$
\begin{aligned}
\frac{de_1^*}{dPR_r} &= \begin{cases} \frac{P_2H-P_1EG}{P_2F-P_1G}, & \text{if } \frac{P_1}{P_2}\in(f_{min},f_{max}),\\ 0, & \text{if } \frac{P_1}{P_2}<f_{min},\\ -\frac{1}{e_2^*\frac{\partial Q}{\partial e_1^*}}, & \text{if } \frac{P_1}{P_2}>f_{max},\end{cases}\\
\frac{de_2^*}{dPR_r} &= \begin{cases} \frac{P_1(EF-H)}{P_2F-P_1G}, & \text{if } \frac{P_1}{P_2}\in(f_{min},f_{max}),\\ -\frac{e_2^*}{e_2^{*2}\frac{\partial Q}{\partial e_2^*}-PR_r}, & \text{if } \frac{P_1}{P_2}<f_{min},\\ 0, & \text{if } \frac{P_1}{P_2}>f_{max},\end{cases}
\end{aligned}
\tag{43}
$$

where the expressions of E, F, G, and H are given in (42).

Proof: See the Appendix.

With the derived $\frac{de_1^*}{dPR_r}$ and $\frac{de_2^*}{dPR_r}$, it is obvious that for $\frac{P_1}{P_2}<f_{min}$ and $\frac{P_1}{P_2}>f_{max}$, both $\frac{de_1^*}{dPR_r}$ and $\frac{de_2^*}{dPR_r}$ are non-negative (noting that $e_2^*\frac{\partial Q}{\partial e_1^*}$ and $e_2^{*2}\frac{\partial Q}{\partial e_2^*}-PR_r$ are negative, see Section III for details). For $\frac{P_1}{P_2}\in(f_{min},f_{max})$, to validate that both $\frac{de_1^*}{dPR_r}$ and $\frac{de_2^*}{dPR_r}$ are positive, 6000 test cases are constructed with system parameters selected randomly and equiprobably from the following sets:

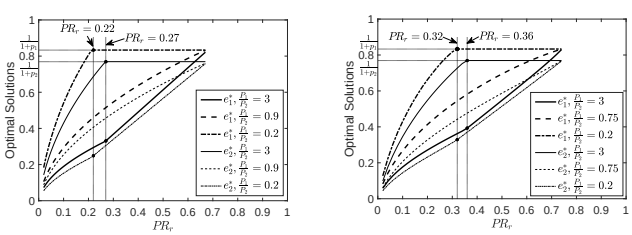
$$
\begin{aligned}
&p_1,p_2\in(0,1),\ PR_r\in(0,PR_{max}],\\
&\frac{P_1}{P_2}\in(f_{min},f_{max}),\ N\in\{1,2,\ldots,10\}.
\end{aligned}
\tag{44}
$$

As a result, we have:

Numerical Fact 5.1: For all constructed 6000 test cases with $\frac{P_1}{P_2} \in (f_{min}, f_{max})$, both $\frac{de_1^*}{dPR_r}$ and $\frac{de_2^*}{dPR_r}$ are always positive.

Based on the analysis above, we conclude that both e_1^* and e_2^* are increasing (not necessarily strictly) in PR_r. To illustrate it, a system with $p_1 = 0.2$ and $p_2 = 0.3$ is exemplified in Fig. 7 for both $N = 1$ and $N = 2$. Note that PR_{max} is 0.6795 and 0.7421 for $N = 1$ and $N = 2$, respectively. As a result, we observe:

- when $\frac{P_1}{P_2} = 3$, for $N = 1$, $\forall PR_r \in (0, 0.27]$ (correspondingly, for $N = 2$, $\forall PR_r \in (0, 0.36]$), $\frac{P_1}{P_2} \in [f_{min}, f_{max}]$ and e_1^* and e_2^* are strictly increasing in PR_r; $\forall PR_r \in (0.27, PR_{max}]$ (correspondingly, for $N = 2$, $\forall PR_r \in (0.36, PR_{max}]$), $\frac{P_1}{P_2} > f_{max}$ and e_1^* is strictly increasing while e_2^* achieves its maximum $\frac{1}{1+p_2}$;
- for $N = 1$, when $\frac{P_1}{P_2} = 0.9$ (correspondingly, for $N = 2$, when $\frac{P_1}{P_2} = 0.75$), since $\forall PR_r \in (0, PR_{max}]$, $\frac{P_1}{P_2} \in [f_{min}, f_{max}]$, e_1^* and e_2^* are strictly increasing in PR_r;
- when $\frac{P_1}{P_2} = 0.2$, for $N = 1$, $\forall PR_r \in (0, 0.22]$ (correspondingly, for $N = 2$, $\forall PR_r \in (0, 0.32]$), $\frac{P_1}{P_2} \in [f_{min}, f_{max}]$ and e_1^* and e_2^* are strictly increasing in PR_r; $\forall PR_r \in (0.22, PR_{max}]$ (correspondingly, for $N = 2$, $\forall PR_r \in (0.32, PR_{max}]$), $\frac{P_1}{P_2} < f_{min}$ and e_2^* is strictly increasing while e_1^* achieves its maximum $\frac{1}{1+p_1}$;
- when $\frac{P_1}{P_2} = 0.2$, the turning points for the optimal solution are $PR_r = 0.22$ ($N = 1$) and $PR_r = 0.32$ ($N = 2$); when $\frac{P_1}{P_2} = 3$, these turning points are $PR_r = 0.27$ ($N = 1$) and $PR_r = 0.36$ ($N = 2$).



(a) $N = 1$ (b) $N = 2$

Fig. 7: The optimal solution, (e_1^*, e_2^*), as a function of PR_r ($p_1 = 0.2$ and $p_2 = 0.3$)

On the basis of the derivative of the optimal solution with respect to PR_r, the sensitivity of the optimal objective value is analyzed. As a result, we have:

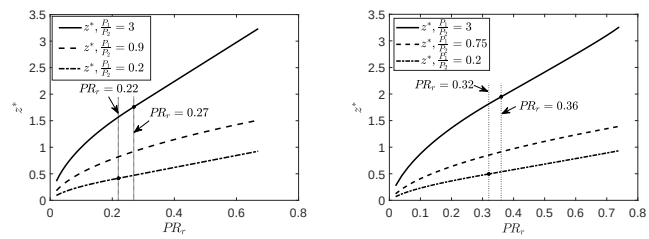
Proposition 5.2: For the system defined by model (i)-(vii), the derivative of the optimal objective value of (P2) with respect to PR_r is

$$\frac{dz^*}{dPR_r} = \begin{cases} -P_1 \frac{1}{e_2^* \frac{\partial Q}{\partial e_1^*}}, & \text{if } \frac{P_1}{P_2} \in [f_{min}, f_{max}], \\ -P_2 \frac{e_2^*}{e_2^{*2} \frac{\partial Q}{\partial e_2^*} - PR_r}, & \text{if } \frac{P_1}{P_2} < f_{min}, \\ -P_1 \frac{1}{e_2^* \frac{\partial Q}{\partial e_1^*}}, & \text{if } \frac{P_1}{P_2} > f_{max}. \end{cases} \tag{45}$$

Proof: Based on Proposition 5.1, it is easy to draw the conclusion. ■

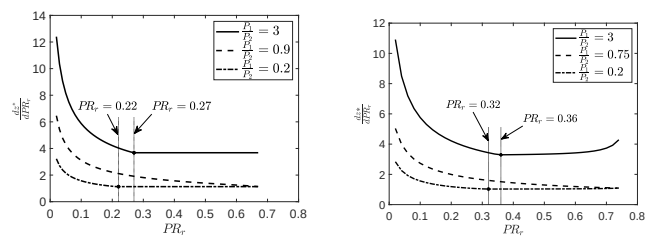
Note that when $\frac{P_1}{P_2} < f_{min}$ or $\frac{P_1}{P_2} > f_{max}$, although $\frac{de_1^*}{dPR_r}$ and $\frac{de_2^*}{dPR_r}$ are discontinuous at some PR_r (see Fig. 7), $\frac{dz^*}{dPR_r}$ is always continuous. This conclusion is in accord with the one for the Bernoulli reliability model.

Since both $\frac{de_1^*}{dPR_r}$ and $\frac{de_2^*}{dPR_r}$ are non-negative and at least one of them is positive for any $PR_r \in (0, PR_{max})$, we have $\frac{dz^*}{dPR_r} > 0$. Furthermore, since the expressions in the right-hand side of (45) are continuous, $\frac{dz^*}{dPR_r}$ is continuous, which implies that z^* is continuously differentiable. Functions z^* and $\frac{dz^*}{dPR_r}$ are shown in Figs. 8 and 9, from which one can also observe that z^* is concave when PR_r is small (i.e., before the turning point) and is slightly convex when PR_r is large (i.e., after the turning point).



(a) $N = 1$ (b) $N = 2$

Fig. 8: The optimal objective value, z^*, as a function of PR_r ($P_2 = 1$, $p_1 = 0.2$, and $p_2 = 0.3$)



(a) $N = 1$ (b) $N = 2$

Fig. 9: The derivative of the optimal objective value, $\frac{dz^*}{dPR_r}$, as a function of PR_r ($P_2 = 1$, $p_1 = 0.2$, and $p_2 = 0.3$)

B. Impacts of Other Parameters

In this subsection, the sensitivity of the optimal objective value, z^*, of (P2) with respect to other parameters (i.e., buffer capacity N, machine power P_i, and breakdown probability p_i, $i = 1, 2$) is analyzed. For simplicity, with a slight abuse of notations, let $\frac{\partial Q}{\partial N}$ and $\frac{\partial Q}{\partial p_i}$, $i = 1, 2$, denote $\frac{\partial Q}{\partial N}\big|_{(e_1,e_2)=(e_1^*,e_2^*)}$ and $\frac{\partial Q}{\partial p_i}\big|_{(e_1,e_2)=(e_1^*,e_2^*)}$, respectively.

First, we study the impact of N on z^* and obtain:

Proposition 5.3: For the system defined by model (i)-(vii) with $N > 1$, the derivative of the optimal objective value of (P2) with respect to the buffer capacity is

$$\frac{dz^*}{dN} = \begin{cases} -P_1 \frac{\frac{\partial Q}{\partial N}}{\frac{\partial Q}{\partial e_1^*}}, & \text{if } \frac{P_1}{P_2} \in [f_{min}, f_{max}], \\ -P_2 \frac{e_2^{*2} \frac{\partial Q}{\partial N}}{e_2^{*2} \frac{\partial Q}{\partial e_2^*} - PR_r}, & \text{if } \frac{P_1}{P_2} < f_{min}, \\ -P_1 \frac{\frac{\partial Q}{\partial N}}{\frac{\partial Q}{\partial e_1^*}} & \text{if } \frac{P_1}{P_2} > f_{max}. \end{cases} \tag{46}$$

Proof: See the Appendix.

To check whether $\frac{dz^*}{dN}$ is negative, 6000 test cases are constructed with system parameters selected randomly and equiprobably from the sets in (37) and the one

$$\frac{P_1}{P_2} \in (0, 10). \tag{47}$$

It is worth noting that $\frac{P_1}{P_2} \in [f_{min}, f_{max}]$, $\frac{P_1}{P_2} < f_{min}$, and $\frac{P_1}{P_2} > f_{max}$ are all included in the constructed test cases. As a result, we have:

Numerical Fact 5.2: For all constructed 6000 test cases with $N > 1$, $\frac{\partial Q}{\partial N}$ is always negative.

Since both $\frac{\partial Q}{\partial e_1^*}$ and $e_2^{*2}\frac{\partial Q}{\partial e_2^*} - PR_r$ are negative, based on Proposition 5.3 and Numerical Fact 5.2, it is concluded that z^* is strictly decreasing in N for $N > 1$, which can be observed in Table I (both e_1^* and e_2^* are decreasing in N and at least one of them is strictly decreasing). Extensive numerical experiments show that even if $N = 1$ is included, z^* is also strictly decreasing in N, which can be observed in Table I as well.

Then, the impact of P_i, $i = 1, 2$, on z^* is considered. Similar to the analysis of the impact of PR_r, the impact on the optimal solution, (e_1^*, e_2^*), is firstly analyzed. As a result, we have:

Proposition 5.4: For the optimal solution, (e_1^*, e_2^*), of (P2), when $\frac{P_1}{P_2} \in [f_{min}, f_{max}]$, e_1^* (correspondingly, e_2^*) is strictly decreasing (correspondingly, increasing) in P_1 and strictly increasing (correspondingly, decreasing) in P_2; when $\frac{P_1}{P_2} \notin [f_{min}, f_{max}]$, as long as the change of P_1 and P_2 is small so that $\frac{P_1}{P_2} < f_{min}$ or $\frac{P_1}{P_2} > f_{max}$ is kept, the optimal solution remains the same.

Proof: See the Appendix.

Based on Proposition 5.4, the impact of P_i, $i = 1, 2$, on the optimal objective value is as follows:

Proposition 5.5: For the system defined by model (i)-(vii), the derivative of the optimal objective value of (P2) with respect to the machine power is

$$\frac{dz^*}{dP_1} = e_1^*, \ \frac{dz^*}{dP_2} = e_2^*. \tag{48}$$

Proof: See the Appendix.

From (48), it follows that $\frac{dz^*}{dP_i} > 0$, $i = 1, 2$, which implies that z^* is strictly increasing in P_1 and P_2, respectively. The conclusion in Proposition 5.5, which is in accord with the one for the Bernoulli model, is hypothesized to hold for general reliability models.

Finally, the sensitivity of z^* with respect to p_i, $i = 1, 2$, is analyzed. As a result, we have:

Proposition 5.6: For the system defined by model (i)-(vii), the derivative of the optimal objective value of (P2) with respect to the machine breakdown probability is

$$\frac{dz^*}{dp_1} = \begin{cases} -P_1 \frac{\frac{\partial Q}{\partial p_1}}{\frac{\partial Q}{\partial e_1^*}}, & \text{if } \frac{P_1}{P_2} \in (f_{min}, f_{max}], \\ -P_1 \frac{1}{(1+p_1)^2} - 2P_2 \frac{e_2^{*2}\frac{\partial Q}{\partial p_1}}{e_2^{*2}\frac{\partial Q}{\partial e_2^*} - PR_r}, & \text{if } \frac{P_1}{P_2} < f_{min}, \\ -P_1 \frac{\frac{\partial Q}{\partial p_1}}{\frac{\partial Q}{\partial e_1^*}}, & \text{if } \frac{P_1}{P_2} > f_{max}, \end{cases}$$

$$\frac{dz^*}{dp_2} = \begin{cases} -P_1 \frac{\frac{\partial Q}{\partial p_2}}{\frac{\partial Q}{\partial e_1^*}}, & \text{if } \frac{P_1}{P_2} \in [f_{min}, f_{max}), \\ -P_2 \frac{e_2^{*2}\frac{\partial Q}{\partial p_2}}{e_2^{*2}\frac{\partial Q}{\partial e_2^*} - PR_r}, & \text{if } \frac{P_1}{P_2} < f_{min}, \\ P_1 \frac{PR_r - 2\frac{\partial Q}{\partial p_2}}{\frac{\partial Q}{\partial e_1^*}} - P_2 \frac{1}{(1+p_2)^2}, & \text{if } \frac{P_1}{P_2} > f_{max}. \end{cases} \tag{49}$$

Proof: See the Appendix.

For $N = 1$, based on the expressions of $\frac{\partial Q}{\partial p_1}$ and $\frac{\partial Q}{\partial p_2}$ in [44], it follows that $\frac{dz^*}{dp_i} < 0$, $i = 1, 2$, which implies that z^* is strictly decreasing in p_1 and p_2, respectively. To investigate the sign of $\frac{dz^*}{dp_i}$ for $N > 1$, we have constructed 6000 test cases with parameters randomly and equiprobably selected from the sets in (37) and (47). As a result, we have:

Numerical Fact 5.3: For all constructed 6000 test cases with $N > 1$, $\frac{dz^*}{dp_i}$, $i = 1, 2$, is always negative.

Based on the above analysis, we draw the conclusion that no matter $N = 1$ or $N > 1$, z^* is strictly decreasing in p_1 and p_2, respectively.

VI. Extensions

Sections II-V analyzed and solved the energy consumption optimization problem in the two-machine geometric serial lines. In this section, some extensions of this problem will be addressed.

A. Problem of Minimizing Energy Consumption per Job

In some production systems, the energy efficiency, i.e., the energy consumption per job (which is called specific energy consumption in some research), rather than the total energy consumption, attracts more attention. To reduce the specific energy consumption, the optimization problem in a two-machine geometric serial line is formulated as follows:

$$\text{(P3) min} \quad y = \frac{1}{PR}\sum_{i=1}^{2} P_i e_i \tag{50}$$

$$\text{s.t.:} \quad PR \geqslant PR_r, \tag{51}$$

$$e_2\big[1 - Q(e_1, e_2, N)\big] = PR, \tag{52}$$

$$0 < e_i \leqslant \frac{1}{1+p_i}, \ i = 1, 2. \tag{53}$$

It is clear that problems (P3) and (P1') are the same except for the objective function. To solve (P3), as in Section II-B, the following problem is introduced:

$$\text{(P4) min} \quad y = \frac{1}{PR}\sum_{i=1}^{2} P_i e_i \tag{54}$$

$$\text{s.t.:} \quad PR = PR_r, \tag{55}$$

$$e_2\big[1-Q(e_1,e_2,N)\big]=PR, \tag{56}$$

$$0<e_i\leqslant\frac{1}{1+p_i},\ i=1,2. \tag{57}$$

Comparing (P4) with (P2), one can see that their optimal solutions are identical and their optimal objective values satisfy $y^*=\frac{z^*}{PR_r}$. Furthermore, we have:

Proposition 6.1: For the system defined by model (i)-(vii), the derivative of the optimal objective value of (P4) with respect to PR_r is

$$\frac{dy^*}{dPR_r}=\begin{cases}-\frac{1}{PR_r^2}\Big(\frac{PR_rP_1}{e_2^*\frac{\partial Q}{\partial e_1^*}}+z^*\Big), & \text{if } \frac{P_1}{P_2}\in[f_{min},f_{max}],\\ -\frac{1}{PR_r^2}\Big(\frac{PR_rP_2e_2^*}{e_2^{*2}\frac{\partial Q}{\partial e_2^*}-PR_r}+z^*\Big), & \text{if } \frac{P_1}{P_2}<f_{min},\\ -\frac{1}{PR_r^2}\Big(\frac{PR_rP_1}{e_2^*\frac{\partial Q}{\partial e_1^*}}+z^*\Big), & \text{if } \frac{P_1}{P_2}>f_{max},\end{cases} \tag{58}$$

where $z^*=P_1e_1^*+P_2e_2^*$.

Proof: Clearly, the derivative of y^* with respect to PR_r is

$$\frac{dy^*}{dPR_r}=\frac{1}{PR_r^2}\Big(PR_r\frac{dz^*}{dPR_r}-z^*\Big). \tag{59}$$

According to (45), it is easy to draw the conclusion. ■

Now we focus on the sign of $\frac{dy^*}{dPR_r}$. For $N=1$, we obtain:

Proposition 6.2: For $N=1$, the derivative of the optimal objective value of (P4) with respect to PR_r, i.e., $\frac{dy^*}{dPR_r}$, is negative.

Proof: See the Appendix.

As for $N>1$, 5000 test cases, with system parameters selected from (37) and (47), are constructed to justify $\frac{dy^*}{dPR_r}<0$. As a result, we have:

Numerical Fact 6.1: For all constructed 5000 test cases with $N>1$, $\frac{dy^*}{dPR_r}<0$ always holds.

Based on Proposition 6.2 and Numerical Fact 6.1, we conclude that the optimal objective value, y^*, of (P4), is strictly decreasing in PR_r, which implies that the objective value of (P3) is decreasing in PR. That is to say, (P3) reaches its optimal objective value if and only if PR achieves its maximum value, PR_{max}, and hence its optimal solution is $(\frac{1}{1+p_1},\frac{1}{1+p_2})$.

B. Problem with Machine Power in Operational and Idle States Distinguished

When a machine is up, it could be either operational (i.e., processing jobs) or idle (i.e., starved or blocked). Considering that in general, the power consumed by a machine in operational state is greater than in idle, it is more practical to distinguish the power of a machine in these states. Denote the power of machine m_i, $i=1,2$, in operational and idle states as P_i^{opr} and P_i^{idl}, respectively. Taking into account that for m_i, the probabilities in operational, idle, and down states are PR, e_i-PR, and $1-e_i$, respectively, the problem of minimizing the total energy consumption of the two-machine geometric line is re-formulated as follows:

$$\text{(P5) min}\quad z=\sum_{i=1}^{2}\big[P_i^{opr}PR+P_i^{idl}(e_i-PR)\big] \tag{60}$$

$$\text{s.t.:}\quad PR\geqslant PR_r, \tag{61}$$

$$e_2\big[1-Q(e_1,e_2,N)\big]=PR, \tag{62}$$

$$0<e_i\leqslant\frac{1}{1+p_i},\ i=1,2. \tag{63}$$

The objective function of (P5) can be separated into two parts, viz.,

$$z_1=\sum_{i=1}^{2}P_i^{idl}e_i,\ z_2=\sum_{i=1}^{2}(P_i^{opr}-P_i^{idl})PR. \tag{64}$$

To figure out the impact of PR on the objective value of (P5), the monotonicity of z_1 and z_2 is analyzed. Based on Theorem 2.1 and Corollary 2.1, and taking into account that (P5) with z replaced by z_1 and (P1') have the same form, z_1 reaches its minimum value when $PR=PR_r$. Meanwhile, it is obvious that z_2 reaches its minimum value when $PR=PR_r$ as well. As a result, z achieves its minimum when the equality holds in (61). Given that $z_2=\sum_{i=1}^{2}(P_i^{opr}-P_i^{idl})PR_r$ is a constant, (P5) is equivalently converted into (P6):

$$\text{(P6) min}\quad z=\sum_{i=1}^{2}P_i^{idl}e_i \tag{65}$$

$$\text{s.t.:}\quad e_2\big[1-Q(e_1,e_2,N)\big]=PR_r, \tag{66}$$

$$0<e_i\leqslant\frac{1}{1+p_i},\ i=1,2. \tag{67}$$

Since (P6) is formally the same as (P2), it can be solved by the method developed in Section IV-B.

Then, under the same assumptions, the problem of minimizing the specific energy consumption is re-formulated as

$$\text{(P7) min}\quad y=\frac{1}{PR}\sum_{i=1}^{2}\big[P_i^{opr}PR+P_i^{idl}(e_i-PR)\big] \tag{68}$$

$$\text{s.t.:}\quad PR\geqslant PR_r, \tag{69}$$

$$e_2\big[1-Q(e_1,e_2,N)\big]=PR, \tag{70}$$

$$0<e_i\leqslant\frac{1}{1+p_i},\ i=1,2. \tag{71}$$

The objective function of (P7) can be re-written as $y=\frac{1}{PR}\sum_{i=1}^{2}P_i^{idl}e_i+\sum_{i=1}^{2}(P_i^{opr}-P_i^{idl})$, where the second term in the right-hand side is a constant. Then, (P7) is equivalently converted into (P8):

$$\text{(P8) min}\quad y=\frac{1}{PR}\sum_{i=1}^{2}P_i^{idl}e_i \tag{72}$$

$$\text{s.t.:}\quad PR\geqslant PR_r, \tag{73}$$

$$e_2\big[1-Q(e_1,e_2,N)\big]=PR, \tag{74}$$

$$0<e_i\leqslant\frac{1}{1+p_i},\ i=1,2. \tag{75}$$

Clearly, (P8) is formally the same as (P3) and thus, its optimal solution is $(\frac{1}{1+p_1},\frac{1}{1+p_2})$ and its optimal production rate is PR_{max}.

C. Problem with General Bounds on Machine Efficiency

In practice, due to physical limitations, the efficiency of machine m_i, $i = 1, 2$, is usually confined to a closed set, $[\underline{e}_i, \overline{e}_i]$, which is a subset of $(0, \frac{1}{1+p_i}]$. In this case, for the system defined by model (i)-(vii), the total energy consumption optimization problem is re-formulated as follows:

$$\text{(P9) min} \quad z = \sum_{i=1}^{2} P_i e_i \tag{76}$$

$$\text{s.t.:} \quad e_2\big[1 - Q(e_1, e_2, N)\big] \geqslant PR_r, \tag{77}$$

$$\underline{e}_i \leqslant e_i \leqslant \overline{e}_i, \ i = 1, 2. \tag{78}$$

Problem (P9) has the same form as the one for two-machine Bernoulli serial lines in [34], where by introducing the following problem and solving it, the optimal solution of the former is obtained.

$$\text{(P10) min} \quad z = \sum_{i=1}^{2} P_i e_i \tag{79}$$

$$\text{s.t.:} \quad e_2\big[1 - Q(e_1, e_2, N)\big] = PR_r, \tag{80}$$

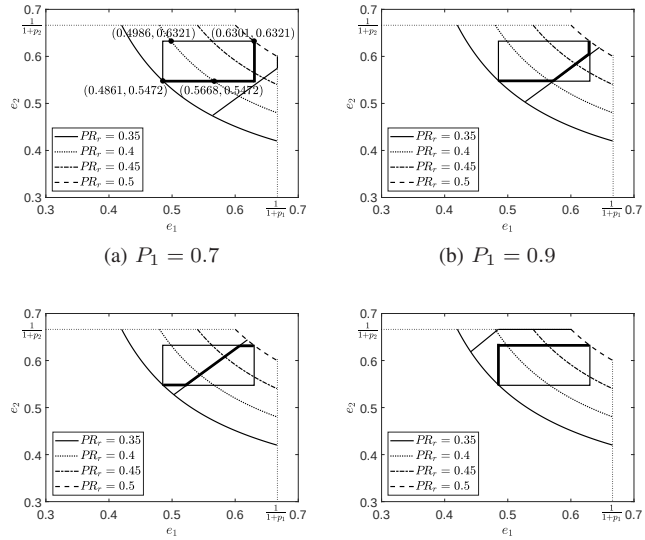
$$\underline{e}_i \leqslant e_i \leqslant \overline{e}_i, \ i = 1, 2. \tag{81}$$

Note that (P10) is the same as (P2) except for constraint (81).

Adopting the approach developed for the energy consumption optimization for the Bernoulli serial line in [34], (P10) can be solved. Specifically, by investigating the relationship between the optimal solution of (P2) and the feasible region of (P10) and taking advantage of the monotonicity of the objective value, the optimal solution of (P10) can be constructed. For this purpose, we first figure out the feasible region of (P10).

From Fig. 10, one can see that constraint (81) characterizes a rectangle area. Since PR is strictly increasing in e_1 and e_2, respectively, it is easy to conclude that for any given N, in the rectangle area, the smallest and largest production rates are, respectively, achieved at the lower-left and the upper-right vertexes of the rectangle (e.g., $(0.4861, 0.5472)$ and $(0.6301, 0.6321)$ for systems with $N = 1$ in Fig. 10). Let $\underline{PR}$ and $\overline{PR}$ denote the smallest and largest production rates, respectively. For example, for systems in Fig. 10, $\underline{PR} = 0.35$ and $\overline{PR} = 0.5$. Assume that $\underline{PR} \leqslant PR_{max}$, otherwise (P10) has no feasible solutions. Clearly, if $PR_r < \underline{PR}$ or $PR_r > \min(\overline{PR}, PR_{max})$, the feasible region is an empty set and thus, (P10) has no feasible solutions; if $\underline{PR} < PR_r < \min(\overline{PR}, PR_{max})$, the feasible region is the intersection of the contour of the production rate (80) and the rectangle area characterized by (81) (e.g., in Fig. 10(a), the feasible region of the problem for $PR_r = 0.4$ is the curve segment inside the rectangle area with endpoints $(0.4986, 0.6321)$ and $(0.5668, 0.5472)$); if $PR_r = \underline{PR}$, the feasible region has only one point, i.e., the lower-left vertex of the rectangle; if $PR_r = \min(\overline{PR}, PR_{max})$, the feasible region has a unique point as well, which is either the upper-right vertex of the rectangle or $(e_1, e_2) = (\frac{1}{1+p_1}, \frac{1}{1+p_2})$, depending on which of $\overline{PR}$ and PR_{max} is smaller.

For the case $\underline{PR} < PR_r < \min(\overline{PR}, PR_{max})$, the feasible range of e_1 corresponding to $\max(\underline{e}_2, e_{2,min}) \leqslant e_2 \leqslant \min(\overline{e}_2, e_{2,max})$ can be identified, where $e_{2,min}$ and $e_{2,max}$ are given in (19) and (23), respectively. Let this range be $[\hat{e}_{1,min}, \hat{e}_{1,max}]$, and



(a) $P_1 = 0.7$ (b) $P_1 = 0.9$

(c) $P_1 = 1.1$ (d) $P_1 = 2$

Fig. 10: Locus of the optimal solution of (P10) for $PR_r \in [\underline{PR}, \min(\overline{PR}, PR_{max})]$ ($p_1 = p_2 = 0.5$, $N = 1$, $P_2 = 1$, $PR_{max} = 0.5556$)

$$\begin{aligned} \hat{\underline{e}}_1 &= \max(e_{1,min}, \underline{e}_1, \hat{e}_{1,min}), \\ \hat{\overline{e}}_1 &= \min(e_{1,max}, \overline{e}_1, \hat{e}_{1,max}). \end{aligned} \tag{82}$$

Then the feasible range of e_1 is

$$\hat{\underline{e}}_1 \leqslant e_1 \leqslant \hat{\overline{e}}_1 \tag{83}$$

and correspondingly, the feasible range of e_2 is

$$\hat{\underline{e}}_2 \leqslant e_2 \leqslant \hat{\overline{e}}_2. \tag{84}$$

Thus, the endpoints of the feasible region of (P10) are $(\hat{\underline{e}}_1, \hat{\overline{e}}_2)$ and $(\hat{\overline{e}}_1, \hat{\underline{e}}_2)$.

Based on the above analysis, the optimal solution of (P10) can be constructed. Let $(\tilde{e}_1^*, \tilde{e}_2^*)$ denote this optimal solution, then we have:

Theorem 6.1: Assume $\underline{PR} \leqslant PR_r \leqslant \min(\overline{PR}, PR_{max})$, then (P10) has a unique optimal solution. If the optimal solution, (e_1^*, e_2^*), of (P2), is a feasible solution of (P10), i.e., $\hat{\underline{e}}_1 \leqslant e_1^* \leqslant \hat{\overline{e}}_1$ (in this case, $\hat{\overline{e}}_2 \geqslant e_2^* \geqslant \hat{\underline{e}}_2$), then

$$(\tilde{e}_1^*, \tilde{e}_2^*) = (e_1^*, e_2^*); \tag{85}$$

if $e_1^* < \hat{\underline{e}}_1$, then

$$(\tilde{e}_1^*, \tilde{e}_2^*) = (\hat{\underline{e}}_1, \hat{\overline{e}}_2); \tag{86}$$

if $e_1^* > \hat{\overline{e}}_1$, then

$$(\tilde{e}_1^*, \tilde{e}_2^*) = (\hat{\overline{e}}_1, \hat{\underline{e}}_2). \tag{87}$$

Proof: Based on the monotonicity of the objective value of (P10) with respect to e_1, it is easy to draw the conclusion. ■

From Theorem 6.1, it is easy to conclude that as shown in Fig. 10, as PR_r increases, the optimal solution of (P10), starting from the lower-left vertex of the rectangle, moves along the piecewise bold lines, which is in accord with the conclusion for two-machine Bernoulli serial lines in [34].

On the basis of Theorem 6.1, the property of the optimal objective value of (P10) is analyzed in the following.

Theorem 6.2: Assume $\underline{PR} \leqslant PR_r \leqslant \min(\overline{PR}, PR_{max})$. The optimal objective value, $\tilde{z}^*$, of (P10), is strictly increasing in PR_r.

Proof: See the Appendix.

Based on Theorems 6.1 and 6.2, the optimal solution of (P9) is constructed. As a result, we have:

Corollary 6.1: Assume $PR_r \leqslant \min(\overline{PR}, PR_{max})$. Then the optimal solution of (P9) is the optimal solution of (P10) with required production rate $\max(\underline{PR}, PR_r)$.

Since it is easy to draw this conclusion from Theorems 6.1 and 6.2, the proof is omitted.

VII. Conclusions and Future Work

In this paper, the energy consumption optimization problem, which minimizes the total or specific energy consumption while maintaining a required production rate, is formulated and solved for the two-machine geometric serial lines. Similar for the Bernoulli lines, we establish two optimality equations and analyze their mathematical properties, based on which an effective and efficient algorithm is developed to solve the energy consumption optimization problem. Moreover, the sensitivity analysis of the optimal solution with system parameters is conducted. The results show that the energy consumption optimization problem for the geometric and Bernoulli lines has some common attributes, which might exist for general reliability models. Finally, some variations of the energy consumption optimization problem are addressed as well.

In the current paper, the repair resources are assumed to be sufficient and their costs are ignored. In the future, the cost of the repair resources will be considered in the problem to balance the resource cost and energy cost. Besides, the energy consumption optimization problem will also be studied in long geometric serial lines and assembly systems. The results obtained for the geometric model will be extended to exponential and non-Markovian, e.g., Weibull, gamma, and log-normal, models. In addition, the energy consumption optimization problem will be investigated for more practical and complex production lines, e.g., for time sensitive lines, for lines with quality inspection, multiple products, and batch processing, etc.

Appendix

Proofs of Theorems

A. Proof of Theorem 2.1

Proof: From (1), one can see that PR is a continuous function of both e_1 and e_2 since the Q-function in (6) is continuous. Therefore, for any $PR_r \in (0, PR_{max})$, (P2) always has at least one feasible solution and thus, it has the optimal solution.

To prove the theorem, we choose PR_{r1} and PR_{r2} such that $0 < PR_{r1} < PR_{r2} < PR_{max}$, and let (P2') and (P2") denote (P2) with PR_r replaced by PR_{r1} and by PR_{r2}, respectively. In addition, denote the optimal solutions of (P2') and (P2") as $(e^*_{1,r1}, e^*_{2,r1})$ and $(e^*_{1,r2}, e^*_{2,r2})$, respectively, and their corresponding optimal values as z^*_{r1} and z^*_{r2}. Construct a solution $(e^*_{1,r2}, \hat{e}_{2,r1})$ of (P2') which satisfies $PR_{r1} = \hat{e}_{2,r1}\big[1 - Q(e^*_{1,r2}, \hat{e}_{2,r1}, N)\big]$. Considering that the production rate of $(e^*_{1,r2}, e^*_{2,r2})$ is PR_{r2} and taking into account the monotonicity of PR with respect to e_2, we have

$$0 < \hat{e}_{2,r1} < e^*_{2,r2} \leqslant \frac{1}{1+p_2}. \tag{88}$$

Clearly, (88) indicates that $(e^*_{1,r2}, \hat{e}_{2,r1})$ is a feasible solution of (P2'). Thus, for the optimal solution $(e^*_{1,r1}, e^*_{2,r1})$ and the feasible solution $(e^*_{1,r2}, \hat{e}_{2,r1})$ of (P2'), and the optimal solution $(e^*_{1,r2}, e^*_{2,r2})$ of (P2"), taking into account (88), we have

$$z^*_{r1} = \sum_{i=1}^{2} P_i e^*_{i,r1} \leqslant P_1 e^*_{1,r2} + P_2 \hat{e}_{2,r1} < \sum_{i=1}^{2} P_i e^*_{i,r2} = z^*_{r2}, \tag{89}$$

which completes the proof. ■

B. Proof of Proposition 4.1

Proof: For $N = 1$, using $f'(e_1) = \frac{u'v - uv'}{v^2}$, it is easy to derive the expression on the right-hand side of (34).

As for $N > 1$, we have

$$\begin{aligned}
f'(e_1) =& \frac{2e_2e_2'\frac{\partial Q}{\partial e_1} + e_2^2\big(\frac{\partial^2 Q}{\partial e_1^2} + \frac{\partial^2 Q}{\partial e_1 \partial e_2}e_2'\big)}{e_2^2\frac{\partial Q}{\partial e_2} - PR_r} - \frac{e_2^2\frac{\partial Q}{\partial e_1}}{\big(e_2^2\frac{\partial Q}{\partial e_2} - PR_r\big)^2} \\
&\cdot \left[2e_2e_2'\frac{\partial Q}{\partial e_2} + e_2^2\Big(\frac{\partial^2 Q}{\partial e_2 \partial e_1} + \frac{\partial^2 Q}{\partial e_2^2}e_2'\Big)\right] \\
=& \frac{e_2e_2'}{(e_2^2\frac{\partial Q}{\partial e_2} - PR_r)^2}\left[\Big(2\frac{\partial Q}{\partial e_1} + e_2\frac{\partial^2 Q}{\partial e_1 \partial e_2}\Big)\right. \\
&\left.\cdot\Big(e_2^2\frac{\partial Q}{\partial e_2} - PR_r\Big) - e_2^2\frac{\partial Q}{\partial e_1}\Big(2\frac{\partial Q}{\partial e_2} + e_2\frac{\partial^2 Q}{\partial e_2^2}\Big)\right] \\
&+ \frac{e_2^2\frac{\partial^2 Q}{\partial e_1^2}}{e_2^2\frac{\partial Q}{\partial e_2} - PR_r} - \frac{e_2^4\frac{\partial Q}{\partial e_1}\frac{\partial^2 Q}{\partial e_1 \partial e_2}}{\big(e_2^2\frac{\partial Q}{\partial e_2} - PR_r\big)^2} \\
=& -\frac{e_2^3\frac{\partial Q}{\partial e_1}}{(e_2^2\frac{\partial Q}{\partial e_2} - PR_r)^3}\left[e_2^3\frac{\partial Q}{\partial e_2}\frac{\partial^2 Q}{\partial e_1 \partial e_2}\right. \\
&\left. - PR_r\Big(2\frac{\partial Q}{\partial e_1} + e_2\frac{\partial^2 Q}{\partial e_1 \partial e_2}\Big) - e_2^3\frac{\partial Q}{\partial e_1}\frac{\partial^2 Q}{\partial e_2^2}\right] \\
&+ \frac{e_2^2\frac{\partial^2 Q}{\partial e_1^2}}{e_2^2\frac{\partial Q}{\partial e_2} - PR_r} - \frac{e_2^4\frac{\partial Q}{\partial e_1}\frac{\partial^2 Q}{\partial e_1 \partial e_2}}{\big(e_2^2\frac{\partial Q}{\partial e_2} - PR_r\big)^2} \\
=& \frac{e_2^2\frac{\partial^2 Q}{\partial e_1^2}}{e_2^2\frac{\partial Q}{\partial e_2} - PR_r} - \frac{e_2^3\frac{\partial Q}{\partial e_1}}{(e_2^2\frac{\partial Q}{\partial e_2} - PR_r)^3}\left[2e_2^3\frac{\partial Q}{\partial e_2}\frac{\partial^2 Q}{\partial e_1 \partial e_2}\right. \\
&\left. - 2PR_r\Big(\frac{\partial Q}{\partial e_1} + e_2\frac{\partial^2 Q}{\partial e_1 \partial e_2}\Big) - e_2^3\frac{\partial Q}{\partial e_1}\frac{\partial^2 Q}{\partial e_2^2}\right],
\end{aligned} \tag{90}$$

which completes the proof. ■

C. Proof of Lemma 4.1

Proof: To prove it, the derivative of $g(e_1)$ with respect to e_1 is derived. From (36), it follows

$$\begin{aligned} g'(e_1) =& \frac{e_2 - PR_r}{(1-e_1)^2} + \frac{e_2'}{1-e_1} \\ &- \frac{(PR_r - e_1e_2)\big[2PR_r(1-e_1) - e_1(PR_r - e_1e_2)\big]}{p_1e_1^3(1-e_1)^2(1-e_2)} \\ &- \frac{e_1(2e_1 - e_1e_2 - PR_r)(PR_r - e_1e_2)e_2'}{p_1e_1^3(1-e_1)(1-e_2)^2}. \end{aligned} \tag{91}$$

Let

$$\begin{aligned} V_g =& p_1^2e_1^4(1-e_1)^2(1-e_2)^2 \\ &\cdot \big[p_2e_2^2(1-e_1)(e_1 - PR_r) + (PR_r - e_1e_2)^2\big], \\ U_g =& V_g g'(e_1). \end{aligned} \tag{92}$$

Taking into account $e_2' = -f(e_1)$ (see equation (32)) and re-arranging the items, we have

$$\begin{aligned} U_g =& p_1^2p_2e_1^4e_2^2(1-e_1)(1-e_2)^2(e_1 - PR_r)(e_2 - PR_r) \\ &- \big[2p_1p_2e_1e_2^2(1-e_1)^2(1-e_2)(e_1 - PR_r)(PR_r - e_1e_2)\big]PR_r \\ &- \big[2p_1e_1(1-e_1)(1-e_2)(PR_r - e_1e_2)^3\big]PR_r \\ &+ p_1p_2e_1^2e_2^2(1-e_1)(1-e_2)(e_1 - PR_r)(PR_r - e_1e_2)^2 \\ &+ p_1e_1^2(1-e_2)(PR_r - e_1e_2)^4 \\ &- p_1^2p_2e_1^4e_2^2(1-e_1)(1-e_2)^3(e_2 - PR_r) \\ &+ p_1p_2e_1^3e_2^2(1-e_1)(1-e_2)^2(e_2 - PR_r)(PR_r - e_1e_2) \\ &+ p_1p_2e_1^2e_2^2(1-e_1)(1-e_2)(e_1 - PR_r)(e_2 - PR_r)(PR_r - e_1e_2) \\ &- p_1p_2e_1^2e_2^2(1-e_1)(1-e_2)^2(PR_r - e_1e_2)^2 \\ &+ p_1e_1^3(1-e_2)(e_2 - PR_r)(PR_r - e_1e_2)\big[p_1e_1(1-e_2)(PR_r - e_1e_2)\big] \\ &+ e_1e_2(1-e_2)(PR_r - e_1e_2)^2\big[p_2e_2(1-e_1)(PR_r - e_1e_2)\big] \\ &+ e_2(e_1 - PR_r)(PR_r - e_1e_2)^2\big[p_2e_2(1-e_1)(PR_r - e_1e_2)\big]. \end{aligned} \tag{93}$$

Clearly, V_g is positive. To prove the lemma, we only need to prove U_g is negative.

For this purpose, the expressions in the brackets of the last three items in (93) are replaced by

$$\begin{aligned} & p_1e_1(1-e_2)(PR_r - e_1e_2) \\ =& p_1p_2e_1e_2(1-e_1)(1-e_2) - p_2e_2(1-e_1)(PR_r - e_1e_2) \\ &+ p_1p_2e_1e_2(PR_r - e_1e_2) \end{aligned} \tag{94}$$

and

$$\begin{aligned} & p_2e_2(1-e_1)(PR_r - e_1e_2) \\ =& p_1p_2e_1e_2(1-e_1)(1-e_2) - p_1e_1(1-e_2)(PR_r - e_1e_2) \\ &+ p_1p_2e_1e_2(PR_r - e_1e_2), \end{aligned} \tag{95}$$

respectively, which are derived from (8) with $PR = PR_r$. In addition, the second and third terms of (93) are, respectively, split into two terms by replacing PR_r outside the brackets by $(PR_r - e_1e_2) + e_1e_2$. As a result, we have

$$\begin{aligned} U_g =& \big[-p_1e_1e_2(1-e_2)(e_1 - PR_r)(PR_r - e_1e_2)^3 \\ &+ p_1p_2e_1e_2^2(e_1 - PR_r)(PR_r - e_1e_2)^3\big] \\ &+ \big[-p_1p_2e_1^3e_2(1-e_1)(1-e_2)(e_2 - PR_r)(PR_r - e_1e_2)^2 \\ &+ p_1^2p_2e_1^4e_2(1-e_2)(e_2 - PR_r)(PR_r - e_1e_2)^2\big] \\ &- 2p_1e_1(1-e_1)(1-e_2)(PR_r - e_1e_2)^4 \\ &+ \big[-p_1e_1^2e_2(1-e_2)^2(PR_r - e_1e_2)^3 \\ &+ p_1p_2e_1^2e_2^2(1-e_2)(PR_r - e_1e_2)^3\big] \\ &+ \Big\{\big[-2p_1e_1^2e_2(1-e_1)(1-e_2)(PR_r - e_1e_2)^3 \\ &+ p_1e_1^2(1-e_2)(PR_r - e_1e_2)^4\big] \\ &+ \big[-2p_1p_2e_1e_2^2(1-e_1)^2(1-e_2)(e_1 - PR_r)(PR_r - e_1e_2)^2 \\ &- 2p_1p_2e_1^2e_2^3(1-e_1)^2(1-e_2)(e_1 - PR_r)(PR_r - e_1e_2) \\ &+ p_1p_2e_1^2e_2^2(1-e_1)(1-e_2)(e_1 - PR_r)(PR_r - e_1e_2)^2 \\ &+ p_1p_2e_1^3e_2^2(1-e_1)(1-e_2)^2(e_2 - PR_r)(PR_r - e_1e_2) \\ &+ p_1p_2e_1^2e_2^2(1-e_1)(1-e_2)(e_1 - PR_r)(e_2 - PR_r)(PR_r - e_1e_2) \\ &+ p_1p_2e_1e_2^2(1-e_1)(1-e_2)(e_1 - PR_r)(PR_r - e_1e_2)^2\big]\Big\} \\ &+ \big[p_1^2p_2e_1^4e_2^2(1-e_1)(1-e_2)^2(e_1 - PR_r)(e_2 - PR_r) \\ &+ p_1^2p_2e_1^4e_2(1-e_1)(1-e_2)^2(e_2 - PR_r)(PR_r - e_1e_2) \\ &- p_1^2p_2e_1^4e_2^2(1-e_1)(1-e_2)^3(e_2 - PR_r)\big] \\ =& -p_1e_1e_2(e_1 - PR_r)(PR_r - e_1e_2)^3(1 - e_2 - p_2e_2) \\ &- p_1p_2e_1^3e_2(1-e_2)(e_2 - PR_r)(PR_r - e_1e_2)^2(1 - e_1 - p_1e_1) \\ &- 2p_1e_1(1-e_1)(1-e_2)(PR_r - e_1e_2)^4 \\ &- p_1e_1^2e_2(1-e_2)(PR_r - e_1e_2)^3(1 - e_2 - p_2e_2) \\ &- \Big\{\big[p_1e_1^2e_2(1-e_1)(1-e_2)(PR_r - e_1e_2)^3 \\ &+ p_1e_1^2(1-e_2)(e_2 - PR_r)(PR_r - e_1e_2)^3\big] \\ &+ \big[p_1p_2e_1e_2^2(1-e_1)^2(1-e_2)(e_1 - PR_r)(PR_r - e_1e_2)^2 \\ &- p_1p_2e_1^2e_2(1-e_2)(e_2 - PR_r)(PR_r - e_1e_2)^3 \\ &- p_1p_2e_1^2e_2(1-e_2)(e_2 - PR_r)^2(PR_r - e_1e_2)^2\big]\Big\} \\ &- p_1^2p_2e_1^4e_2(1-e_1)(1-e_2)^3(e_2 - PR_r)^2 \\ =& -p_1e_1e_2(e_1 - PR_r)(PR_r - e_1e_2)^3(1 - e_2 - p_2e_2) \\ &- p_1p_2e_1^3e_2(1-e_2)(e_2 - PR_r)(PR_r - e_1e_2)^2(1 - e_1 - p_1e_1) \\ &- 2p_1e_1(1-e_1)(1-e_2)(PR_r - e_1e_2)^4 \\ &- p_1e_1^2e_2(1-e_2)(PR_r - e_1e_2)^3(1 - e_2 - p_2e_2) \\ &- p_1e_1^2e_2(1-e_1)(1-e_2)(PR_r - e_1e_2)^3 \\ &- p_1p_2e_1e_2^2(1-e_1)^2(1-e_2)(e_1 - PR_r)(PR_r - e_1e_2)^2 \\ &- p_1e_1^2(1-e_2)(e_2 - PR_r)(PR_r - e_1e_2)^3(1 - e_2 - p_2e_2) \\ &+ p_1e_1^2(1-e_2)(e_2 - PR_r)\big[p_2e_2(e_2 - PR_r)(PR_r - e_1e_2)^2 \\ &- p_1p_2e_1^2e_2(1-e_1)(1-e_2)^2(e_2 - PR_r) - e_2(PR_r - e_1e_2)^3\big]. \end{aligned} \tag{96}$$

Since $PR_r < e_i \leqslant \frac{1}{1+p_i}$, $i = 1, 2$, and $PR_r - e_1e_2 > 0$ (following from (8)), it is easy to check that all items except the last one are negative. To validate $U_g < 0$, let $U_{g,s}$ denote the expression in the bracket of the last item of (96). Re-write

$$\begin{aligned} U_{g,s} =& p_2e_2(e_2 - PR_r)(PR_r - e_1e_2)^2 \\ &- p_1p_2e_1e_2(1-e_1)(1-e_2)(e_1 - PR_r)(e_2 - PR_r) \\ &- \big[p_1p_2e_1e_2(1-e_1)(1-e_2)\big](e_2 - PR_r)(PR_r - e_1e_2) \\ &- e_2(PR_r - e_1e_2)^3 \end{aligned} \tag{97}$$

and replace the expression in the bracket of the third item of the above equation by

$$\begin{aligned} & p_1p_2e_1e_2(1-e_1)(1-e_2) \\ =& \big[p_1e_1(1-e_2) + p_2e_2(1-e_1) - p_1p_2e_1e_2\big](PR_r - e_1e_2), \end{aligned}$$

(98)

we have

$$\begin{aligned}
U_{g,s} =& p_2e_2(e_2-PR_r)(PR_r-e_1e_2)^2 \\
&- p_1p_2e_1e_2(1-e_1)(1-e_2)(e_1-PR_r)(e_2-PR_r) \\
&- p_1e_1(1-e_2)(e_2-PR_r)(PR_r-e_1e_2)^2 \\
&- p_2e_2(1-e_1)(e_2-PR_r)(PR_r-e_1e_2)^2 \\
&+ p_1p_2e_1e_2(e_2-PR_r)(PR_r-e_1e_2)^2 - e_2(PR_r-e_1e_2)^3 \\
=& - e_1(e_2-PR_r)(PR_r-e_1e_2)^2(1-e_2-p_2e_2) \\
&+ e_1(1-e_2)(e_2-PR_r)(PR_r-e_1e_2)^2 \\
&- p_1p_2e_1e_2(1-e_1)(1-e_2)(e_1-PR_r)(e_2-PR_r) \\
&- p_1e_1(e_2-PR_r)(PR_r-e_1e_2)^2(1-e_2-p_2e_2) \\
&- e_2(PR_r-e_1e_2)^3 \\
=& - e_1(e_2-PR_r)(PR_r-e_1e_2)^2(1-e_2-p_2e_2) \\
&+ (e_1-PR_r)(e_2-PR_r)\big[(PR_r-e_1e_2)^2 \\
&- p_1p_2e_1e_2(1-e_1)(1-e_2)\big] - PR_r(PR_r-e_1e_2)^3 \\
&- p_1e_1(e_2-PR_r)(PR_r-e_1e_2)^2(1-e_2-p_2e_2).
\end{aligned} \tag{99}$$

For the expression in the bracket of the second term of the right-hand side of (99), it can be re-written as

$$\begin{aligned}
&\frac{(PR_r-e_1e_2)}{p_1e_1(1-e_2)+p_2e_2(1-e_1)-p_1p_2e_1e_2} \\
&\cdot\Big\{p_1p_2e_1e_2(1-e_1)(1-e_2) \\
&-\big[p_1e_1(1-e_2)+p_2e_2(1-e_1)-p_1p_2e_1e_2\big]^2\Big\} \\
=&\frac{(PR_r-e_1e_2)}{p_1e_1(1-e_2)+p_2e_2(1-e_1)-p_1p_2e_1e_2} \\
&\cdot\Big\{-p_1^2e_1^2(1-e_2)(1-e_2-p_2e_2)-p_2e_2(1-e_1-p_1e_1) \\
&\cdot\big[p_1e_1(1-e_2)+p_2e_2(1-e_1)-p_1p_2e_1e_2\big]\Big\},
\end{aligned} \tag{100}$$

which is clearly negative. Considering that the other terms of the right-hand side of (99) are all negative, we conclude $U_{g,s}$ is negative, which implies U_g is negative and completes the proof. ■

D. Proof of Lemma 4.2

Proof: To prove it, the derivative of $h(e_1)$ is derived. From (36), it follows

$$\begin{aligned}
h'(e_1) =& \frac{1}{1-e_2}+\frac{(e_1-PR_r)e_2'}{(1-e_2)^2} \\
&+\frac{(PR_r-e_1e_2)\big[-2e_2(1-e_1)+(PR_r-e_1e_2)\big]}{p_2e_2^2(1-e_1)^2(1-e_2)} \\
&+\frac{(PR_r-e_1e_2)\big[e_2(PR_r-e_1e_2)-2(1-e_2)PR_r\big]e_2'}{p_2e_2^3(1-e_1)(1-e_2)^2}.
\end{aligned} \tag{101}$$

Let

$$\begin{aligned}
V_h =& p_1p_2e_1^2e_2^2(1-e_1)^2(1-e_2)^2 \\
&\cdot\big[p_2e_2^2(1-e_1)(e_1-PR_r)+(PR_r-e_1e_2)^2\big], \quad (102) \\
U_h =& V_h h'(e_1).
\end{aligned}$$

Taking into account $e_2' = -f(e_1)$ and re-arranging the items, we have

$$\begin{aligned}
U_h =& p_1p_2e_1^2e_2^2(1-e_1)^2(1-e_2)(PR_r-e_1e_2)^2 \\
&+ p_1p_2^2e_1^2e_2^4(1-e_1)^3(1-e_2)(e_1-PR_r) \\
&- 2p_1p_2e_1^2e_2^3(1-e_1)^2(1-e_2)(e_1-PR_r)(PR_r-e_1e_2) \\
&+ p_1p_2e_1^2e_2^2(1-e_1)(1-e_2)(e_1-PR_r)(PR_r-e_1e_2)^2 \\
&+ 2p_2e_1e_2^2(1-e_1)(1-e_2)(PR_r-e_1e_2)^3 \\
&+ 2p_2e_2(1-e_1)(1-e_2)(PR_r-e_1e_2)^4 \\
&- p_2e_2^2(1-e_1)(PR_r-e_1e_2)^4 \\
&- p_1p_2^2e_1^2e_2^4(1-e_1)^2(1-e_2)(e_1-PR_r)(e_2-PR_r) \\
&+ 2p_1p_2e_1^3e_2^2(1-e_1)(1-e_2)^2(e_2-PR_r)(PR_r-e_1e_2) \\
&+ 2p_1p_2e_1^2e_2(1-e_1)(1-e_2)^2(e_2-PR_r)(PR_r-e_1e_2)^2 \\
&- p_1p_2e_1^2e_2^2(1-e_1)(1-e_2)(e_2-PR_r)(PR_r-e_1e_2)^2 \\
&- e_1e_2(1-e_1)(PR_r-e_1e_2)^2\big[p_1e_1(1-e_2)(PR_r-e_1e_2)\big] \\
&- e_1(e_2-PR_r)(PR_r-e_1e_2)^2\big[p_1e_1(1-e_2)(PR_r-e_1e_2)\big] \\
&- p_2e_2^3(1-e_1)(e_1-PR_r)(PR_r-e_1e_2)\big[p_2e_2(1-e_1)(PR_r-e_1e_2)\big].
\end{aligned} \tag{103}$$

Clearly, V_h is positive. To prove the lemma, we only need to prove U_h is positive.

For this purpose, the expressions in the brackets of the last three items in (103) are replaced by (94) and (95), respectively.

As a result, we have

$$
\begin{aligned}
U_h =&\big[p_2e_1e_2^2(1-e_1)^2(PR_r-e_1e_2)^3-p_1p_2e_1^2e_2^2(1-e_1)(PR_r-e_1e_2)^3\big]\\
&+\big[p_2e_1e_2^2(1-e_1)(1-e_2)(PR_r-e_1e_2)^3\\
&-p_2e_2^2(1-e_1)(PR_r-e_1e_2)^4\big]\\
&+p_1p_2e_1^2e_2(1-e_1)(1-e_2)^2(e_2-PR_r)(PR_r-e_1e_2)^2\\
&+p_2e_2(1-e_1)(1-e_2)(PR_r-e_1e_2)^4\\
&+\big[p_2e_1e_2(1-e_1)(e_2-PR_r)(PR_r-e_1e_2)^3\\
&-p_1p_2e_1^2e_2(e_2-PR_r)(PR_r-e_1e_2)^3\big]\\
&+\Big\{\big[p_1p_2e_1^2e_2^2(1-e_1)^2(1-e_2)(PR_r-e_1e_2)^2\\
&+p_1p_2e_1^2e_2^2(1-e_1)(1-e_2)(e_1-PR_r)(PR_r-e_1e_2)^2\\
&-p_1p_2e_1^2e_2^2(1-e_1)(1-e_2)(e_2-PR_r)(PR_r-e_1e_2)^2\big]\\
&-p_1p_2e_1^2e_2^2(1-e_1)^2(1-e_2)(PR_r-e_1e_2)^2\\
&-p_1p_2e_1^2e_2^3(1-e_1)^2(1-e_2)(e_1-PR_r)(PR_r-e_1e_2)\\
&+p_1p_2e_1^3e_2^2(1-e_1)(1-e_2)^2(e_2-PR_r)(PR_r-e_1e_2)\Big\}\\
&+\big[p_1p_2^2e_1^2e_2^4(1-e_1)^3(1-e_2)(e_1-PR_r)\\
&-p_1p_2^2e_1e_2^4(1-e_1)^2(1-e_2)(e_1-PR_r)(PR_r-e_1e_2)\\
&-p_1p_2^2e_1e_2^4(1-e_1)(e_1-PR_r)(PR_r-e_1e_2)^2\\
&+p_1p_2e_1e_2^3(1-e_1)(1-e_2)(e_1-PR_r)(PR_r-e_1e_2)^2\\
&-p_1p_2^2e_1^2e_2^4(1-e_1)^2(1-e_2)(e_1-PR_r)(e_2-PR_r)\big]\\
&+\big[p_2e_1e_2^2(1-e_1)(1-e_2)(PR_r-e_1e_2)^3\\
&+p_2e_2(1-e_1)(1-e_2)(PR_r-e_1e_2)^4\big]\\
&+\big[p_1p_2e_1^2e_2(1-e_1)(1-e_2)^2(e_2-PR_r)(PR_r-e_1e_2)^2\\
&-p_1p_2e_1^2e_2(1-e_1)(1-e_2)(e_2-PR_r)(PR_r-e_1e_2)^2\\
&+p_1p_2e_1^3e_2^2(1-e_1)(1-e_2)^2(e_2-PR_r)(PR_r-e_1e_2)\\
&-p_1p_2e_1^2e_2^3(1-e_1)^2(1-e_2)(e_1-PR_r)(PR_r-e_1e_2)\big]\\
=&p_2e_1e_2^2(1-e_1)(PR_r-e_1e_2)^3(1-e_1-p_1e_1)\\
&+p_2e_2^2(1-e_1)(e_1-PR_r)(PR_r-e_1e_2)^3\\
&+p_1p_2e_1^2e_2(1-e_1)(1-e_2)^2(e_2-PR_r)(PR_r-e_1e_2)^2\\
&+p_2e_2(1-e_1)(1-e_2)(PR_r-e_1e_2)^4\\
&+p_2e_1e_2(e_2-PR_r)(PR_r-e_1e_2)^3(1-e_1-p_1e_1)\\
&+\big[p_1p_2e_1e_2^3(1-e_1)(e_1-PR_r)(PR_r-e_1e_2)^2(1-e_2-p_2e_2)\\
&+p_1p_2^2e_1e_2^4(1-e_1)^3(1-e_2)(e_1-PR_r)^2\big]\\
&+\big[p_2e_2^2(1-e_1)(e_1-PR_r)(PR_r-e_1e_2)^3\\
&+p_2e_2(1-e_1)(PR_r-e_1e_2)^4\big]\\
&-\big[p_1p_2e_1e_2^2(1-e_1)(e_1-PR_r)^2(PR_r-e_1e_2)^2\\
&+p_1p_2e_1e_2^2(1-e_1)(e_1-PR_r)(PR_r-e_1e_2)^3\big]\\
=&p_2e_1e_2^2(1-e_1)(PR_r-e_1e_2)^3(1-e_1-p_1e_1)\\
&+p_2e_2^2(1-e_1)(e_1-PR_r)(PR_r-e_1e_2)^3\\
&+p_1p_2e_1^2e_2(1-e_1)(1-e_2)^2(e_2-PR_r)(PR_r-e_1e_2)^2\\
&+p_2e_2(1-e_1)(1-e_2)(PR_r-e_1e_2)^4\\
&+p_2e_1e_2(e_2-PR_r)(PR_r-e_1e_2)^3(1-e_1-p_1e_1)\\
&+p_2e_2(1-e_1)(PR_r-e_1e_2)^4\\
&+p_1p_2e_1e_2^3(1-e_1)(e_1-PR_r)(PR_r-e_1e_2)^2(1-e_2-p_2e_2)\\
&+p_2e_2^2(1-e_1)(e_1-PR_r)(PR_r-e_1e_2)^3(1-e_1-p_1e_1)\\
&+p_2e_2^2(1-e_1)(e_1-PR_r)\big[e_1(PR_r-e_1e_2)^3\\
&+p_1p_2e_1e_2^2(1-e_1)^2(1-e_2)(e_1-PR_r)\\
&-p_1e_1(e_1-PR_r)(PR_r-e_1e_2)^2\big].
\end{aligned}
\tag{104}
$$

Similar in the proof of Lemma 4.1 (see Appendix C), it is easy to see that all items of the right-hand side of the above equation except the last one are positive. As for the last item, comparing the expression in the bracket with that in the bracket of the last item of (96), it is not hard to prove its positiveness. Thus, U_h is positive, which completes the proof. ■

E. Proof of Proposition 5.1

Proof: From the expression of PR_r, we have

$$PR_r = e_2^*\big[1-Q(e_1^*,e_2^*,N)\big]. \tag{105}$$

Take the total derivative of both sides of the above equation with respect to e_1^*, e_2^*, and PR_r, and re-arrange the terms, we obtain

$$-e_2^*\frac{\partial Q}{\partial e_1^*}\,de_1^* + \Big(1-Q-e_2^*\frac{\partial Q}{\partial e_2^*}\Big)\,de_2^* = dPR_r. \tag{106}$$

In the following, we consider the cases $\frac{P_1}{P_2}\in(f_{min},f_{max})$, $\frac{P_1}{P_2}<f_{min}$, $\frac{P_1}{P_2}>f_{max}$ separately.

For the case $\frac{P_1}{P_2}\in(f_{min},f_{max})$, considering that

$$f(e_1^*) = \frac{e_2^*\frac{\partial Q}{\partial e_1^*}}{e_2^*\frac{\partial Q}{\partial e_2^*}-(1-Q)} = \frac{P_1}{P_2}, \tag{107}$$

(106) can be rewritten as

$$P_1\frac{de_1^*}{dPR_r} + P_2\frac{de_2^*}{dPR_r} = P_1E. \tag{108}$$

Meanwhile, based on $f(e_1^*)=\frac{P_1}{P_2}$, we have

$$\frac{P_1}{P_2} = \begin{cases} \frac{p_2e_2^{*2}(PR_r-e_1^*e_2^*)^2+p_1p_2e_1^{*2}e_2^{*2}(1-e_2^*)(e_2^*-PR_r)}{p_1e_1^{*2}(PR_r-e_1^*e_2^*)^2+p_1p_2e_1^{*2}e_2^{*2}(1-e_1^*)(e_1^*-PR_r)}, & \text{if } N=1,\\ \frac{e_2^{*2}\frac{\partial Q}{\partial e_1^*}}{e_2^{*2}\frac{\partial Q}{\partial e_2^*}-PR_r}, & \text{if } N>1. \end{cases} \tag{109}$$

For $N=1$, re-arranging the above equation results in

$$
\begin{aligned}
&P_1\big[p_1e_1^{*2}(PR_r-e_1^*e_2^*)^2+p_1p_2e_1^{*2}e_2^{*2}(1-e_1^*)(e_1^*-PR_r)\big]\\
=&P_2\big[p_2e_2^{*2}(PR_r-e_1^*e_2^*)^2+p_1p_2e_1^{*2}e_2^{*2}(1-e_2^*)(e_2^*-PR_r)\big].
\end{aligned}
\tag{110}
$$

Take the total derivative of both sides of the above equation with respect to e_1^*, e_2^*, and PR_r, and re-arrange the terms, we obtain

$$
\begin{aligned}
&de_1^*\Big\{\frac{2P_2}{e_1^*}\big[p_2e_2^{*2}(PR_r-e_1^*e_2^*)^2+p_2e_1^*e_2^{*3}(PR_r-e_1^*e_2^*)\big]\\
&-P_1p_1e_1^{*2}e_2^*\big[2(PR_r-e_1^*e_2^*)+p_2e_2^*(e_1^*-PR_r)-p_2e_2^*(1-e_1^*)\big]\Big\}\\
&+de_2^*\Big\{-\frac{2P_1}{e_2^*}\big[p_1e_1^{*2}(PR_r-e_1^*e_2^*)^2+p_1e_1^{*3}e_2^*(PR_r-e_1^*e_2^*)\big]\\
&+P_2p_2e_1^*e_2^{*2}\big[2(PR_r-e_1^*e_2^*)+p_1e_1^*(e_2^*-PR_r)-p_1e_1^*(1-e_2^*)\big]\Big\}\\
=&dPR_r\Big\{-P_1p_1e_1^{*2}\big[2(PR_r-e_1^*e_2^*)-p_2e_2^{*2}(1-e_1^*)\big]\\
&+P_2p_2e_2^{*2}\big[2(PR_r-e_1^*e_2^*)-p_1e_1^{*2}(1-e_2^*)\big]\Big\}.
\end{aligned}
\tag{111}
$$

Combining with (108) and (111) results in

$$
\begin{aligned}
\frac{de_1^*}{dPR_r} =& \frac{P_2H-P_1EG}{P_2F-P_1G},\\
\frac{de_2^*}{dPR_r} =& \frac{P_1(EF-H)}{P_2F-P_1G}.
\end{aligned}
\tag{112}
$$

For $N>1$, the derivatives $\frac{de_1^*}{dPR_r}$ and $\frac{de_2^*}{dPR_r}$ can be obtained similarly. Different from $N=1$, (110) and (111) are replaced by

$$P_1\Big(e_2^{*2}\frac{\partial Q}{\partial e_2^*}-PR_r\Big) = P_2e_2^{*2}\frac{\partial Q}{\partial e_1^*} \tag{113}$$

and

$$\begin{aligned} &de_1^*\Big(e_2^{*2}\frac{\partial^2 Q}{\partial e_1^*\partial e_2^*}-\frac{P_2}{P_1}e_2^{*2}\frac{\partial^2 Q}{\partial e_1^{*2}}\Big)+de_2^*\Big(2e_2^*\frac{\partial Q}{\partial e_2^*}\\ &+e_2^{*2}\frac{\partial^2 Q}{\partial e_2^{*2}}-2\frac{P_2}{P_1}e_2^*\frac{\partial Q}{\partial e_1^*}-\frac{P_2}{P_1}e_2^{*2}\frac{\partial^2 Q}{\partial e_1^*\partial e_2^*}\Big)=dPR_r, \end{aligned} \tag{114}$$

respectively. The forms of $\frac{de_1^*}{dPR_r}$ and $\frac{de_2^*}{dPR_r}$ in (112) still hold for $N>1$.

As for the cases $\frac{P_1}{P_2}<f_{min}$ and $\frac{P_1}{P_2}>f_{max}$, $f(e_1^*)=\frac{P_1}{P_2}$ does not hold anymore. Assume that $\frac{P_1}{P_2}<f_{min}$, then $e_1^*=\frac{1}{1+p_1}$. That is to say, (106) is simplified as

$$\Big(1-Q-e_2^*\frac{\partial Q}{\partial e_2^*}\Big)\,de_2^*=dPR_r, \tag{115}$$

which implies that

$$\begin{aligned} \frac{de_1^*}{dPR_r}&=0,\\ \frac{de_2^*}{dPR_r}&=\frac{1}{1-Q-e_2^*\frac{\partial Q}{\partial e_2^*}}=-\frac{e_2^*}{e_2^{*2}\frac{\partial Q}{\partial e_2^*}-PR_r}. \end{aligned} \tag{116}$$

Similarly, for $\frac{P_1}{P_2}>f_{max}$, $e_2^*=\frac{1}{1+p_2}$ and (106) is simplified as

$$-e_2^*\frac{\partial Q}{\partial e_1^*}\,de_1^*=dPR_r, \tag{117}$$

which implies that

$$\frac{de_1^*}{dPR_r}=-\frac{1}{e_2^*\frac{\partial Q}{\partial e_1^*}},\ \frac{de_2^*}{dPR_r}=0. \tag{118}$$

■

F. Proof of Proposition 5.3

Proof: For Equation (105), taking the total derivative of both sides with respect to e_1^*, e_2^*, and N, we have

$$e_2^*\frac{\partial Q}{\partial e_1^*}\,de_1^*+\Big[e_2^*\frac{\partial Q}{\partial e_2^*}-(1-Q)\Big]\,de_2^*+e_2^*\frac{\partial Q}{\partial N}\,dN=0. \tag{119}$$

In the following, we prove the proposition for the three cases one by one.

First, we prove it for $\frac{P_1}{P_2}\in[f_{min},f_{max}]$. Taking into account (107), (119) can be re-written as

$$P_1\,de_1^*+P_2\,de_2^*=-P_1\frac{\frac{\partial Q}{\partial N}}{\frac{\partial Q}{\partial e_1^*}}\,dN. \tag{120}$$

Thus,

$$\frac{dz^*}{dN}=P_1\frac{de_1^*}{dN}+P_2\frac{de_2^*}{dN}=-P_1\frac{\frac{\partial Q}{\partial N}}{\frac{\partial Q}{\partial e_1^*}}. \tag{121}$$

As for $\frac{P_1}{P_2}\notin[f_{min},f_{max}]$, e_1^* or e_2^* is constant and (119) can be simplified. Specifically, if $\frac{P_1}{P_2}<f_{min}$, the optimal solution is $e_1^*=\frac{1}{1+p_1}$ and $e_2^*=e_{2,min}$, where e_1^* is a constant. In this case, (119) is rewritten as

$$\Big[e_2^*\frac{\partial Q}{\partial e_2^*}-(1-Q)\Big]\,de_2^*=-e_2^*\frac{\partial Q}{\partial N}\,dN. \tag{122}$$

And then, we have

$$\frac{dz^*}{dN}=P_2\frac{de_2^*}{dN}=-P_2\frac{e_2^{*2}\frac{\partial Q}{\partial N}}{e_2^{*2}\frac{\partial Q}{\partial e_2^*}-PR_r}. \tag{123}$$

Similarly, when $\frac{P_1}{P_2}>f_{max}$, e_2^* is a constant, thus (119) is simplified as

$$e_2^*\frac{\partial Q}{\partial e_1^*}\,de_1^*=-e_2^*\frac{\partial Q}{\partial N}\,dN, \tag{124}$$

which leads to

$$\frac{dz^*}{dN}=P_1\frac{de_1^*}{dN}=-P_1\frac{\frac{\partial Q}{\partial N}}{\frac{\partial Q}{\partial e_1^*}}. \tag{125}$$

■

G. Proof of Proposition 5.4

Proof: When $\frac{P_1}{P_2}\in[f_{min},f_{max}]$, taking into account (33) and the monotonicity of $f(e_1)$, it is easy to conclude that e_1^* is decreasing in P_1 and thus, e_2^* is increasing. Similarly, it can be proved that e_1^* and e_2^* are, respectively, increasing and decreasing in P_2.

As for the case $\frac{P_1}{P_2}\notin[f_{min},f_{max}]$, clearly, $\frac{P_1}{P_2}$ is either smaller than f_{min} or greater than f_{max}. Thus, from the conclusion in Section IV-B, the optimal solution is still $(e_{1,max},e_{2,min})$ for the former case and $(e_{1,min},e_{2,max})$ for the latter one, which completes the proof. ■

H. Proof of Proposition 5.5

Proof: Take the total derivative of both sides of (105) with respect to e_1^* and e_2^* and re-arrange the terms, we have

$$e_2^*\frac{\partial Q}{\partial e_1^*}\,de_1^*=\Big(1-Q-e_2^*\frac{\partial Q}{\partial e_2^*}\Big)\,de_2^*. \tag{126}$$

For $\frac{P_1}{P_2}\in[f_{min},f_{max}]$, combining (126) with $f(e_1^*)=\frac{e_2^*\frac{\partial Q}{\partial e_1^*}}{e_2^*\frac{\partial Q}{\partial e_2^*}-(1-Q)}=\frac{P_1}{P_2}$, we obtain

$$P_1\,de_1^*+P_2\,de_2^*=0. \tag{127}$$

Then, the derivative of z^* with respect to P_1 is derived:

$$\frac{dz^*}{dP_1}=e_1^*+P_1\frac{de_1^*}{dP_1}+P_2\frac{de_2^*}{dP_1}=e_1^*. \tag{128}$$

Similarly, the derivative of z^* with respect to P_2 for $\frac{P_1}{P_2}\in[f_{min},f_{max}]$ is obtained:

$$\frac{dz^*}{dP_2}=e_2^*. \tag{129}$$

For the case that $\frac{P_1}{P_2}\notin[f_{min},f_{max}]$, the optimal solution, which is $(e_{1,max},e_{2,min})$ or $(e_{1,min},e_{2,max})$, stays unchanged when P_i changes slightly, $i=1,2$. Thus, the derivatives of z^* with respect to P_1 and P_2 are as follows:

$$\begin{aligned} \frac{dz^*}{dP_1}&=e_1^*+P_1\frac{de_1^*}{dP_1}+P_2\frac{de_2^*}{dP_1}=e_1^*,\\ \frac{dz^*}{dP_2}&=e_2^*+P_1\frac{de_1^*}{dP_2}+P_2\frac{de_2^*}{dP_2}=e_2^*, \end{aligned} \tag{130}$$

which is consistent with the case that $\frac{P_1}{P_2}\in[f_{min},f_{max}]$. ■

I. Proof of Proposition 5.6

Proof: Since the expressions of $\frac{dz^*}{dp_1}$ and $\frac{dz^*}{dp_2}$ can be deduced in the same way, in the following, we only provide the derivation of $\frac{dz^*}{dp_1}$.

Taking the total derivative of both sides of (105) with respect to e_1^*, e_2^*, and p_1 and re-arranging the terms, we have

$$e_2^*\frac{\partial Q}{\partial e_1^*}\,de_1^*+\Big[e_2^*\frac{\partial Q}{\partial e_2^*}-(1-Q)\Big]\,de_2^*=-e_2^*\frac{\partial Q}{\partial p_1}\,dp_1. \quad (131)$$

For $\frac{P_1}{P_2}\in(f_{min},f_{max})$, since $f(e_1^*)=\frac{e_2^*\frac{\partial Q}{\partial e_1^*}}{e_2^*\frac{\partial Q}{\partial e_2^*}-(1-Q)}=\frac{P_1}{P_2}$, we obtain

$$\frac{dz^*}{dp_1}=P_1\frac{de_1^*}{dp_1}+P_2\frac{de_2^*}{dp_1}=-P_1\frac{\frac{\partial Q}{\partial p_1}}{\frac{\partial Q}{\partial e_1^*}}. \quad (132)$$

When $\frac{P_1}{P_2}<f_{min}$, since $e_1^*=\frac{1}{1+p_1}$, we have

$$e_2^*\frac{\partial Q}{\partial e_1^*}\,de_1^*=e_2^*\frac{\partial Q}{\partial e_1^*}\frac{de_1^*}{dp_1}\,dp_1=e_2^*\frac{\partial Q}{\partial p_1}\,dp_1. \quad (133)$$

Based on the above equation, (131) can be rewritten as

$$e_2^*\frac{\partial Q}{\partial p_1}\,dp_1+\Big[e_2^*\frac{\partial Q}{\partial e_2^*}-(1-Q)\Big]\,de_2^*=-e_2^*\frac{\partial Q}{\partial p_1}\,dp_1, \quad (134)$$

from which we obtain

$$\frac{de_2^*}{dp_1}=\frac{-2e_2^*\frac{\partial Q}{\partial p_1}}{e_2^*\frac{\partial Q}{\partial e_2^*}-(1-Q)}. \quad (135)$$

Considering that $\frac{dz^*}{dp_1}=P_1\frac{de_1^*}{dp_1}+P_2\frac{de_2^*}{dp_1}$ and $\frac{de_1^*}{dp_1}=-\frac{1}{(1+p_1)^2}$, the expression of $\frac{dz^*}{dp_1}$ for $\frac{P_1}{P_2}<f_{min}$ is obtained:

$$\frac{dz^*}{dp_1}=-P_1\frac{1}{(1+p_1)^2}-2P_2\frac{e_2^{*2}\frac{\partial Q}{\partial p_1}}{e_2^{*2}\frac{\partial Q}{\partial e_2^*}-PR_r}. \quad (136)$$

For $\frac{P_1}{P_2}>f_{max}$, since e_2^* equals $\frac{1}{1+p_2}$ which is a constant, (131) is rewritten as

$$e_2^*\frac{\partial Q}{\partial e_1^*}\,de_1^*=-e_2^*\frac{\partial Q}{\partial p_1}\,dp_1. \quad (137)$$

Then, we have

$$\frac{dz^*}{dp_1}=P_1\frac{de_1^*}{dp_1}=-P_1\frac{\frac{\partial Q}{\partial p_1}}{\frac{\partial Q}{\partial e_1^*}}. \quad (138)$$

As far as $\frac{P_1}{P_2}=f_{min}$ is concerned, let $\hat{p}_1$ denote the value of p_1 satisfying $\frac{P_1}{P_2}=f_{min}$. Since the difference of left and right limits of $\hat{p}_1$, i.e.,

$$\begin{aligned}&-P_1\frac{\frac{\partial Q}{\partial p_1}}{\frac{\partial Q}{\partial e_1^*}}-\Big[-P_1\frac{1}{(1+p_1)^2}-2P_2\frac{e_2^{*2}\frac{\partial Q}{\partial p_1}}{e_2^{*2}\frac{\partial Q}{\partial e_2^*}-PR_r}\Big]\\ =&P_1\frac{1}{(1+p_1)^2}+P_2\frac{e_2^{*2}\frac{\partial Q}{\partial p_1}}{e_2^{*2}\frac{\partial Q}{\partial e_2^*}-PR_r},\end{aligned} \quad (139)$$

is positive (noting that $\frac{\partial Q}{\partial p_1}$ and $e_2^{*2}\frac{\partial Q}{\partial e_2^*}-PR_r$ are negative, see the report [44]), $\frac{dz^*}{dp_1}$ does not exist. While for $\frac{P_1}{P_2}=f_{max}$, it is clear that the corresponding left and right limits are equal to each other, which implies that $\frac{dz^*}{dp_1}=-P_1\frac{\frac{\partial Q}{\partial p_1}}{\frac{\partial Q}{\partial e_1^*}}$. ■

J. Proof of Proposition 6.2

Proof: As before, we prove it by considering the cases $\frac{P_1}{P_2}\in[f_{min},f_{max}]$, $\frac{P_1}{P_2}<f_{min}$, and $\frac{P_1}{P_2}>f_{max}$ separately.

First, we consider the case that $\frac{P_1}{P_2}>f_{max}$. From Section IV-B, it follows that, for $\frac{P_1}{P_2}>f_{max}$, the optimal solution of (P2) is $(e_{1,min},e_{2,max})$. Noting that (P4) and (P2) have the same optimal solution and taking into account (23) and (24), we have

$$y^*=\frac{P_1(1+p_2)}{1+p_1p_2}+\frac{P_2}{(1+p_2)PR_r}. \quad (140)$$

Clearly, $\frac{dy^*}{dPR_r}<0$. For the case $\frac{P_1}{P_2}<f_{min}$, the proof is similar and omitted here.

As for the case $\frac{P_1}{P_2}\in[f_{min},f_{max}]$, from (8), it follows

$$\frac{\partial Q}{\partial e_1}=-\frac{(PR_r-e_1e_2)^2+p_1e_1^2(1-e_2)(e_2-PR_r)}{p_1e_1^2e_2(1-e_1)(1-e_2)}. \quad (141)$$

Thus, based on (58), we have

$$\begin{aligned}\frac{dy^*}{dPR_r}=&-\frac{1}{PR_r^2}\Big[PR_rP_1/\Big(e_2^*\frac{\partial Q}{\partial e_1}\Big|_{(e_1,e_2)=(e_1^*,e_2^*)}\Big)+z^*\Big]\\ =&\frac{1}{PR_r^2}\Big[\frac{P_1p_1e_1^{*2}(1-e_1^*)(1-e_2^*)PR_r}{(PR_r-e_1^*e_2^*)^2+p_1e_1^{*2}(1-e_2^*)(e_2^*-PR_r)}\\ &-P_1e_1^*-P_2e_2^*\Big]\\ =&\frac{P_1e_1^*}{p_2e_2^*PR_r^2}\Big[\frac{p_1p_2e_1^*e_2^*(1-e_1^*)(1-e_2^*)PR_r}{(PR_r-e_1^*e_2^*)^2+p_1e_1^{*2}(1-e_2^*)(e_2^*-PR_r)}\\ &-p_2e_2^*-\frac{P_2p_2e_2^{*2}}{P_1e_1^*}\Big].\end{aligned} \quad (142)$$

Letting $U=p_2e_2^*PR_r^2\big[(PR_r-e_1^*e_2^*)^2+p_1e_1^{*2}(1-e_2^*)(e_2^*-PR_r)\big]$ and taking into account (110), the equation above can be rewritten as

$$\begin{aligned}\frac{dy^*}{dPR_r}=&\frac{P_1e_1^*}{U}\Big\{p_1p_2e_1^*e_2^*(1-e_1^*)(1-e_2^*)(PR_r-e_1^*e_2^*)\\ &+p_1p_2e_1^{*2}e_2^{*2}(1-e_1^*)(1-e_2^*)\\ &-p_2e_2^*\big[(PR_r-e_1^*e_2^*)^2+p_1e_1^{*2}(1-e_2^*)(e_2^*-PR_r)\big]\\ &-p_1e_1^*\big[(PR_r-e_1^*e_2^*)^2+p_2e_2^{*2}(1-e_1^*)(e_1^*-PR_r)\big]\Big\}.\end{aligned} \quad (143)$$

In addition, from (8), it follows

$$\begin{aligned}&p_1p_2e_1^*e_2^*(1-e_1^*)(1-e_2^*)\\ =&\big[p_1e_1^*(1-e_2^*)+p_2e_2^*(1-e_1^*)-p_1p_2e_1^*e_2^*\big](PR_r-e_1^*e_2^*),\end{aligned} \quad (144)$$

which results in

$$\begin{aligned}\frac{dy^*}{dPR_r} =& \frac{P_1 e_1^*}{U}\Big\{\big[p_1 e_1^*(1-e_2^*) + p_2 e_2^*(1-e_1^*) - p_1 p_2 e_1^* e_2^*\big] \\ & \cdot (PR_r - e_1^* e_2^*)^2 + p_1 p_2 e_1^{*2} e_2^{*2}(1-e_1^*)(1-e_2^*) \\ & - p_2 e_2^*\big[(PR_r - e_1^* e_2^*)^2 + p_1 e_1^{*2}(1-e_2^*)(e_2^* - PR_r)\big] \\ & - p_1 e_1^*\big[(PR_r - e_1^* e_2^*)^2 + p_2 e_2^{*2}(1-e_1^*)(e_1^* - PR_r)\big]\Big\} \\ =& \frac{P_1 e_1^*}{U}\big[-(p_1+p_2) e_1^* e_2^*(PR_r - e_1^* e_2^*)^2 - p_1 p_2 e_1^* e_2^* \\ & \cdot (PR_r - e_1^* e_2^*)^2 + p_1 p_2 e_1^{*2} e_2^{*2}(1-e_1^*)(1-e_2^*) \\ & - p_1 p_2 e_1^* e_2^{*2}(1-e_1^*)(e_1^* - PR_r) \\ & - p_1 p_2 e_1^{*2} e_2^*(1-e_2^*)(e_2^* - PR_r)\big] \\ =& -\frac{P_1 e_1^*}{U}\big[(p_1+p_2) e_1^* e_2^*(PR_r - e_1^* e_2^*)^2 \\ & + p_1 p_2 e_1^* e_2^*(e_1^* - PR_r)(e_2^* - PR_r)\big].\end{aligned} \tag{145}$$

Clearly, the sum of terms in the bracket of the last expression of (145) is positive. Taking into account that $U > 0$, it is concluded that $\frac{dy^*}{dPR_r} < 0$, which completes the proof. ■

K. Proof of Theorem 6.2

Proof: From Fig. 10, one can see that as PR_r increases, the optimal solution of (P10) moves along its locus starting from the lower-left vertex of the rectangle characterized by (81). Since the locus consists of edges of the rectangle and the locus of the optimal solution of (P2), we only need to prove that, as PR_r increases, the objective value of (P10) increases along each edge of the rectangle and the locus of the optimal solution of (P2).

For the former, without loss of generality, we take the bottom horizontal edge of the rectangle as an example. Clearly, $\tilde{e}_2^*$ is fixed and $\tilde{z}^*$ is strictly increasing in $\tilde{e}_1^*$, which is strictly increasing in PR_r. On the other edges, it can be proved similarly. As for the latter, it is also true from Theorem 2.1. Thus, the proof is completed. ■

References

[1] C. Ho, P. Rao, N. Iloeje, A. Marschilok, B. Liaw, S. Kaur, J. Slaughter, K. Hertz, L. Wendt, S. Supekar, and M. Montes, "Energy storage for manufacturing and industrial decarbonization (energy storm)," September 2022. [Online]. Available: https://www.osti.gov/biblio/1887337

[2] J. Li, J. R. Morrison, M. T. Zhang, M. Nakano, S. Biller, and B. Lennartson., "Editorial: Automation in green manufacturing," *IEEE Transactions on Automation Science and Engineering*, vol. 10, no. 1, pp. 1–4, 2013.

[3] J. A. Buzacott and J. G. Shanthikumar, *Stochastic Models of Manufacturing Systems*. Englewood Cliffs, N. J.: Prentice Hall, 1993.

[4] S. B. Gershwin, *Manufacturing Systems Engineering*. Englewood Cliffs, N. J.: PTR Prentice Hall, 1994.

[5] J. Li and S. M. Meerkov, *Production Systems Engineering*. New York: Springer, 2009.

[6] J. Li, D. E. Blumenfeld, N. Huang, and J. M. Alden, "Throughput analysis of production systems: Recent advances and future topics," *International Journal of Production Research*, vol. 47, no. 14, pp. 3823–3851, 2009.

[7] C. T. Papadopoulos, J. Li, and M. E. J. O'Kelly, "A classification and review of timed Markov models of manufacturing systems," *Computers & Industrial Engineering*, vol. 128, pp. 219–244, 2019.

[8] M. Subramaniyan, A. Skoogh, J. Bokrantz, M. A. Sheikh, M. Thürer, and Q. Chang, "Artificial intelligence for throughput bottleneck analysis – state-of-the-art and future directions," *Journal of Manufacturing Systems*, vol. 60, pp. 734–751, 2021.

[9] F.-Y. Yan, J.-Q. Wang, Y. Li, and P.-H. Cui, "An improved aggregation method for performance analysis of Bernoulli serial production lines," *IEEE Transactions on Automation Science and Engineering*, vol. 18, no. 1, pp. 114–121, 2021.

[10] L. Liu, C.-B. Yan, and J. Li, "Modeling, analysis, and improvement of batch-discrete manufacturing systems: A systems approach," *IEEE Transactions on Automation Science and Engineering*, vol. 19, no. 3, pp. 1567–1585, 2022.

[11] Y. Kang and F. Ju, "Integrated analysis of productivity and machine condition degradation: A geometric-machine case," in *2016 IEEE International Conference on Automation Science and Engineering (CASE 2016)*, 2016, pp. 1128–1133.

[12] F. Wang, F. Ju, and N. Kang, "Transient analysis and real-time control of geometric serial lines with residence time constraints," *IISE Transactions*, vol. 51, no. 7, pp. 709–728, 2019.

[13] F. Ju, J. Li, G. Xiao, J. Arinez, and W. Deng, "Modeling, analysis, and improvement of integrated productivity and quality system in battery manufacturing," *IIE Transactions*, vol. 47, no. 12, pp. 1313–1328, 2015.

[14] C.-B. Yan and Q. Zhao, "Analytical approach to estimate efficiency of series machines in production lines," *IEEE Transactions on Automation Science and Engineering*, vol. 15, no. 3, pp. 1027–1040, 2018.

[15] Y. Li, Y. He, Y. Wang, P. Yan, and X. Liu, "A framework for characterising energy consumption of machining manufacturing systems," *International Journal of Production Research*, vol. 52, no. 2, pp. 314–325, 2014.

[16] S. Wang, X. Wang, J. Yu, S. Ma, and M. Liu, "Bi-objective identical parallel machine scheduling to minimize total energy consumption and makespan," *Journal of Cleaner Production*, vol. 193, pp. 424–440, 2018.

[17] J. F. Wang, Z. C. Fei, Q. Chang, Y. Fu, and S. Q. Li, "Energy-saving operation of multistage stochastic manufacturing systems based on fuzzy logic," *International Journal of Simulation Modelling*, vol. 18, no. 1, pp. 138–149, 2019.

[18] L. Yang, J. Deuse, and P. Jiang, "Multi-objective optimization of facility planning for energy intensive companies," *Journal of Intelligent Manufacturing*, vol. 24, no. 6, pp. 1095–1109, 2013.

[19] J. L. Rivera and T. Reyes-Carrillo, "A life cycle assessment framework for the evaluation of automobile paint shops," *Journal of Cleaner Production*, vol. 115, pp. 75–87, 2016.

[20] C. A. Guerrero, J. Wang, J. Li, J. Arinez, S. Biller, N. Huang, and G. Xiao, "Production system design to achieve energy savings in an automotive paint shop," *International Journal of Production Research*, vol. 49, no. 22, pp. 6769–6785, 2011.

[21] Y. Alaouchiche, Y. Ouazene, and F. Yalaoui, "Energy-efficient buffer allocation problem in unreliable production lines," *The International Journal of Advanced Manufacturing Technology*, vol. 114, no. 9, pp. 2871–2885, 2021.

[22] A. Hasanbeigi, M. Arens, and L. Price, "Alternative emerging ironmaking technologies for energy-efficiency and carbon dioxide emissions reduction: A technical review," *Renewable and Sustainable Energy Reviews*, vol. 33, pp. 645–658, 2014.

[23] F. Pusavec and J. Kenda, "The transition to a clean, dry, and energy efficient polishing process: an innovative upgrade of abrasive flow machining for simultaneous generation of micro-geometry and polishing in the tooling industry," *Journal of cleaner production*, vol. 76, pp. 180–189, 2014.

[24] M. Mashaei and B. Lennartson, "Energy reduction in a pallet-constrained flow shop through on-off control of idle machines," *IEEE Transactions on Automation Science and Engineering*, vol. 10, no. 1, pp. 45–56, 2013.

[25] N. Frigerio and A. Matta, "Energy-efficient control strategies for machine tools with stochastic arrivals," *IEEE Transactions on Automation Science and Engineering*, vol. 12, no. 1, pp. 50–61, 2015.

[26] Y. Li, J.-Q. Wang, and Q. Chang, "Event-based production control for energy efficiency improvement in sustainable multistage manufacturing systems," *Journal of Manufacturing Science and Engineering*, vol. 141, no. 2, 2019.

[27] A. Loffredo, N. Frigerio, E. Lanzarone, and A. Matta, "Energy-efficient control policy for parallel and identical machines with availability constraint," *IEEE Robotics and Automation Letters*, vol. 6, no. 3, pp. 5713–5719, 2021.

[28] X. Wang, Y. Dai, and Z. Jia, "Energy-efficient on/off control in serial production lines with Bernoulli machines," *Flexible Services and Manufacturing Journal, online published*, pp. 1–26, 2022.

[29] G. Chen, L. Zhang, J. Arinez, and S. Biller, "Energy-efficient production systems through schedule-based operations," *IEEE Transactions on Automation Science and Engineering*, vol. 10, no. 1, pp. 27–37, 2013.

[30] C. K. Pang and C. V. Le, "Optimization of total energy consumption in flexible manufacturing systems using weighted p-timed petri nets and

dynamic programming," *IEEE Transactions on Automation Science and Engineering*, vol. 11, no. 4, pp. 1083–1096, 2013.

[31] X. Li, K. Xing, M. C. Zhou, X. Wang, and Y. Wu, "Modified dynamic programming algorithm for optimization of total energy consumption in flexible manufacturing systems," *IEEE Transactions on Automation Science and Engineering*, vol. 16, no. 2, pp. 691–705, 2018.

[32] W. Su, X. Xie, J. Li, and L. Zheng, "Improving energy efficiency in Bernoulli serial lines: An integrated model," *International Journal of Production Research*, vol. 54, no. 11, pp. 3414–3428, 2016.

[33] C.-B. Yan, X. Cheng, F. Gao, and X. Guan, "Formulation and solution methodology for reducing energy consumption in two-machine Bernoulli serial lines," *IEEE Transactions on Automation Science and Engineering*, vol. 19, no. 1, pp. 522–530, 2022.

[34] C.-B. Yan, "Analysis and optimization of energy consumption in two-machine Bernoulli lines with general bounds on machine efficiency," *IEEE Transactions on Automation Science and Engineering*, vol. 18, no. 1, pp. 151–163, 2021.

[35] X. Cheng, C.-B. Yan, and F. Gao, "Energy cost optimisation in two-machine Bernoulli serial lines under time-of-use pricing," *International Journal of Production Research*, vol. 60, no. 13, pp. 3948–3964, 2022.

[36] Z. Pei, P. Yang, Y. Wang, and C.-B. Yan, "Energy consumption control in the two-machine Bernoulli serial production line with setup and idleness," *International Journal of Production Research*, vol. 61, no. 9, pp. 2917–2936, 2023.

[37] W. Su, X. Xie, J. Li, L. Zheng, and S. C. Feng, "Reducing energy consumption in serial production lines with Bernoulli reliability machines," *International Journal of Production Research*, vol. 55, no. 24, pp. 7356–7379, 2017.

[38] C.-B. Yan and Z. Zheng, "Problem formulation and solution methodology for energy consumption optimization in Bernoulli serial lines," *IEEE Transactions on Automation Science and Engineering*, vol. 18, no. 2, pp. 776–790, 2021.

[39] ——, "An effective and efficient divide-and-conquer algorithm for energy consumption optimisation problem in long Bernoulli serial lines," *International Journal of Production Research*, vol. 59, no. 23, pp. 7018–7036, 2021.

[40] H. Dong and J. Li, "Energy and productivity analysis in serial production lines with setups," *IEEE Robotics and Automation Letters*, vol. 7, no. 3, pp. 7108–7115, 2022.

[41] P. Yang and Z. Pei, "Energy-saving manufacturing system design with two geometric machines," *Sustainability*, vol. 14, no. 18, p. 11448, 2022.

[42] C.-B. Yan and L. Liu, "Energy consumption optimization in two-machine geometric serial lines," in *2020 IEEE International Conference on Automation Science and Engineering (CASE 2020)*, 2020, pp. 830–835.

[43] J. Li, "Production variability in manufacturing systems: A systems approach," Ph.D. dissertation, University of Michigan, 2000.

[44] C.-B. Yan and L. Liu, "Derivatives of key functions w.r.t. machine efficiencies in two-machine geometric line," 2020. [Online]. Available: http://gr.xjtu.edu.cn/web/chaoboyan/reports

Chao-Bo Yan (Senior Member, IEEE) received his B.S. degree from Xi'an Jiaotong University, Xi'an, China, in 2004, and M.S. and Ph.D. degrees in Control Theory and Engineering from Tsinghua University, Beijing, China, in 2007 and 2012, respectively.

He was a Visiting Scholar with the University of Michigan, Ann Arbor, MI, USA, from 2009 to 2010, and a Post-Doctoral Research Fellow from 2012 to 2014. He is currently a Professor with the School of Automation Science and Engineering, Faculty of Electronic and Information Engineering, Xi'an Jiaotong University. His research interests are system optimization and control, and cyber-physical systems (CPS) theory and its applications to manufacturing, logistics, and inventory systems.

Dr. Yan serves as Senior Editor of the IEEE ROBOTICS AND AUTOMATION LETTERS and Editor on the Conference Editorial Board (CEB) of the IEEE Conference on Automation Science and Engineering (IEEE CASE). He served as the Publication Chair and General Co-Chair of IEEE CASE 2017 and 2022, respectively, as Associate Editor of the IEEE TRANSACTIONS ON AUTOMATION SCIENCE AND ENGINEERING and the IEEE ROBOTICS AND AUTOMATION LETTERS, and as Guest Editor of the INTERNATIONAL JOURNAL OF PRODUCTION RESEARCH and IISE TRANSACTIONS.

Lingchen Liu (Graduate Student Member, IEEE) received the B.Eng. degree from the School of Electronics and Information Technology, Sun Yat-Sen University, Guangzhou, China, in 2019. She is currently pursuing the Ph.D. degree with the School of Automation Science and Engineering, Xi'an Jiaotong University, Xi'an, China. Her research interests include modeling, analysis, and optimization of production systems.