西安交通大學 XI'AN JIAOTONG UNIVERSITY

Financial Engineering

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Finance and Financial Engineering

Finance is a field that is concerned with the allocation (investment) of assets and liabilities over space and time, often under conditions of **risk** or **uncertainty**. Finance can also be defined as the art of **money** management.

Financial Engineering represents the emerging discipline wherein mathematical tools are used to model financial markets and solve problems in finance， also named as：

- Computational Finance
- Financial Mathematics
- Mathematical Finance
- Quantititave Finance[1]

[1] （https://iafe.org）

❑ Money-driven interdisciplinary study

Mathematics includes the study of such topics as quantity, structure, space and change.

Finance： subject studying money

Financial Engineering： study finance by Math.

Start： 1900s emerged as a discipline:1970s

Problems in Financial Engineering： Pricing and optimal investment

- Asset pricing
- Modern portfolio theory
- Risk measure
- Derivatives development

❑ Financial engineers

Financial engineers typically work in investment banks, insurance companies, hedge funds, commercial banks, regulatory agencies corporate treasuries.

- Investment Banking
- Corporate Strategic Planning
- Risk Management
- Primary and Derivatives Securities Valuation
- Financial Information Systems Management
- Portfolio Management
- Security Trading

[1]https://www.topaccountingdegrees.org/faq/what-is-financial-engineering/

❑ Financial engineers : prerequisite

Generally, Financial Engineers are strong on the following fields:

(1) Finance Preparation:

Fundamentals of Corporate Finance, Options, Futures, and Other Derivatives, Investments, Intro to Financial Account （CFA Level 1.）

(2) Math Preparation :

Calculus，Linear Algebra，Partial Differential Equations，Statistics，Numerical Analysis， optimization

(3) Programming Preparation:

C++, Paython, Matlab

❑ Courses in Business school (USA) :

- University of California, Berkeley
  Haas School of Business,
  Master in Financial Engineering
  http://haas.berkeley.edu/MFE/index.html
- Carnegie Mellon University
  Graduate School of Business
  Master of Science in Computational Finance
  http://student-2k.gsia.cmu.edu/mscf/

Financial Engineering

❑ Courses in industrial engineering school (USA):

⇨ Princeton University
Department of Operations Research & Financial Engineering
M.S.E. in Operations Research and Financial Engineering
http://www.orfe.princeton.edu/graduate/index.html

⇨ Columbia University Department of Industrial Engineering and Operations Research M.S. in Financial Engineering
http://www.ieor.columbia.edu/finance.html

⇨ Cornell University School of Operations Research and Industrial Engineering Master of Engineering in Financial Engineering
http://www.orie.cornell.edu/me...ables/financial1.html

⇨ University of Michigan, Ann Arbor College of Engineering
Master of Science in Financial Engineering
http://interpro.engin.umich.edu/fep/

❑ Courses in Mathematics school (USA):

- Stanford University The Departments of Mathematics and Statistics
  MS in Financial Mathematics
  http://math.stanford.edu/FinMath/
- University of Chicago Department of Mathematics
  Master of Science in Financial Mathematics
  http://finmath.uchicago.edu/
- New York University Department of Mathematics
  Master of Science in Mathematics in Finance
  http://math.nyu.edu/financial_mathematics/
- University of Southern California Department of Mathematics
  Master of Science in Mathematical Finance
  http://www.usc.edu/dept/LAS/CAMS/MF/

Financial Engineering

Courses in Canada

⇨ University of Toronto
Masters Program in Mathematical Finance
http://www.math.toronto.edu/finance/

⇨ York University
Graduate Diploma in Financial Engineering
http://www.yorku.ca/fineng/

Courses in Great Britain

⇨ University of Oxford
Oxford Centre for Industrial and Applied Mathematics
Postgraduate Diploma in Mathematical Finance
http://www.conted.ox.ac.uk/courses/mathsfinance/

⇨ The University of Edinburgh
Management School
MSc in Financial Mathematics
http://www.cpa.ed.ac.uk/prosp/...ncialMathematics.html

❑ Courses in Singapore:

⇨ National University of Singapore
Centre for Financial Engineering
Master of Science in Financial Engineering
http://cfe.nus.edu.sg/msc_fe.htm

⇨ Nanyang Technological University
Nanyang Business School.
Master of Science in Financial Engineering
http://nbs.ntu.edu.sg/Programmes/Graduate/MFE/

❑ Courses in China:

➪ Dudan University
School of Management
https://www.fdsm.fudan.edu.cn/En/

➪ Central University of Finance and Economics
http://en.cufe.edu.cn/

➪ Shanghai University of Finance and Economics
http://english.sufe.edu.cn/

➪ Renmin University of China
Business School
http://en.rmbs.ruc.edu.cn/

❑ Courses in my university:

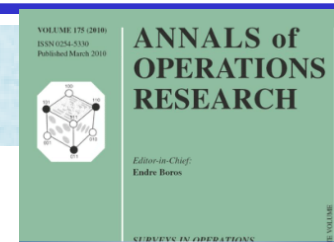
➪ Xi’an Jiaotong University

Department of Mathematics and Statistics

Research Center for optimization and finance engineering

http://en.xjtu.edu.cn/

http://xiammt.xjtu.edu.cn/yjst/zyhjsylhjryjzx.htm



❑ Outline

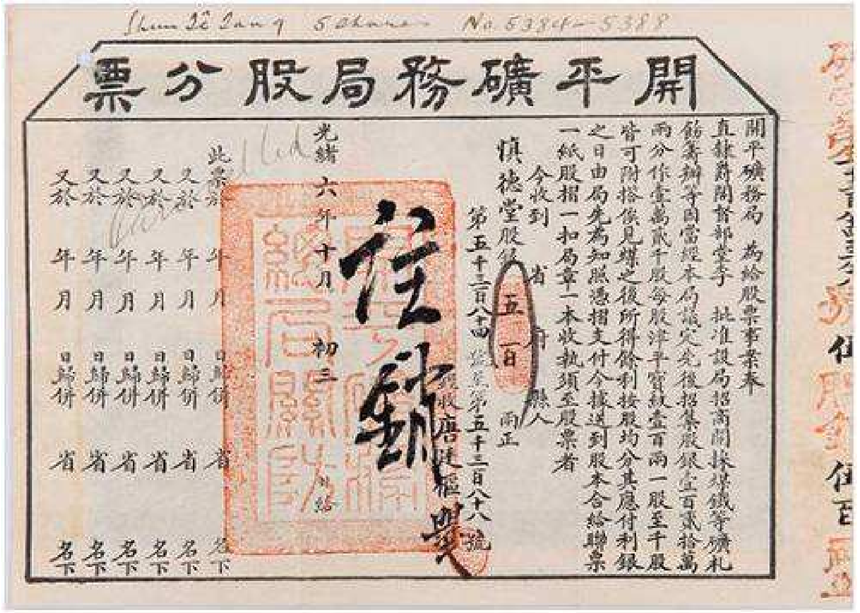
Today we will study:

- Securities
- Risk measure
- Portfolio selection

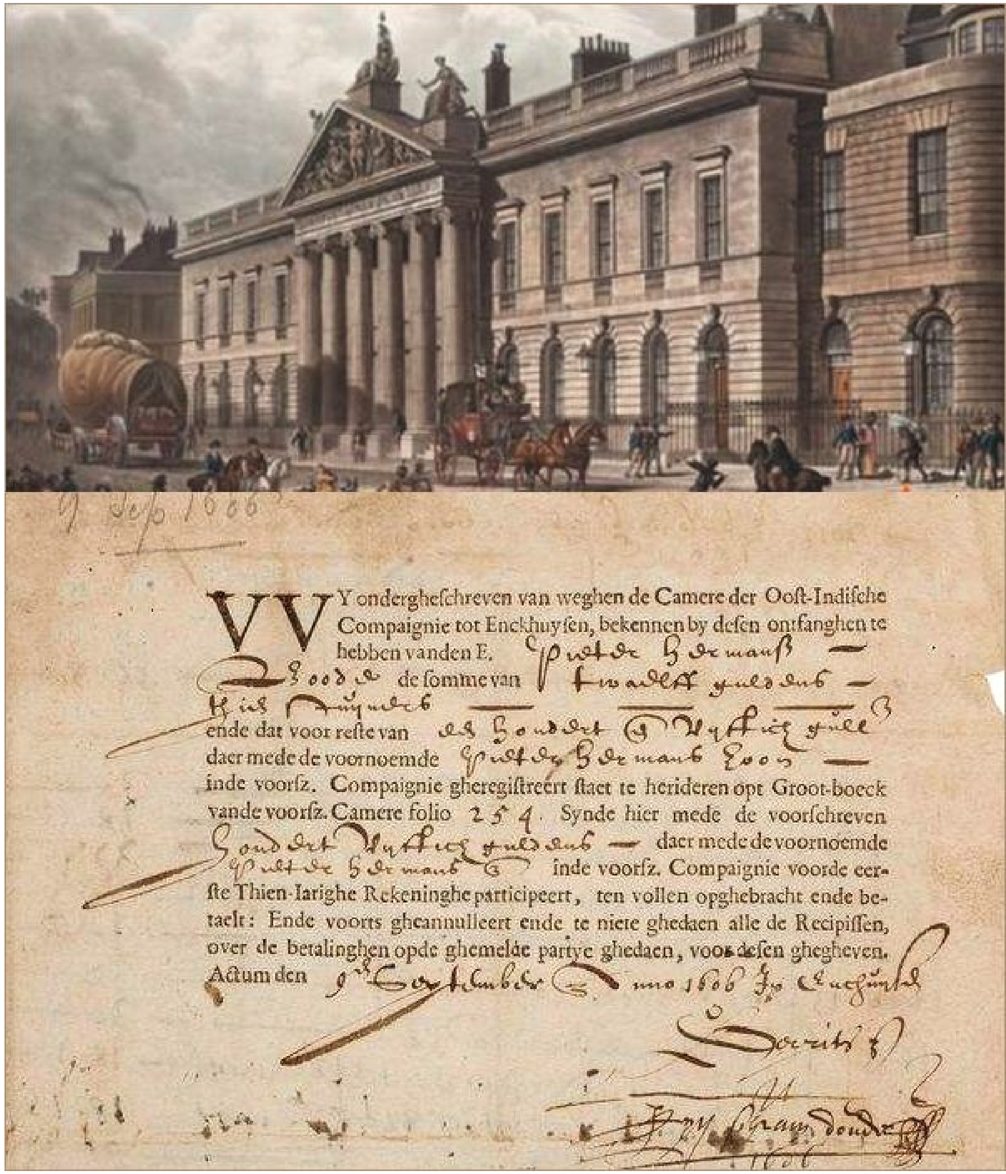
What is a security?

➢A security is a fungible, negotiable instrument representing financial value.

➢Securities are broadly categorized into debt and equity securities such as bonds and common stocks, respectively.



China 1870s



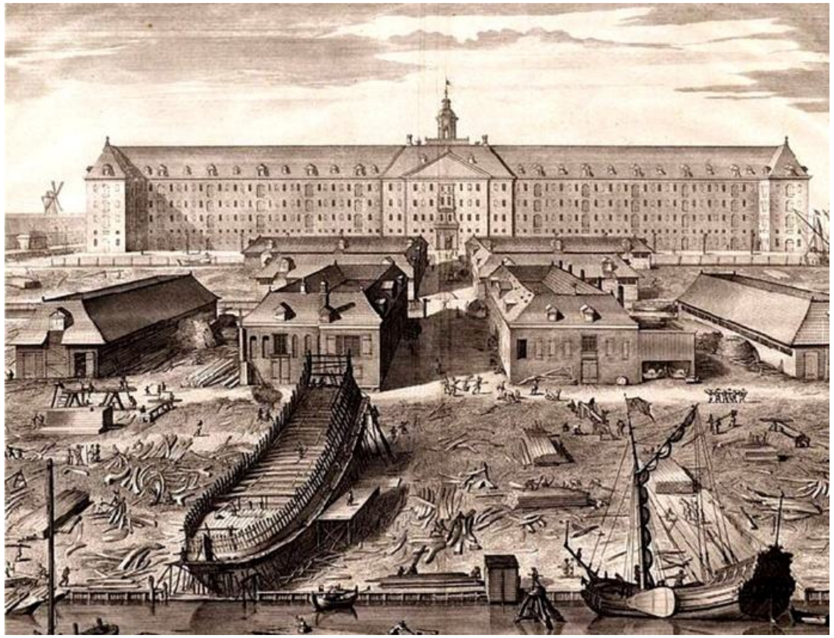
Netherland 1600s

What's the purpose of securities?

For the Issuer

Rise New Capital: Depending on the pricing and market demand, securities might be an attractive option

Repackaging: Achieve regulatory capital efficiencies.



Dutch East India Company

Investment in colony

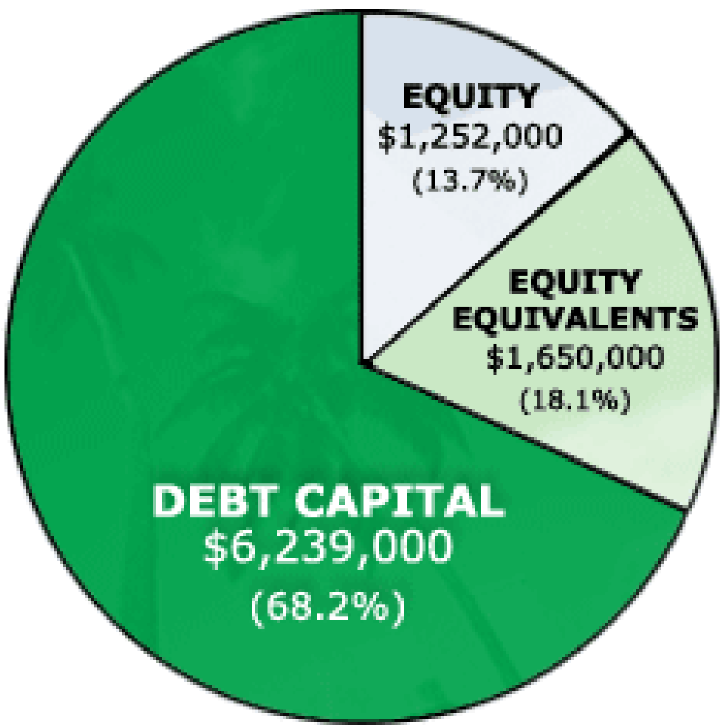
What's the purpose of securities?

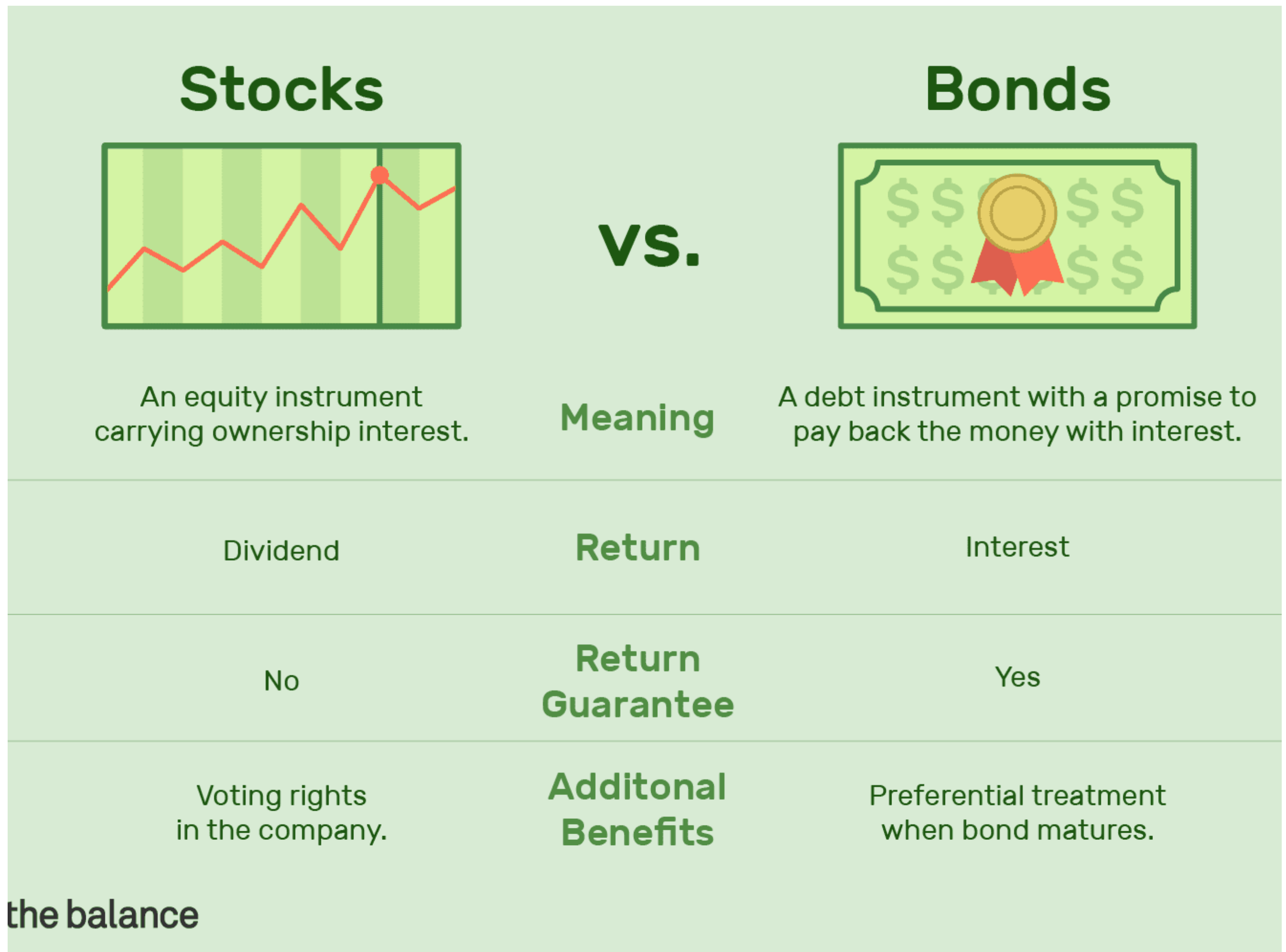
For the Holder

Investment: Debt securities generally offer a higher rate of interest than bank deposits, and equities may offer the prospect of capital growth.

Collateral: Purchasing securities with borrowed money secured by other securities.

Traditionally, securities are divided into debt securities and equity.



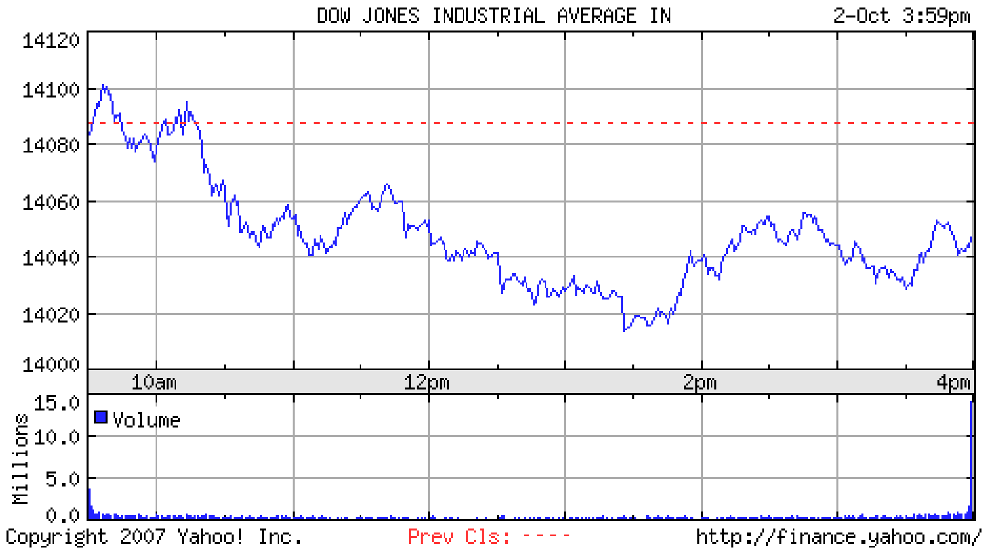


⇨ Debt securities may be called debentures, bonds, notes or commercial paper depending on their maturity and certain other characteristics.

⇨ The holder of a debt security is typically entitled to the payment of principal and interest, together with other contractual rights under the terms of the issue, such as the right to receive certain information.

⇨ Debt securities are generally issued for a fixed term and redeemable by the issuer at the end of that term.

Equity

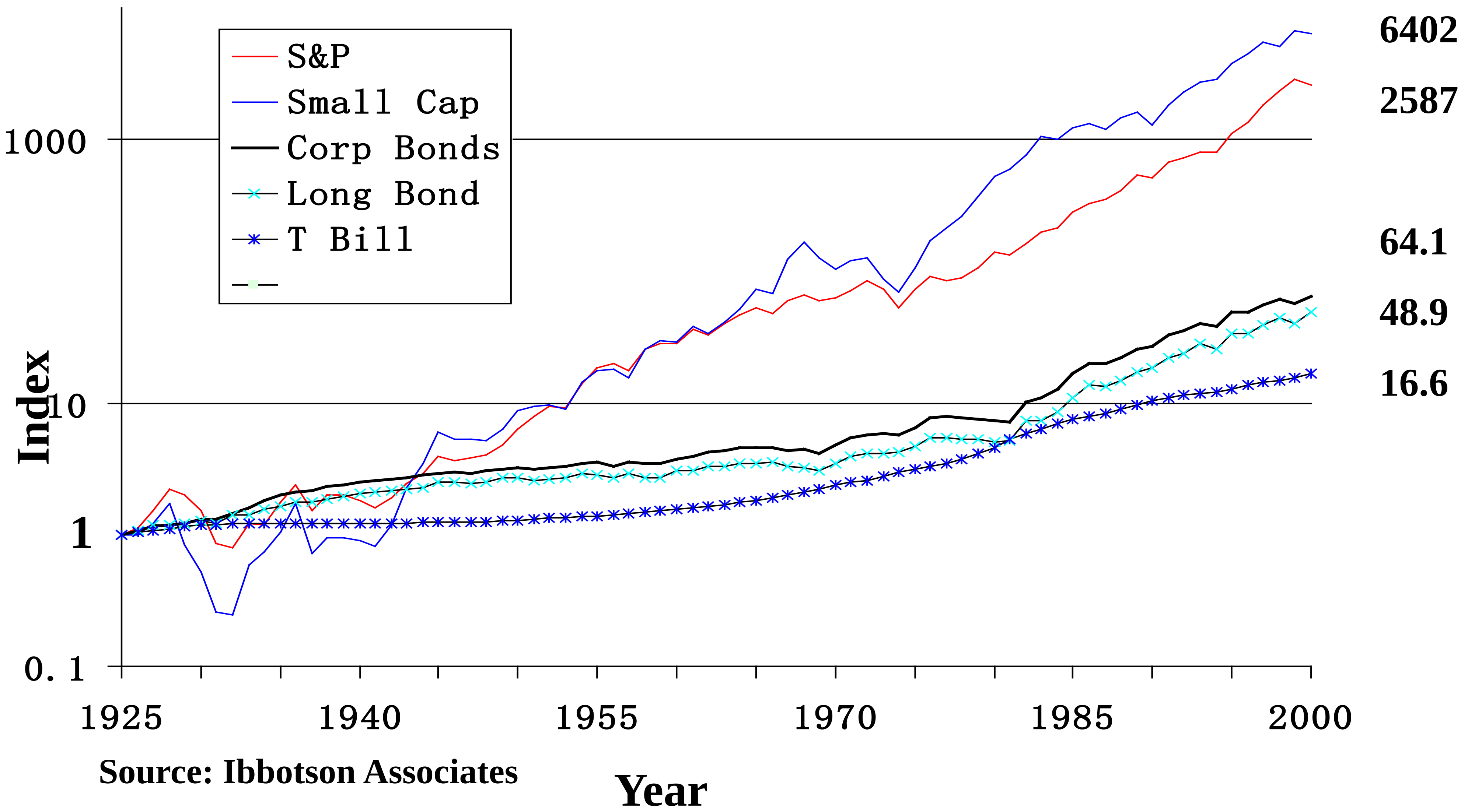


- An equity security is a share in the capital stock of a company (typically common stock, although preferred equity is also a form of capital stock).

- The holder of an equity is a shareholder, owning a share, or fractional part of the issuer. Unlike debt securities, which typically require regular payments (interest) to the holder, equity securities are not entitled to any payment.

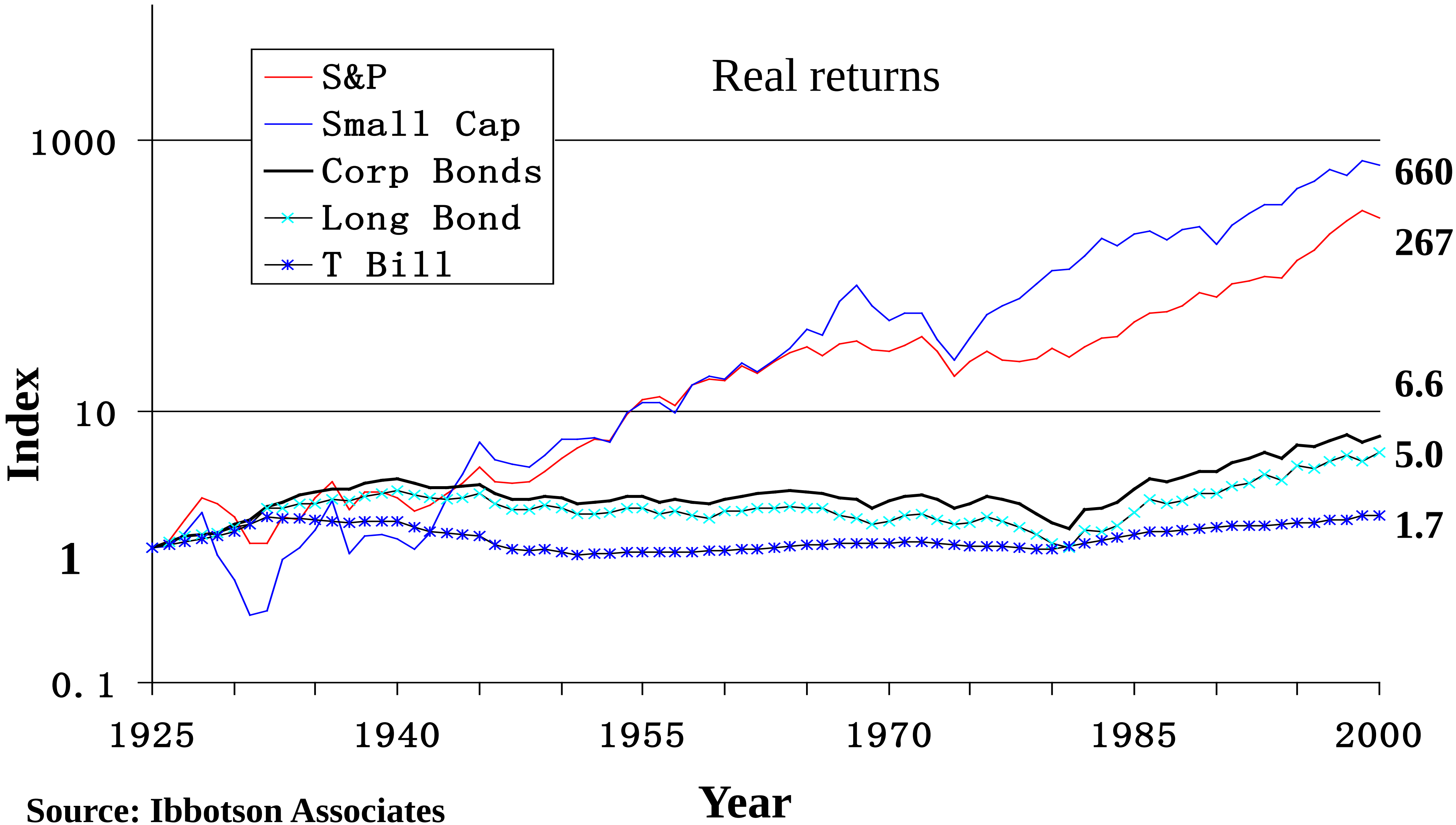
- Equity also enjoys the right to profits and capital gain.

The Value of an Investment of $1 in 1926



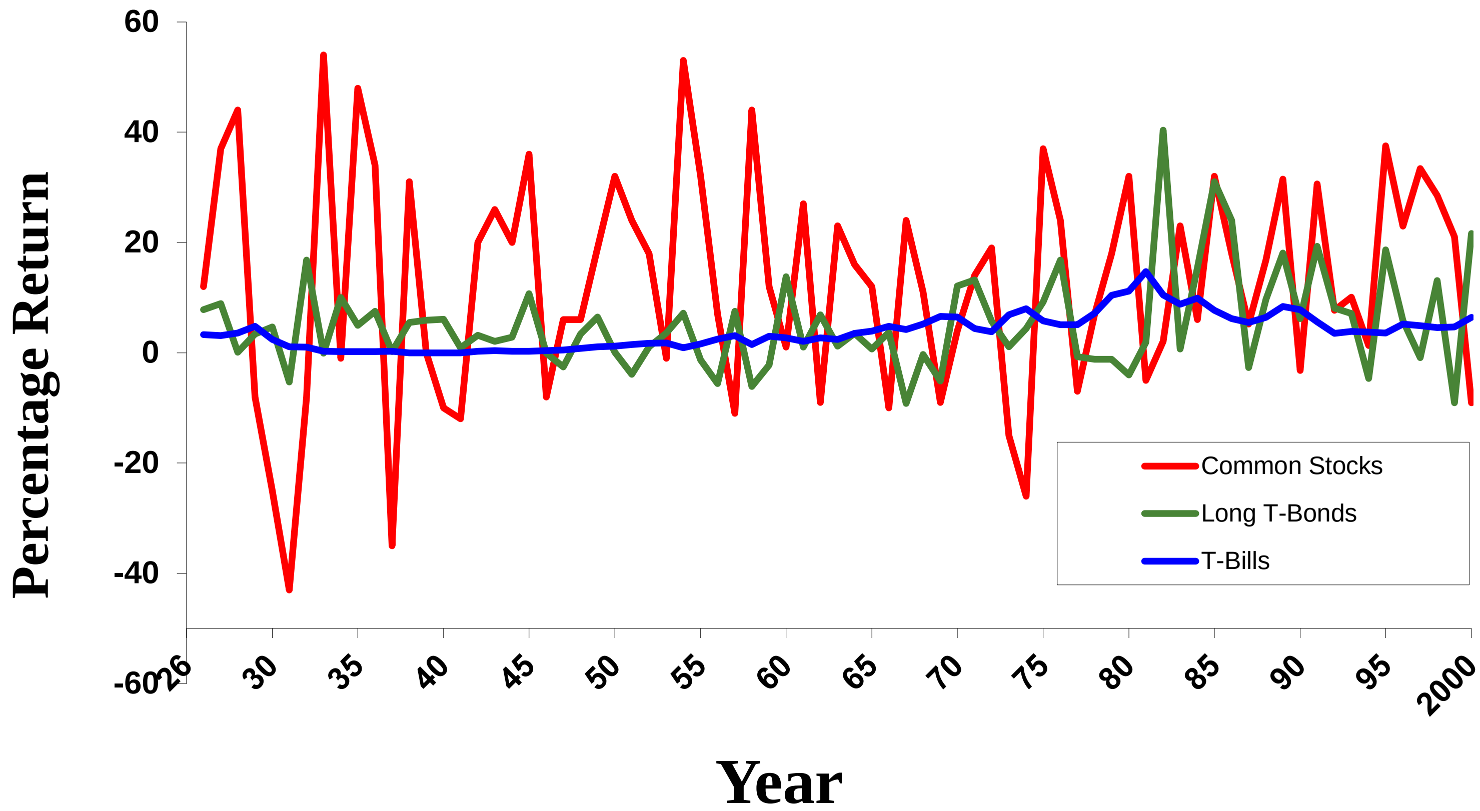
Source: Ibbotson Associates

The Value of an Investment of $1 in 1926



Source: Ibbotson Associates

Rates of Return 1926-2000



Source: Ibbotson Associates

Index of Chinese market

Year

Source: google finance

Index of Chinese market: SHA000001

Year

Source: SINA FINANCE

Financial Engineering

- Returns on investment are uncertain (risky)

- We model uncertainty of future returns using

  -Expected return: the return you expect to receive on average => NOT ENOUGH!

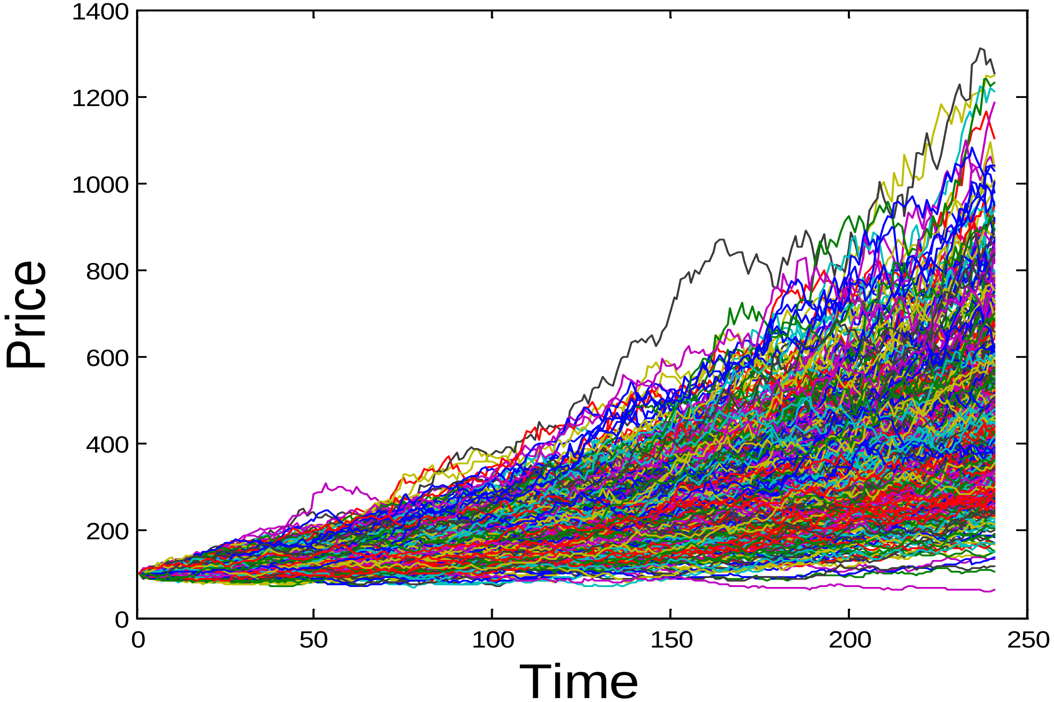
  -Volatility (standard deviation): degree of dispersion of future returns => RISK

❑ Risks

Risk：
Intentional interaction with uncertainty

Financial risk：
Danger or possibility that shareholders, investors, or other financial stakeholders will lose money. Uncertainty (volatility) of future price, interest rate or return rate

Original:
Asymmetry and incompleteness of information



Simulations of a piece process

Portfolio selection → reduce non systemic risk
Tools for portfolio selection： mathematical models

Financial decision making: from risk measure to portfolio selection

Content

Introduction

According to the type of financial securities considered:

- **Operational risk:** Changes in the value of the portfolio due to poor management or maintenance
- **Liquidity risk:** Difficult or impossible to sell and redeem your holdings
- **Exchange rate risk:**Changes in foreign investment return caused by changes in exchange rate
- **Credit risk:** The holder of the security cannot perform his/her obligations
- **Market risk:** Changes in portfolio return caused by change of market state
- etc.

The greater the risk, the greater the return or loss

How to control risk? How to balance the benefits and risks?

Introduction

Early method:

- ☞ focuses on qualitative research, risk only plays an auxiliary, explanatory role

Simple indicators widely used in empirical research:

- ☞ volatility, for single security's return
- ☞ duration or payback period, for fixed income securities, (valid) period
- ☞ Beta factor, for a portfolio,
- ☞ Convexity, first order Delta, second order Gamma, for derivative financial product,

Introduction

Definition of risk:

- Uncertainty of future investment results due to one or more uncertain factors

Risk measure:

- Some quantitative method for uncertainty of future investment results

Mathematically:

The risk can be viewed as a random variable X, defined in a probability space (Ω, κ, P):

- X represents the investment results. random return: $(X \geq 0)$, or random loss $(X < 0)$
- V: The set of random variable X, such as $L^P(\Omega, F, P)$
- Risk Measure $\rho : \mathrm{V} \rightarrow \mathrm{R}$. ρ corresponds to different forms of risk measures

Risk measure based on moment information

Risk measure based on the moment information of the return distribution

—The first important contribution in the quantification of risk measures

MV Model (Markowitz, 1952)

$$\text{variance}: \quad E\left[\left(X - E[X]\right)^2\right]$$

Reasonability:

- The variance describes how real random return deviates from its mean
- For normally distributed return, mean and variance determine the distribution
- Many utility functions can be approximated by quadratic functions of mean and variance of returns

Portfolio selection theory

Figure: Henry. M. Markowitz

- "*One day in 1950, in the library of the Business School of the University of Chicago, I was check- ing out the possibility of writing my Ph.D. dissertation ... to 'stock market'*. H. Markowitz" (OR 2002, Vol. 50).

- Previous theory on investment: J. Williams (1938) *The Theory of Investment Value*. The value of a stock is the expected presented value of its future dividends.
- An old saying "*not to put all one's eggs in one basket*"=>Diversification of the risk of a portfolio.
- Markowitz realized that the theory lacks an analysis of the impact of risk. This insight led to the development of his seminal theory of portfolio allocation under uncertainty, "Portfolio Selection", published in 1952 by *The Journal of Finance*.

Risk measure based on moment information

MV different forms of the model:

-Portfolio with n securities:

-Investment weights vector $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}} \in \mathbb{R}^n$,

-Random return vector $R = (R_1, R_2, \cdots, R_n)$,

-Mean vector $r = (r_1, r_2, \cdots, r_n)^{\mathrm{T}}$

-Covariance matrix $V = (\sigma_{ij})$,σ_{ij}: covariance between R_i and R_j,

Portfolio returns:

$$X = \sum_{i=1}^{n} \omega_i R_i \Longrightarrow E[X] = \sum_{i=1}^{n} \omega_i r_i = r^T \omega, \quad \omega^T V \omega$$

$$\begin{aligned} \max \quad & r^T \omega \\ \text{s.t.} \quad & e^T \omega = 1, \\ & \omega^T V \omega = (\leqslant) \bar{\rho}, \end{aligned} \qquad \begin{aligned} \min \quad & \omega^T V \omega \\ \text{s.t.} \quad & e^T \omega = 1, \\ & r^T \omega = (\geqslant) \bar{r} \end{aligned} \qquad \begin{aligned} \max \quad & r^T \omega - \lambda \frac{1}{2} \omega^T V \omega \\ \text{s.t.} \quad & e^T \omega = 1, \\ & e = (1, 1, \cdots, 1)^T \in \mathbb{R}^n, \end{aligned}$$

Risk measure based on moment information

Consider other constraints?

- ➢ **Market friction:** transaction cost (Atkinson & Alvarez, 2001)
- ➢ **Multi-stage MV Model:** Steinbach (2001)

Deficiency of MV model:

① For large-scale portfolios, $\sum$: computationally expensive and difficult to estimate accurately. The effect of estimation error Chen & Zhao (2003, 2004)

② The distribution of returns are often obvious non-normal, fat-tailed, left-skewed. Only one or two order moments cannot fully reflect the randomness of income.

③ The quadratic utility function implied by the MV model is irrational. Exceeding a certain critical point will lead to increasing risk aversion level and negative marginal utility at some sharp points.

Risk measure based on moment information

To overcome the first deficiency: how to effectively solve large-scale MV model

using the factor model (Perold, 1981)

$$R_i = \alpha_i + \beta_{i1} F_1 + \cdots + \beta_{ik} F_k + \varepsilon_i, \quad i = 1, 2, \cdots, n,$$

Where F_k is the k-th random factor, ε_i is the random error term with $E(\varepsilon_i) = 0$, ε_i is unrelated to $F_k\,(k = 1, 2, \cdots, \kappa), \varepsilon_i\,(j \neq i)$.

Let

$$\sigma_i^2 = E[\varepsilon_i^2], \quad f_{rs} = \mathrm{cov}[F_r, F_s].$$

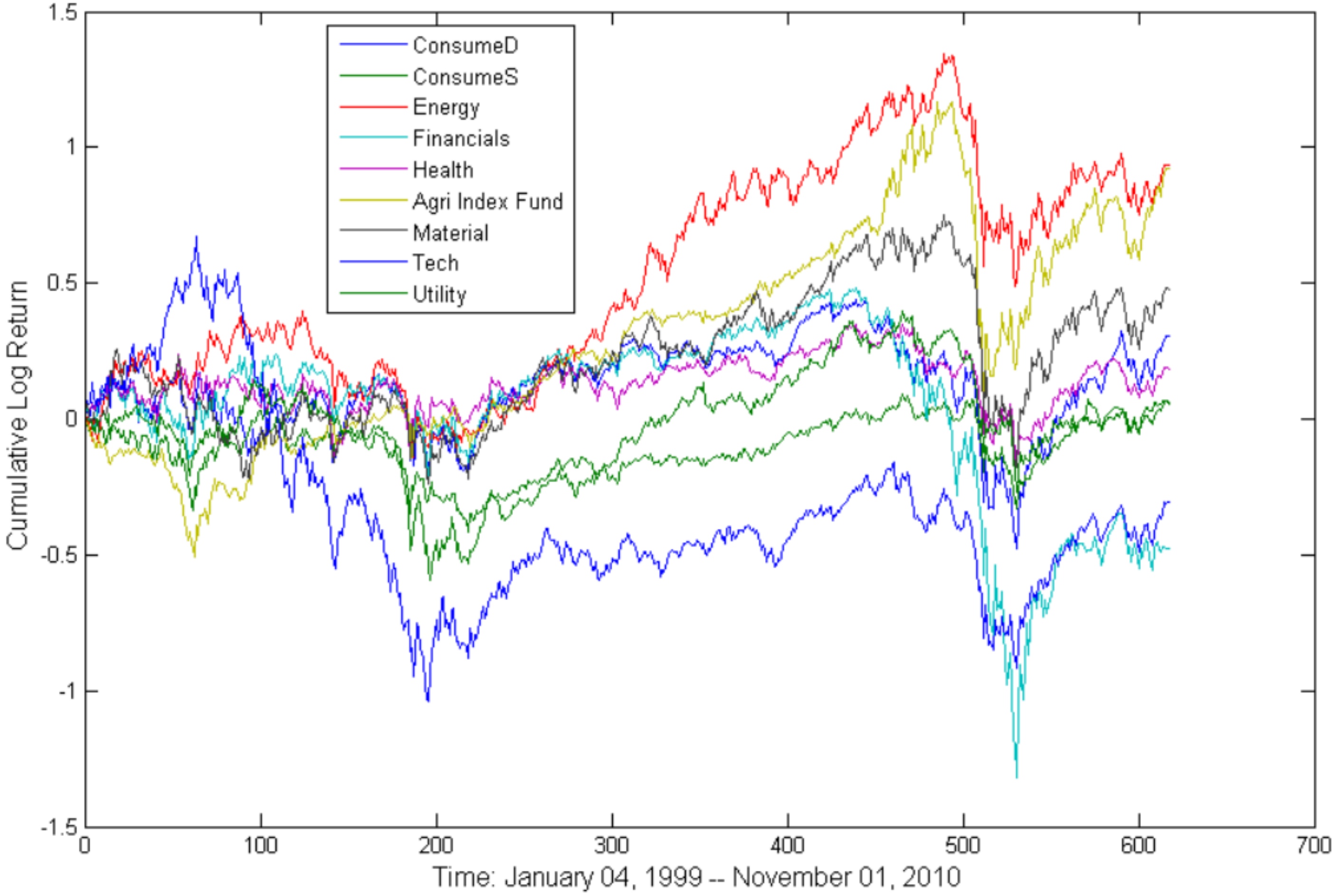
$$\omega^T V \omega \to \sum_{i=1}^{n} \sum_{j=1}^{n} \widehat{\sigma}_{ij} \omega_i \omega_j = \sum_{i=1}^{n} \sigma_i^2 \omega_i^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{r=1}^{k} \sum_{s=1}^{k} f_{rs} \beta_{ir} \beta_{js} \omega_i \omega_j.$$

Risk measure based on moment information

MV model can be formulated as

$$\begin{aligned}
\min \quad & \sum_{i=1}^{n} \sigma_i^2 \omega_i^2 + \sum_{r=1}^{k} \sum_{s=1}^{k} f_{rs} y_r y_s \\
\text{s.t.} \quad & r^T \omega \geqslant \bar{r}, \\
& \sum_{i=1}^{n} \beta_{jk} \omega_j - y_k = 0, \quad k = 1, 2, \cdots, \kappa, \\
& \sum_{j=1}^{n} \omega_j = 1, \\
& \omega_j \geqslant 0, \quad j = 1, 2, \cdots, n,
\end{aligned}$$

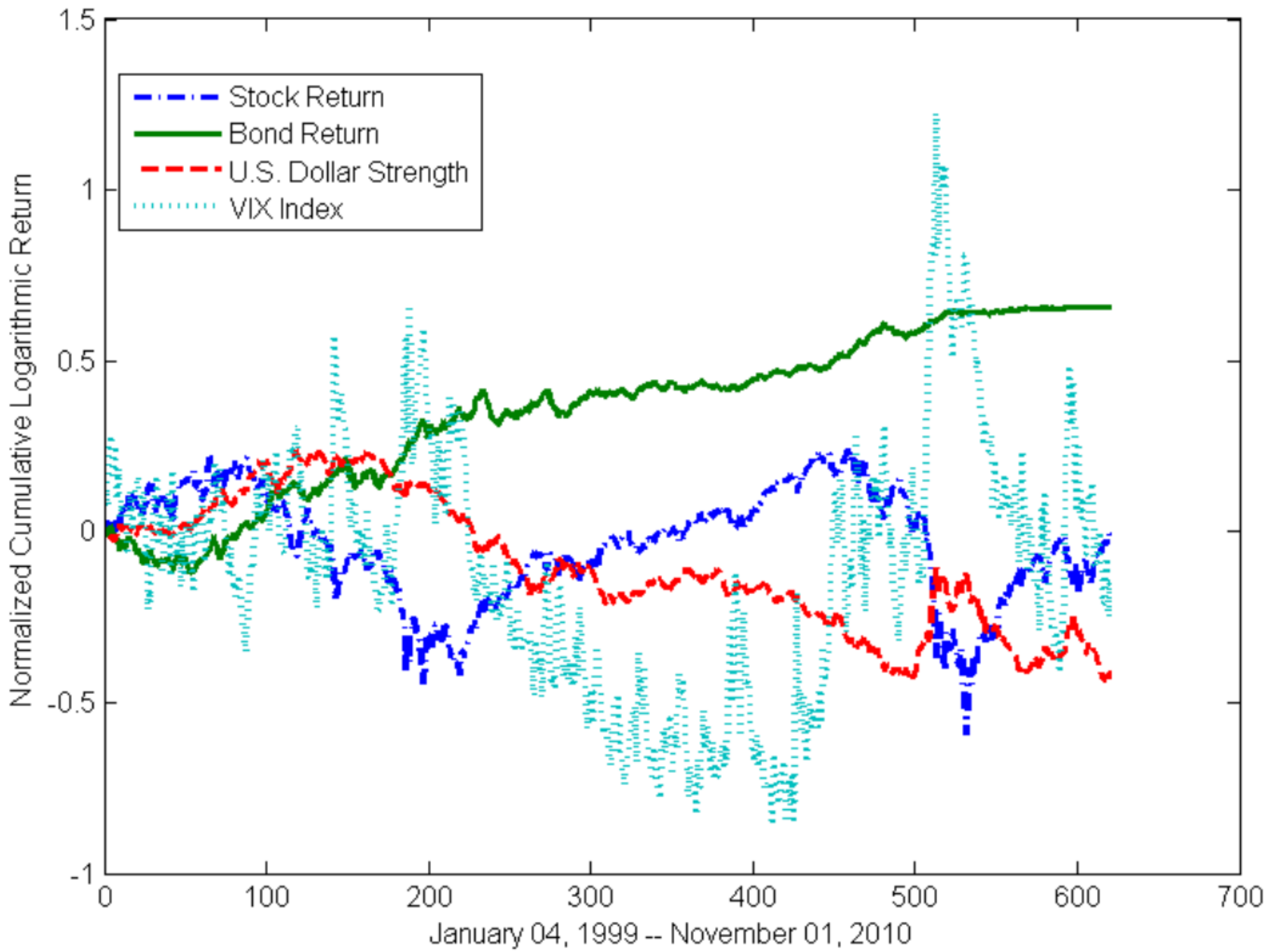
When k is large, $B = (\beta_{jk}) \in \mathbb{R}^{n \times k}$ is sparse. And k is usually far smaller than n. Solve efficiently using sparse optimization techniques.



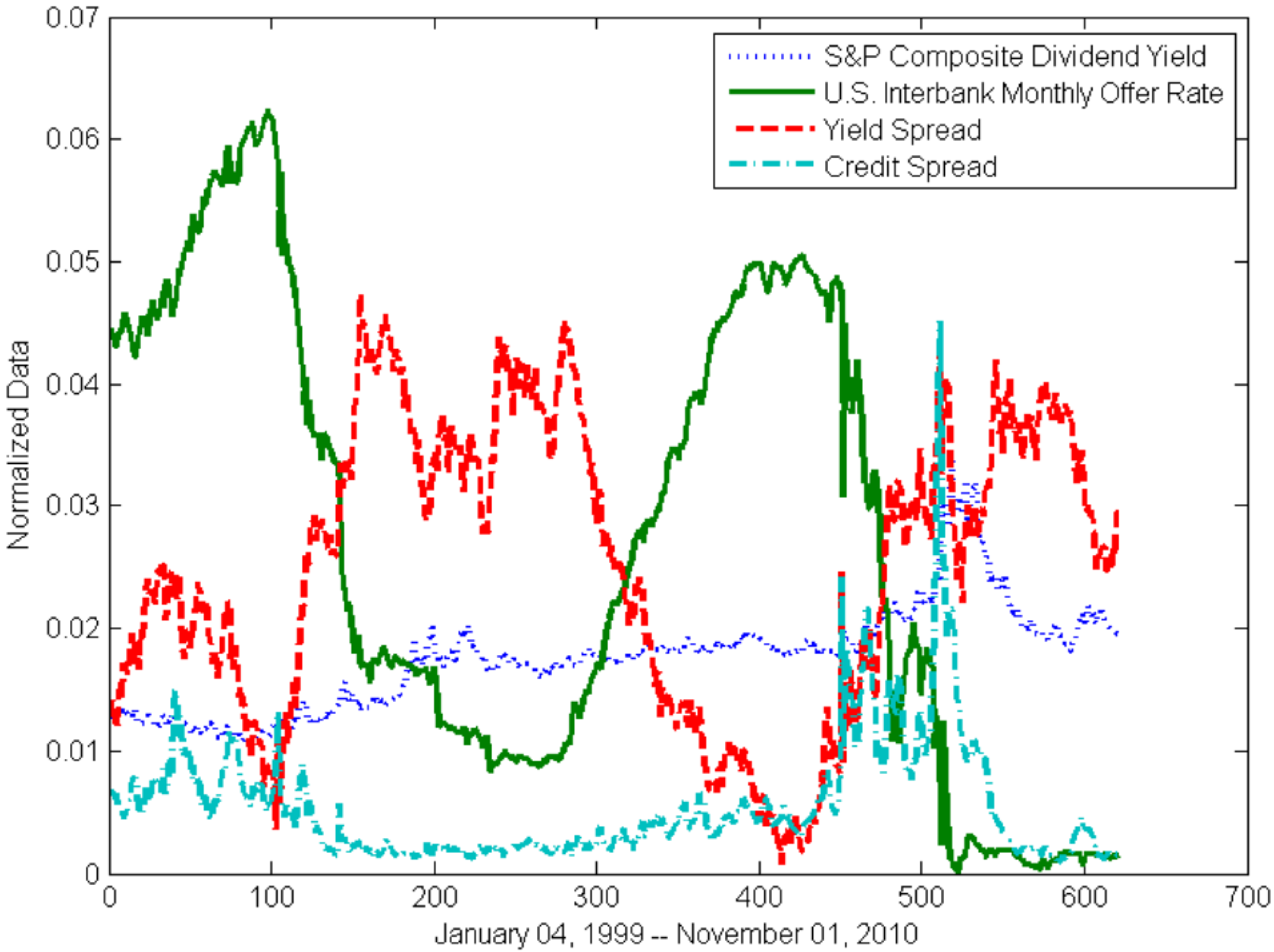
Indicators in different sectors

Economic Indicators

- Current stock market return (STK): Log return on the S&P 500 Price Index.
- Current bond market return (BND): Log return on the 10 Year U.S. Treasury Bond.
- Current currency strength (USD): Log changes in the Dollar Index.
- Volatility (VIX): Measured as the standard deviation of short term stock returns.
- Dividend yield (EDY): S&P 500 Aggregate Dividend Yield.
- Interest rate (UIR) : U.S. Interbank Offer Rate.
- Yield spread (TYS): 10 year U.S. Treasury Bond - 3 Month T-Bill.
- Credit spread (UCS): U.S. Corporate BAA - U.S. Corporate AAA.



- Stocks and VIX move in opposite directions (correlation: -75%).
- Stocks and bonds sometimes move together.
- Most of the time, stocks and the U.S. currency move in opposite directions.



Interest rate and yield spread move in opposite directions.
Credit spread is usually high, when interest rate is low.

Risk measure based on moment information

The tight decomposition of variance-covariance matrices, (Konno &Suzuki, 1992)

Let $(r_{1t}, r_{2t}, \cdots, r_{mt}),\ t = 1, 2, \cdots, T$ be T independent samples of $R = (R_1, R_2, \cdots, R_n)^T$,

$$\widehat{r}_i = \frac{1}{T}\sum_{t=1}^{T} r_{it}, \quad \widehat{\sigma}_{ij} = \frac{1}{T}\sum_{t=1}^{T}(r_{it} - \widehat{r}_i)(r_{jt} - \widehat{r}_j), \quad i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, n$$

Where $\widehat{r}_i$, $\widehat{\sigma}_{ij}$ is an unbiased estimate of r_j, σ_{ij}

$$\omega^T V \omega \to \sum_{i=1}^{n}\sum_{j=1}^{n}\widehat{\sigma}_{ij}\omega_i\omega_j = \sum_{i=1}^{n}\sum_{j=1}^{n}\Big\{\frac{1}{T}\sum_{t=1}^{T}(r_{it} - \widehat{r}_i)(r_{jt} - \widehat{r}_j)\Big\}\omega_i\omega_j$$

$$= \frac{1}{T}\sum_{t=1}^{T}\Big\{\sum_{i=1}^{n}(r_{it} - \widehat{r}_i)\omega_i\Big\}^2.$$

Risk measure based on moment information

Let

$$z_t = \sum_{i=1}^{n} (r_{it} - \widehat{r}_i)\omega_i, \quad t = 1, 2, \cdots, T,$$

MV model can be formulated as

$$\begin{aligned}
\min \quad & \sum_{t=1}^{T} z_t^2 \\
\text{s.t.} \quad & r^T \omega \geqslant \overline{r}, \\
& z_t = \sum_{j=1}^{n} (r_{jt} - \widehat{r}_j)\omega_j, \quad t = 1, 2, \cdots, T, \\
& \sum_{j=1}^{n} \omega_j = 1, \\
& \omega_j \geqslant 0, \quad j = 1, 2, \cdots, n.
\end{aligned}$$

Risk measure based on moment information

Absolute deviation measure, (Konno & Yamazaki, 1991)

$$AD = E[|X - E[X]|]$$

Normal distribution

$$AD = \sqrt{\frac{2}{\pi}}\sqrt{E[(X - E(X))^2]}$$

MAD model

$$\begin{aligned}
\min \quad & E\Big[\Big|\sum_{j=1}^{n} R_j\omega_j - E\Big[\sum_{j=1}^{n} R_j\omega_j\Big]\Big|\Big] \\
\text{s.t.} \quad & r^T\omega \geqslant \overline{r}, \\
& e^T\omega = 1, \\
& \omega_j \geqslant 0,\ j = 1, 2, \cdots, n.
\end{aligned}$$

Risk measure based on moment information

Based on historical data

$$AD = \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{j=1}^{n}(r_{jt}-r_j)\omega_j\right|,$$

implies

$$\begin{aligned}
\min \quad & \sum_{t=1}^{T}\left|\sum_{j=1}^{n}(r_{jt}-r_j)\omega_j\right| \\
\text{s.t.} \quad & \sum_{j=1}^{n} r_j\omega_j \geqslant \bar{r}, \\
& \sum_{j=1}^{n}\omega_j = 1, \\
& \omega_j \geqslant 0, \quad j = 1, 2, \ldots, n,
\end{aligned}$$

Risk measure based on moment information

and

$$
\begin{aligned}
\min \quad & \frac{1}{T}\sum_{t=1}^{T} y_t \\
\text{s.t.} \quad & y_t - \sum_{j=1}^{n} (r_{jt} - r_j)\omega_j \geqslant 0, \quad t = 1, 2, \cdots, T, \\
& y_t + \sum_{j=1}^{n} (r_{jt} - r_j)\omega_j \geqslant 0, \quad t = 1, 2, \cdots, T, \\
& \sum_{j=1}^{n} r_j \omega_j \geqslant \overline{r}, \\
& \sum_{j=1}^{n} \omega_j = 1, \\
& \omega_j \geqslant 0, \quad j = 1, 2, \ldots, n.
\end{aligned}
$$

Risk measure based on moment information

To overcome the second deficiency: use higher order moment information

Skewness (third order moment):

$$\kappa(X) = \frac{E\left[\left(X - E(X)\right)^3\right]}{E\left[\left(X - E(X)\right)^2\right]^{\frac{3}{2}}}.$$

When the mean and variance are the same, investors will choose a portfolio with larger third-order moment, and even place the third-order moment in a more important position.

Third-order center moment (Konno *et al*, 1993)

$$\gamma\left[R(X)\right] = E\left[\left(X - E(X)\right)^3\right],$$

$$v_{ijk} = E\left[(R_i - r_i)(R_j - r_j)(R_k - r_k)\right].$$

Risk measure based on moment information

MVS (mean-variance-skewness) model

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} v_{ijk}\omega_i\omega_j\omega_k \\
\text{s.t.} \quad & \omega^T V\omega = \overline{\rho}, \\
& r^T\omega = \overline{r}, \\
& e^T\omega = 1, \\
& \omega \geqslant 0.
\end{aligned}
$$

Risk measure based on moment information

Higher order moments?

Forth order moments: kurtosis $\Longrightarrow$minimize kurtosis

Better way to characterize the skewed, high kurtosis and fat-tail distribution?

- Generalized error distribution
- Extreme value distribution $\rightarrow$EVT
- Stable distribution, its characteristic function is

$$\phi_R(t) = \begin{cases} \exp\{-\gamma^\tau |t|^\tau (1 - i\eta\,\mathrm{sgn}(t)\tan(\frac{\pi\tau}{2})) + i\delta t\}, \tau \neq 1, \\ \exp\{-\gamma|t|(1 - i\eta\frac{2}{\pi}\,\mathrm{sgn}(t)\log(t)) + i\delta t\}, \quad \tau = 1, \end{cases}$$

where $\tau \in (0, 2]$ is the stable indicator, i.e., kurtosis parameter $\eta \in [-1, 1]$ is the skewness parameter; $\delta \in \mathbb{R}$ is the location parameter;$\gamma \in \mathbb{R}^+$ is the dispersion parameter; $\tau = 2,\ \eta = 0$: normal;$\tau < 2$: high kurtosis and fat-tailed $\eta > 0\,(\eta < 0)$ right (left) skewed.

Risk measure based on moment information

Skewed distribution

Skewed normal distribution SN:

$$x \sim SN_n(\varsigma, \Omega, \alpha),$$

density

$$f(x) = 2\phi_n(z - \varsigma; \Omega)\Phi\left[\alpha^T \omega^{-1}(z - \varsigma)\right], \quad x \in \mathbb{R}^n,$$

where $\phi_n(\cdot)$ is the density of n-dimensional normal distribution, $\Phi(\cdot) \sim N(0,1)$; $\varsigma = (\varsigma_1, \varsigma_2, \cdots, \varsigma_n)^T$ is location parameter; $\omega = \text{diag}\,(\sigma_{11}, \sigma_{22}, \cdots, \sigma_{nn})$ is scalar parameter; $\alpha \in \mathbb{R}^n$ is shape parameter; $\alpha = 0$: normal, the absolute value of α is bigger, the skewness is larger; $\alpha \to \infty$, $f(\cdot) \to$ is half-normal density function.

Risk measure based on moment information

Skewed t distribution

$$S_t : x \sim S_{t_n}(\varsigma, \Omega, \alpha, \gamma), \quad x = \varsigma + V^{-\frac{1}{2}} Z, \quad Z \sim SN_n(0, \Omega, \alpha), \quad V \sim \frac{\varphi_\gamma^2}{\gamma},$$

is dependent from Z.

General skewed distribution

Perturbation of skewness factor to a symmetric distribution

$$f(x) = 2 f_0(x) G\big[\omega(x)\big], \quad x \in \mathbb{R}^n$$

f is a n-dimensional density function, if the density f_0 is symmetry;
G is a 1-dimensional distribution function, satisfying $G(-x) = 1 - G(x)$;
$\omega(x) : R^n \to R,\ \omega(-x) = -\omega(x)$;
$G[\omega(x)]$ is the skewness factor bringed by $f_0(x)$.
Different f_0, G and $\omega(x)$ correspond to different skewed distribution.

Experiment 1			
Gamble 1A		Gamble 1B	
Winnings	Chance	Winnings	Chance
$1 million	100%	$1 million	89%
		Nothing	1%
		$5 million	10%

Experiment 2			
Gamble 2A		Gamble 2B	
Winnings	Chance	Winnings	Chance
Nothing	89%	Nothing	90%
$1 million	11%		
		$5 million	10%

Experiment 1				Experiment 2			
Gamble 1A		Gamble 1B		Gamble 2A		Gamble 2B	
Winnings	Chance	Winnings	Chance	Winnings	Chance	Winnings	Chance
$1 million	89%	$1 million	89%	Nothing	89%	Nothing	89%
$1 million	11%	Nothing	1%	$1 million	11%	Nothing	1%
		$5 million	10%			$5 million	10%

A = B

Allais paradox

Risk measure based on moment information

Downside risk measure

- Variance: view biased smaller than or larger than expected value the same – both large return and big loss are risk
- Investors: care about big losses rather than large return: Real risk

Roy's safety first technique (Roy, 1952)

Downside risk measure avoid normality: partial variance, semi-variance

- Below-target semivariance

$$E[(\max(0, T - X))^2]$$

  T is the target return

- Below-mean semivariance

we choose $T = E[x]$

Risk measure based on moment information

- Combined semivariance (Hamza & Janssen, 1998)

$$\alpha E\Big[\big(\min\big(0, E[x]-x\big)\big)^2\Big] + \beta E\Big[\big(\max\big(0, E[x]-x\big)\big)^2\Big], \quad \alpha, \beta > 0.$$

- LPM: lower partial moment (Bawa 1975; Fishburn 1977)

$$LPM(\alpha, T) = E\Big[\big(\max(0, T-X)\big)^{\alpha}\Big],$$

$$\alpha \begin{cases} < 1, \text{risk preference}, \\ = 1, \quad \text{risk neutral}, \\ > 1, \quad \text{risk aversion}, \end{cases}$$

Different from variance and semi-variance, LPM corresponds to a series of utility functions.

Risk measure based on moment information

MADS (mean-absolute deviation-skewness) model

Third-order lower partial moment

$$\gamma_-(x) = E\big[g\big(x - E[x]\big)\big], \quad g(u) = \begin{cases} 0, \ u \geqslant 0, \\ u^3, u < 0, \end{cases}$$

$$\begin{aligned}
\min \quad & E\Big[g\Big(\sum_{j=1}^{n}(R_j - r_j)\omega_j\Big)\Big] \\
\text{s.t.} \quad & E\Big[\Big|\sum_{j=1}^{n} R_j\omega_j - E\Big[\sum_{j=1}^{n} R_j\omega_j\Big]\Big|\Big] \leqslant \overline{\omega}, \\
& r^T\omega = \overline{r}, \\
& e^T\omega = 1,
\end{aligned}$$

Risk measure based on moment information

$$g(\cdot) \to G(u) = -|u - \rho_1|_- - \alpha|u - \rho_2|_-,$$

$$\rho_2 < \rho_1 < 0, \quad \alpha > 0, \quad |v|_- = \begin{cases} 0, & v \geqslant 0, \\ -v, & v < 0. \end{cases}$$

$$\begin{aligned} \min \quad & E\Big[\Big|\sum_{j=1}^{n} R_j\omega_j - \rho_1\Big|_-\Big] + \alpha E\Big[\Big|\sum_{j=1}^{n} R_j\omega_j - \rho_2\Big|_-\Big], \\ \text{s.t.} \quad & E\Big[\Big|\sum_{j=1}^{n} R_j\omega_j - E\Big[\sum_{j=1}^{n} R_j\omega_j\Big]\Big|\Big] \leqslant \overline{\omega}, \\ & r^T\omega = \overline{r}, \quad e^T\omega = 1, \quad \omega \geqslant 0. \end{aligned}$$

Risk measure based on moment information

Based on T historical data, $r_{jt},\ j=1,\ldots,n,\quad t=1,\ldots,T \Longrightarrow$

$$
\begin{aligned}
\min \quad & \frac{1}{T-1}\Big(\sum_{t=1}^{T} u_t + \alpha \sum_{t=1}^{T} v_t\Big) \\
\text{s.t.} \quad & u_t + \sum_{j=1}^{n} r_{jt}\omega_j \geqslant \rho_1, \quad t = 1, 2, \cdots, T, \\
& v_t + \sum_{j=1}^{n} r_{jt}\omega_j \geqslant \rho_2, \quad t = 1, 2, \cdots, T, \\
& \varsigma_t - \eta_t - \sum_{j=1}^{n} r_{jt}\omega_j = \overline{r}, \quad t = 1, 2, \cdots, T, \\
& \sum_{t=1}^{T} (\varsigma_t + \eta_t) \leqslant \overline{\omega}, \quad \sum_{j=1}^{n} r_j\omega_j = \overline{r}, \\
& \sum_{j=1}^{n} \omega_j = 1, \quad \omega \geqslant 0, \quad i = 1, 2, \cdots, n, \\
& u_t \geqslant 0, \quad v_t \geqslant 0, \quad \varsigma_t \geqslant 0, \quad \eta_t \geqslant 0, \quad t = 1, 2, \cdots, T.
\end{aligned}
$$

Risk measure based on moment information

co-LPM→GCLPM (generalized or asymetric co-LPM)

$$GCLPM_n(\tau, R_i, R_j) = \int_{-\infty}^{\tau} \int_{-\infty}^{+\infty} (\tau - R_i)^{n-1} (\tau - R_j) \mathrm{d}F(R_i, R_j),$$

$$GCLPM_n(\tau, R_i, R_j) \neq GCLPM_n(\tau, R_j, R_i)$$

If $R_i = R_j$, $LPM_n(\tau, R_i)$

Discrete case

$$GCLPM_n(\tau, R_i, R_j) = \frac{1}{T-1} \sum_{t=1}^{T} \left[\max\left(0, (\tau - R_{it})\right)\right]^{n-1} (\tau - R_{jt})$$

→portfolio optimization

Risk measure based on moment information

A general formulation? (Kijima & Ohnishi, 1993)

$$\sigma_k(x;f) = \left\{E\left[f\left(x - E[x]\right)^k\right]\right\}^{\frac{1}{k}}, \quad k \geqslant 1,$$

$$f(y) = |y| \to \left\{E\left[f\left(x - E[x]\right)^k\right]\right\}^{\frac{1}{k}},$$

when $k = 1$: absolute deviation; when $k = 2$ standard deviation.

when $k = \infty$, L_∞ risk measure: $\max_{1 \leqslant j \leqslant n} E[x_j - Ex_j]$, x_j is the j-th component of x

$$\to f(y) = \begin{cases} c_+ x, x \geqslant 0, \\ c_- x, x < 0, \end{cases}$$

$c_+ = 0,\ c_- = -1$: different kinds of LPM measure

Risk measure based on moment information

Exponentially weighted mean square risk

$$E\big[\omega(X)(X-T)^2\big],$$

T: target return rate, $\omega(X) > 0$: weight function.

When target is the expected return rate

$$T = E[X], \quad \begin{cases} w(X) \equiv 1, & \text{variance,} \\ w(X) = \begin{cases} 1, X < E(X), \\ 0, X > E(X), \end{cases} & \text{semivariance,} \end{cases}$$

The choice of $\omega(X) > 0$: $w(X) = \exp\big(-\theta(X-T)\big),\ \theta > 0$

Constructing asymmetric risk measures from the perspective of approximating utility functions (King, 1993)

Risk measure based on moment information

Application

funds performance evaluation, asset ranking, corresponding investment optimization method

typical indicator

Sharpe ratio $$\Phi(X) = \frac{E[X]}{E[(X - E[X])^2]^{\frac{1}{2}}}$$

Treynor ratio $$\Phi(X) = \frac{E[X] - R_f}{\beta},$$

R_f: risk-free return rate, $\beta = \frac{\mathrm{cov}(X, R_M)}{\mathrm{Var}(R_M)}$

Lower Partial Variance Indicator

$$\Phi(X) = \frac{E[X] - R_f}{\sqrt{E[\min\{X - E[X], 0\}]^2}}$$

Risk measure based on moment information

Two-sided performance indicators

The Sortino-Satchell ratio $$\Phi^q(X) = \frac{E[X]}{E^{\frac{1}{q}}[(X^-)^q]}$$

The Stable ratio $$\Phi_\alpha^p(X) = \frac{E[X]}{A(p,\alpha)^{\frac{1}{p}} E[|X|^p]^{\frac{1}{p}}}$$

$$A(p;\alpha) = \frac{\sqrt{\pi}\,\Gamma(1-\frac{p}{2})}{2^p\,\Gamma(\frac{(1+p)}{2})\Gamma(1-\frac{1-p}{\alpha})}$$

α, stable indicator, $0 \leq p \leqslant \alpha$

Risk measure based on moment information

The Rachev ratio

$$\rho(X) = \frac{\mathrm{CVaR}_{(1-\alpha)\%}(r_f - X)}{\mathrm{CVaR}_{(1-\beta)\%}(X - r_f)}, \quad \mathrm{OR} = \frac{E[X \mid X \geqslant -\mathrm{VaR}_{(1-\alpha)}]}{E[-X \mid X \leqslant -\mathrm{VaR}_{\beta}]}$$

The Generalized Rachev ratio

$$\rho(X) = \frac{E[(X^+)^\gamma \mid X \geqslant -\mathrm{VaR}_{(1-\alpha)}]}{E[-(X^-)^\delta \mid X \leqslant -\mathrm{VaR}_{\beta}]}$$

The Farinelli—Tibiletti ratio

$$\Phi_b^{p,q}(X) = \frac{E^{\frac{1}{p}}[\{(X-b)^+\}^p]}{E^{\frac{1}{q}}[\{(X-b)^-\}^q]}$$

Risk measure based on moment information

Reward $\rightarrow$ upside variability (portfolio managers view)—"good"
Risk $\rightarrow$ downside risk (risk managers view)—"bad"

Choose proper p, q to reflect the importance of the data in the left tail or right tail biased from the benchmark

- p, q is larger, tail effect is more important,
  p, q is smaller, tail effect is less important,
- when p or q is smaller than 1, opposite effect.

For asset allocation or portfolio selection problem: $\max$ "Φ"

Stochastic dominance

Stochastic dominance criteria

General method for comparing uncertain and stochastic phenomena.

SD—stochastic dominance

- Hardy (1934), Marshall & Olkin (1979) et al.
- majorization theory
- Fishburn (1977), Rothschild & Stiglitz (1970) et al.
- General distribution, widely used in economic and financial theoretical research,
- Review literature Bawa (1982), Levy (1992)

idea:

- Compare pointwise the recursive distribution functions defined by the cumulative distribution function of random variables

Stochastic dominance

Let the probability measure of X be P_X

$$F_X^{(1)}(\eta) = F_X(\eta) = \int_{-\infty}^{\eta} P_X(d\varsigma) = P\{X \leqslant \eta\}, \quad \forall\, \eta \in \mathbb{R}$$

Cumulative distribution function

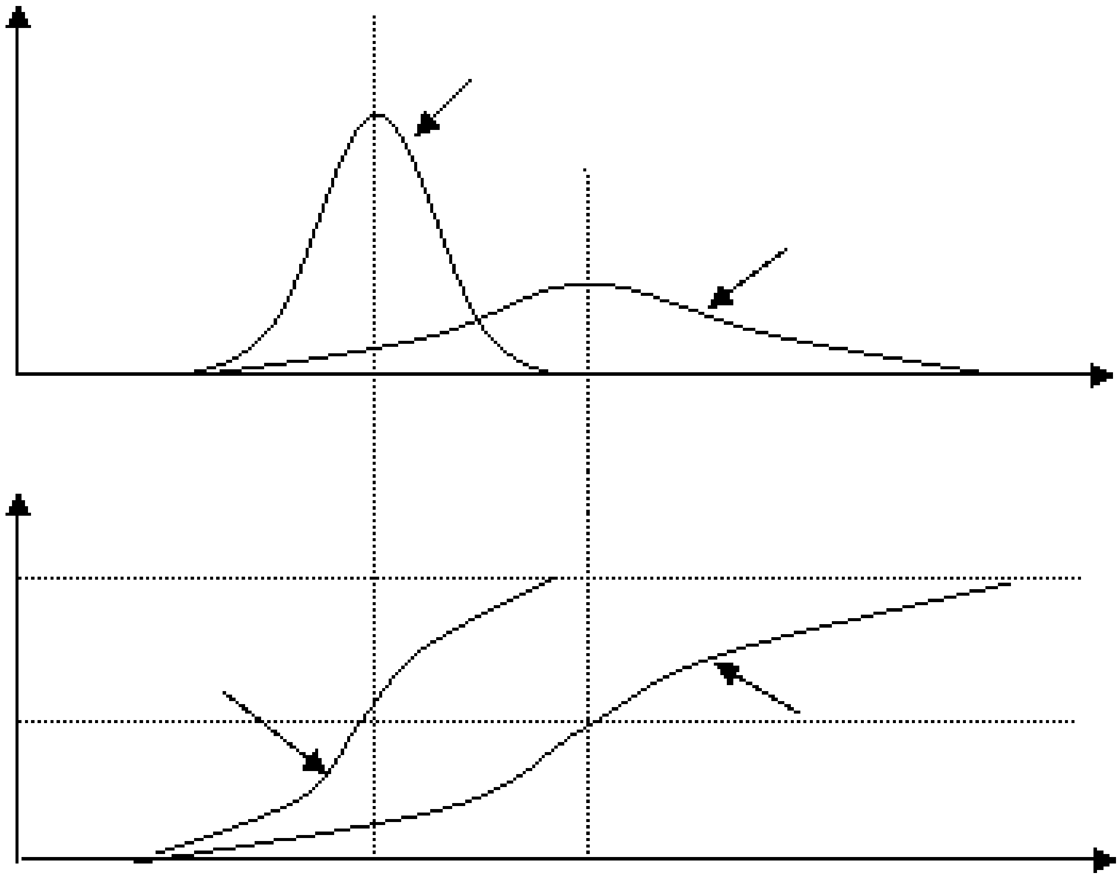
$$F_x^{(k)}(\eta) = \int_{-\infty}^{\eta} F_x^{(k-1)}(\xi)\mathrm{d}\xi, \quad k = 2, 3, \cdots, \quad \forall\, \eta \in \mathbb{R}$$

Definition:We call X dominates Y in the kth order, if

$$F_x^{(k)}(\eta) \leqslant F_y^{(k)}(\eta), \quad \forall\, \eta \in \mathbb{R}$$

and for some η, the inequality holds strictly.

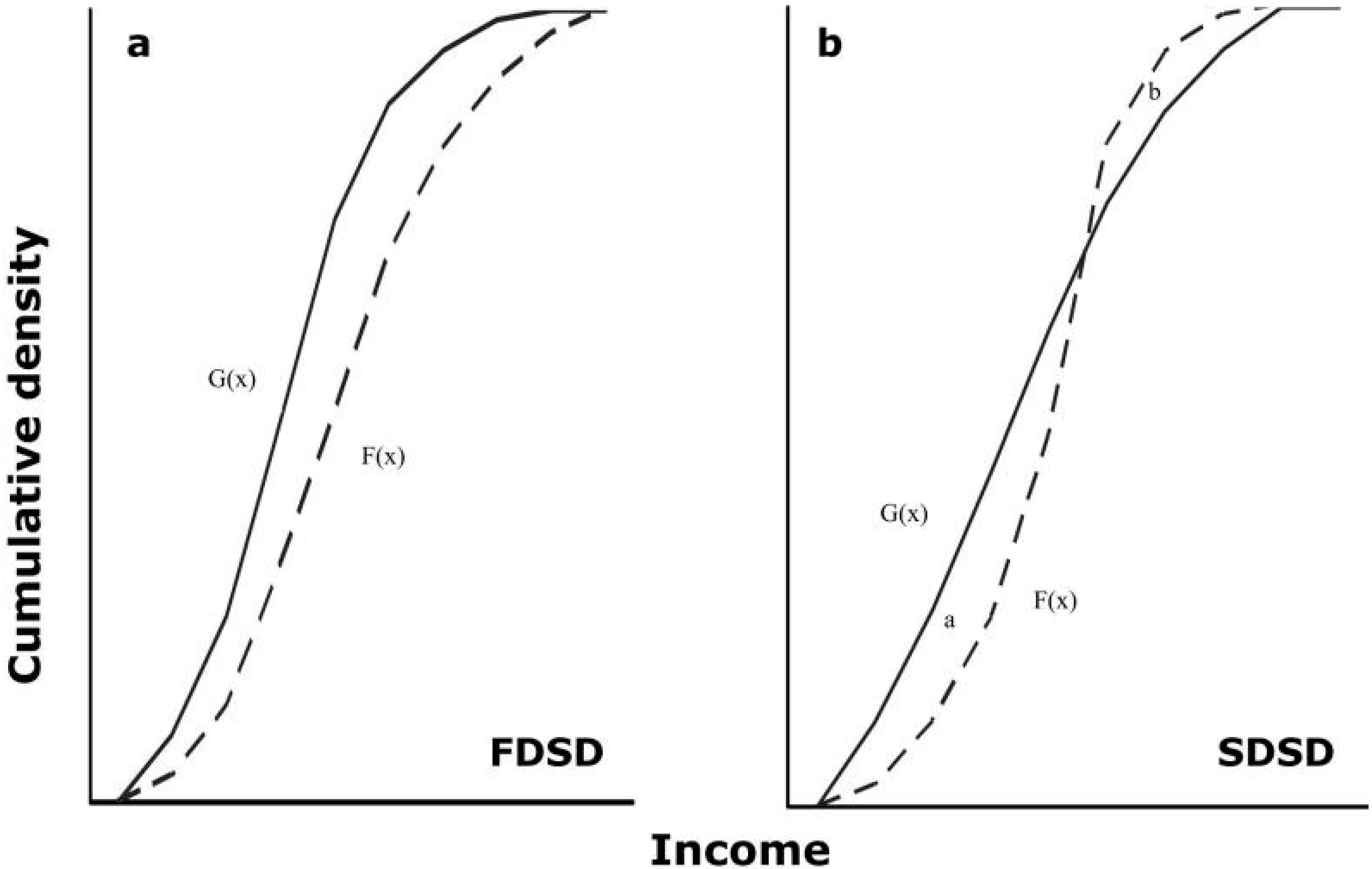
X dominates Y in the $k-1$th order $\Longrightarrow$ X dominates Y in the kth order



Density

Distribution

First order stochastic dominance



First order second order
stochastic dominance

Stochastic dominance

Equivalent Definition Based on Utility Function

- U_1: all the utility functions satisfying $U' \geqslant 0$
- U_2: all the utility functions satisfying $u' \geqslant 0$ and $u" \leqslant 0$
- U_3: all the utility functions satisfying $u' \geqslant 0, u" \leqslant 0$ and $u''' \geqslant 0$

Generally,

- U_n: All the utility functions whose even-order derivatives are negative and odd-order derivatives are positive

Recursive definition:

$$U_n = \left\{u \in U_{n-1} : (-1)^n u^{(n)} \leqslant 0\right\}, \quad U^{(0)} = U$$

Definition: We call x dominates Y in the nth order, if

$$Eu(X) \geqslant Eu(Y), \quad \forall\, u \in U_n$$

and for some $u* \in U_n$, the inequality holds strictly.

Stochastic dominance

FSD

$$F_X(\eta) \leqslant F_Y(\eta), \quad \forall\, \eta \Leftrightarrow Eu(X) \geqslant Eu(Y), \quad \forall\, u \in U_1$$

SSD

$$\int_{-\infty}^{\eta} F_X(t)\mathrm{d}t \leqslant \int_{-\infty}^{\eta} F_Y(t)\mathrm{d}t, \quad \forall\, \eta \Leftrightarrow Eu(X) \geqslant Eu(Y), \quad \forall\, u \in U_2$$

TSD

$$\int_{-\infty}^{\eta}\int_{-\infty}^{v} F_X(t)\mathrm{d}t\mathrm{d}v \leqslant \int_{-\infty}^{\eta}\int_{-\infty}^{v} F_Y(t)\mathrm{d}t\mathrm{d}v$$

$$\Leftrightarrow Eu(X) \geqslant Eu(Y), \quad \forall\, u \in U_3$$

且

$$E_{F_X}(X) \leqslant E_{F_Y}(Y)$$

Stochastic dominance

FSD:

Utility function are non-decreasing, investors pursue return (wealth) maximization

SSD:

Utility function are non-decreasing, investors pursue return (wealth) maximization and are risk averse

TSD:

In addition to the assumptions of SSD, investors are required to have decreasing absolute risk aversion

Stochastic dominance

Advantages of SD:

- Multi-criteria model, axiomatized form
- No need to make any assumptions about the probability distribution of returns
  Phenomenon comparison, theoretical research
- No need to specify the form of investor's utility function

Disadvantages of SD:

- Its definition does not provide a simple calculation method
- Need to compare all possible choices one by one, infinitely many, difficult to apply

Stochastic dominance

Relationship with the mean-risk model (Ogryczak & Ruszczynski, 1999, 2001)

When using a semi-variance relative to a fixed target return as a risk measure, mean-risk model is consistent with SD.

kth order center semi-deviation:

$$\delta_X^k = \left\{ E\Big(\max\big(0, E[X] - X\big)\Big)^k \right\}^{\frac{1}{k}} = \left\{ \int_{-\infty}^{E[X]} \big(E[X] - \varsigma\big)^k p_x(\mathrm{d}\varsigma) \right\}^{\frac{1}{k}}$$

absolute semi-deviation:

$$\delta_x = \int_{-\infty}^{E[X]} \big(E[X] - \varsigma\big) p_x(\mathrm{d}\varsigma) = \frac{1}{2} \int_{-\infty}^{\infty} \big|\varsigma - E[X]\big| p_x(\mathrm{d}\varsigma)$$

Stochastic dominance

Standard semi-deviation:

$$\delta_x = \left\{ \int_{-\infty}^{E[X]} \big(E[X] - \varsigma\big)^2 p_x(\mathrm{d}\varsigma) \right\}^{\frac{1}{2}}$$

Mean-risk control:

$$X \succ_{\frac{\mu}{r}} Y \Leftrightarrow E(X) \geqslant E(Y) \ \ \& \ \ r_X \leqslant r_Y,$$

$$X \succ_{\frac{\mu}{r}} Y \Leftrightarrow E(X) - \lambda r_X \geqslant E[Y] - \lambda r_Y, \quad \forall\, \lambda \geqslant 0.$$

Theorem: If $X \succ_{SSD} rY$,then $E(X) \geqslant E(Y)$ and

$$E(X) - \delta_X \geqslant E[Y] - \delta_Y \big(E(X) - \sigma_X \geqslant E[Y] - \sigma_Y\big)$$

When $E(X) > E(Y)$, the second inequality holds strictly.

Stochastic dominance

For the maximum $X \in \mathbb{Q}$ such that

$$E(X) - \lambda\delta_X\big(E(X) - \lambda\sigma_x\big), \quad 0 < \lambda \leqslant 1$$

, it is efficient with respect to SSD.

Notation

$$L_k = L_k = L_k(\Omega, \kappa, p) : E\Big[|X|^k\Big] < \infty$$

Theorem: Let $k \geqslant 1$ and $X, Y \in L_k$, if $X \succ_{(k+1)} Y$, then $E(X) \geqslant E(Y)$, and

$$E(X) - \delta_x^{(k)} \geqslant E[Y] - \delta_Y^{(k)}$$

When $E(X) \geqslant E(Y)$, the second inequality holds strictly.

Stochastic dominance

If for some $k \geqslant 1$, $X \succ_{(k+1)} Y$, then $E(X) \geqslant E(Y)$ and for all $m \geqslant k$ satisfying $E\{|x|^m\} < \infty$,

$$E(X) - \delta_x^{(m)} \geqslant E[Y] - \delta_Y^{(m)}$$

Definition:For some nonnegative α, if

$$X \succ_{(k)} Y \Rightarrow E(X) \geqslant E[Y], \quad \text{且} \quad E[X] - \alpha Y_x \geqslant E[Y] - \alpha Y_Y$$

then we say mean-risk model is α-consistent with SD in kth order.

Stochastic dominance

- In $L = k$, $\delta^{(k)}$ mean-risk model is 1-consistent with SD in all $1, 2, \cdots, k+1$ order

From these results, we know

- Downside risk measure is better than and can replace the classical variance risk measure.
- Not Necessarily! Grootveld & Hallerbach (1999) in the mean-risk framework, only a few of the underlying risk measures are better than the variance.

Stochastic dominance

Remarks:

- Use SD for research on uncertainties in economy, finance, etc.
- Find the relationship between SD with different orders and other types of risk measures to demonstrate the risk
- The rationality of measurement, guide the choice of optimal portfolio
- Discuss tractable algorithms of calculating SD under specific condition, e.g., generalized error distribution, stable distribution, skewed distribution
- Based on SD, explore problems that cannot be solved under other types of risk measures

  The problem of complex portfolio selection based on nonlinear utility function

VaR

Value-at-Risk

—The second important approach in risk measurement

- VaR quantifies market risk with multiple sources into a single number
- At a given confidence level, what is the maximum loss that an investor may suffer during a certain investment period?
- How much of the investor's total investment is at risk?

Definition: for a given time interval and probability level $k\,(0 < k < 1)$, VaR_k represents the minimum loss occurring with probability $1-k$; or the maximum loss occurring with probability k.

$$\mathrm{VaR}_k = -F_X^{-1}(k),$$

F_X^{-1} is the inverse function of X' distribution function, F_X.

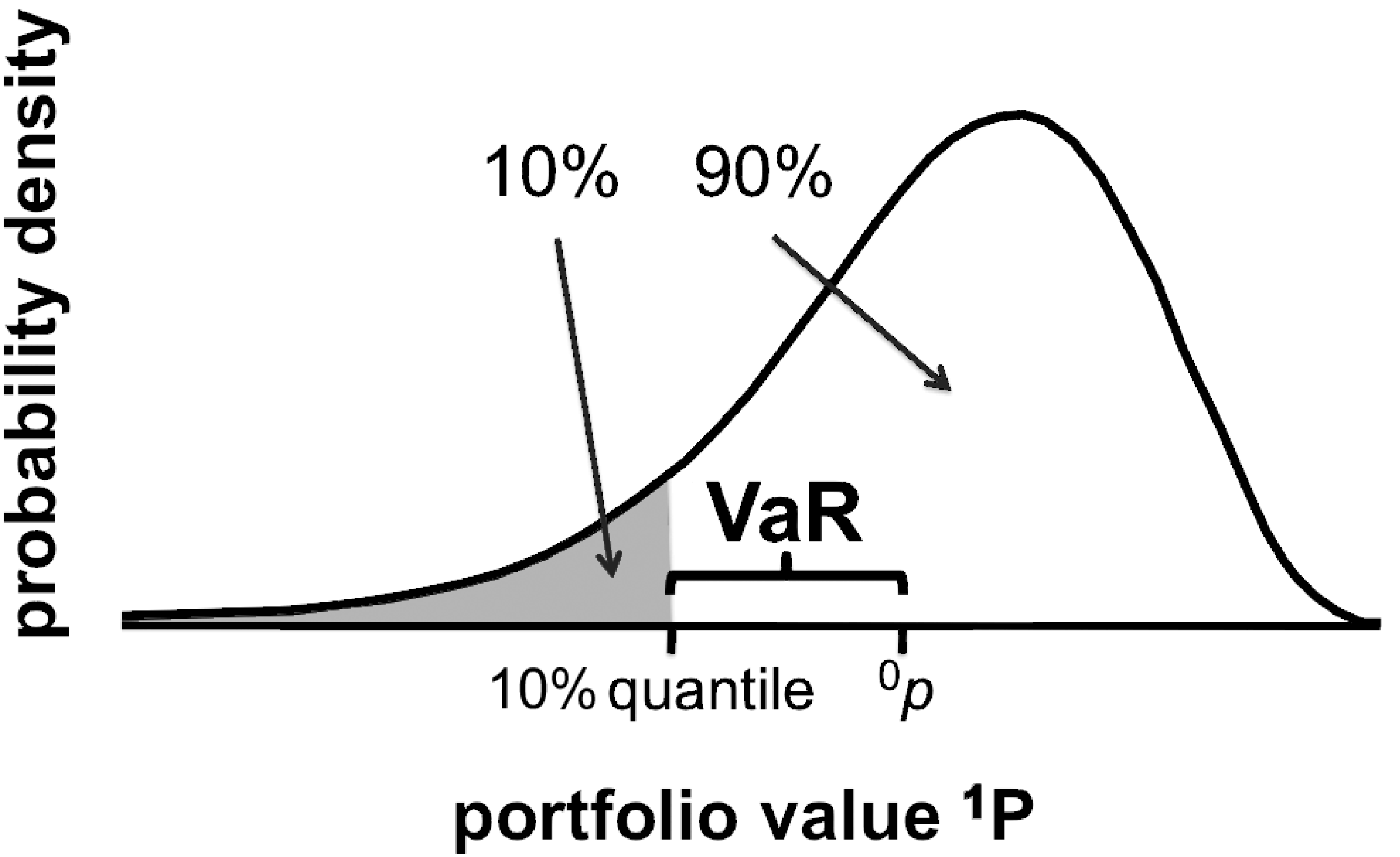


Exhibit: A portfolio’s 90% VaR is the amount of money such that there is a 90% probability of the portfolio losing less than that amount of money

the 90% quantile of ^1L.

VaR

Dual

Bankruptcy risk (risk of ruin): measures the probability of a company's bankruptcy or the occurrence of a catastrophic event: the maximum probability of a certain loss.

For multivariate distributions, F_X^{-1} is undefined at some values of k (Rockafellar & Vryasev, 2000)

$$\mathrm{VaR}_k = \inf\{-F_X^{-1}(k)\}$$

Computation of VaR (Penza & Bansal, 2001)

- Historical simulation method: A simple empirical method that does not require any assumptions about the distribution of market factors and simulates the future returns of the portfolio directly based on historical data collected from market.

VaR

- Monte Carlo simulation method: Use statistical method to estimate the parameters of the market factor, and then simulate the scenarios of market factors..

Increase computational efficiency: scenario, simulation (Jamshudian & Zhu, 1996)

Reduce estimated variance: importance sampling, stratified sampling (Glasserman, 2000)

These two methods can deal with non-linear financial securities such as options. The disadvantages are the large amount of calculation and low efficiency.

- Analytical method: Assume that the change of the market factor follows multivariate normal distribution, or other distribution

Portfolio value function Market factor model

VaR

δ–GARCH normal model: use GARCH to describe the change of market factor

$$r_t = \mu + \eta_t, \qquad \eta_t \,|\, \Omega_{t-1} \sim N(0, h_t),$$
$$\ln h_t = \alpha + \beta \ln h_{t-1} + \varphi \Big[\frac{|\eta_{t-1}|}{\sqrt{h_{t-1}}} - \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}} \Big] + \gamma \frac{\eta_{t-1}}{\sqrt{h_{t-1}}}$$

r_t: return rate, μ: expected return rate, Ω_{t-1}: information available at time period $t-1$, α, β, φ, γ: parameters in EGARCH model.

-$\ln()$ $\rightarrow$ guarantees the variance $h_t > 0$

-The sharp decline in stock prices will produce greater volatility than their sharp rise

VaR

The disadvantages of above analytical methods: assumed distributions are too simple and special

Under normal distribution: VaR based investment selection model is equivalent to MV model

Other new computation methods:

- Semi-parametric method using high-order information such as skewness and kurtosis, (Li, 1999)
- The calculation of VaR with high kurtosis and fat-tail distribution (Zou Xinyue & Lv Xian, 2003)
- Use generalized error distribution (Tian Guo et al., 2003)

Better computation method?

Based on stable distribution, skewed distribution, etc.

VaR

Remarks:

The advantages of VaR:

① The definition is simple and intuitive, easy to understand

② Theoretically, it can measure various portfolios including complex financial derivatives

③ Lower partial risk measure

Disadvantages of VaR:

① The value of VaR relies on the selection of parameters such as holding period and confidence level. Very sensitive.

② Can't measure loss over VaR

③ People concern more about the risk from abnormal situations; VaR often underestimates the real risk

VaR

④ VaR is generally non-convex to the investment weight! Non-smooth when using a limited number of scenario.

The corresponding portfolio optimization problems are non-smooth and non-convex, with multiple local extrema. Not easy to apply. Hard to be applied in large-scale portfolio optimization problems.

Considering the simplest VaR based optimal portfolio selection problem:

$$
\begin{aligned}
\min \quad & \mathrm{VaR}_{\alpha}(-w^T R) \leftrightarrow \max \quad \mathrm{VaR}_{1-\alpha}(w^T R) \\
\text{s.t.} \quad & w^T r \geqslant \bar{r}, \\
& w^T e = 1, \\
& w \geqslant 0,
\end{aligned}
$$

r is approximated by N scenarios, $R^1, R^2, \cdots, R^N$, with equal appearing probability.

VaR

Definition: $M_{[k:N]}(u^1, u^2, \cdots, u^N)$ is the k-th maximum component in $u^1, u^2, \cdots, u^N$

$$\mathrm{VaR}_\alpha(-w^T R) = M_{[[\alpha N]:N]}(-w^T R^1, -w^T R^2, \cdots, -w^T R^N)$$

(VaR—Opt)

$$\begin{aligned} \min \quad & M_{[[\alpha N]:N]}(-w^T R^1, -w^T R^2, \cdots, -w^T R^N) \\ \text{s.t.} \quad & w^T \widehat{r} \geqslant \overline{r}, \\ & w^T e = 1, \\ & w \geqslant 0, \end{aligned}$$

$\widehat{r} = \frac{1}{N} \sum_{i=1}^{N} R^i$, expected return rate vector

VaR

VaR based optimal portfolio selection problem :

Given cut-off point c and index set I:

$$
\begin{aligned}
P(c,I): \quad \min_{w,a,z} & \left\{ a + \frac{1}{(1-\alpha)N} \left[\sum_{i \notin I} z_i + \sum_{i \in I} (c-a) \right] \right\} \\
\text{s.t.} \quad & z_i \geqslant -w^T R^i - a, \quad i \notin I, \\
& -w^T R^i \geqslant c, \quad i \in I, \\
& c \geqslant a, \\
& w^T \widehat{r} \geqslant \overline{r}, \\
& w^T e = 1, \\
& z^i \geqslant 0, \quad i \notin I, \\
& x \geqslant 0
\end{aligned}
$$

VaR

For fixed α and x:

$$I(x,a) = \{i : -w^T R^i > a\} \subseteq \{1, 2, \cdots, N\}$$

Theorem: Suppose x^* is the minimal solution of (VaR-Opt), a^* is the minimum value, then, x^* and a^* are the optimal solution of the LP problem $P(a^*, I(x^*, a^*))$.Moreover, for each fix point (x, a) such that x and a are the optimal solution of $P(a, I(x, a))$, x is the local minimal point of problem (VaR-Opt).

VaR

⑤ Only when x follows elliptical distribution, VaR is sub-additive

- In the elliptical case, the optimal portfolio of VaR based model is consistent with the MV model
- VaR does not satisfy sub-additivity: the decentralization of the portfolio may lead to an increase in risk
- VaR from different sources can't be add together—-unusual

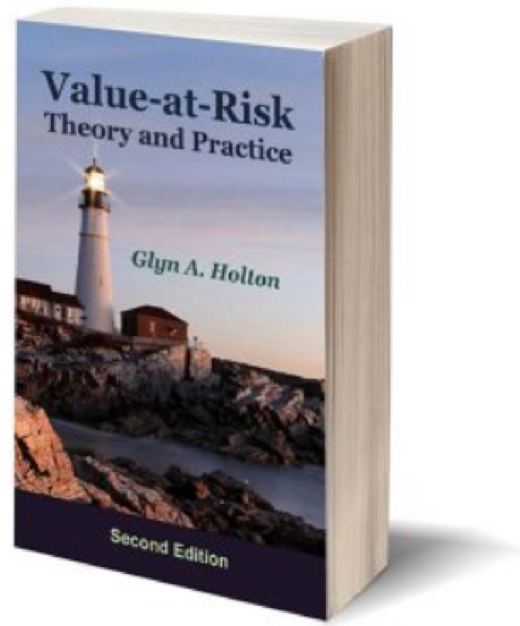
In a word:

VaR is not a really reasonable, good risk measure.

"No More VaR" (J. of Banking & Finance, 2002.26 Special Issue)

More Value-at-Risk Resources

For a deeper discussion of value-at-risk, or for worked examples of actual value-at-risk measures, see my book *Value-at-Risk: Theory and Practice*. I distribute the latest edition free online at http://value-at-risk.net. The book contains about 160 exercises you can practice on, with solutions provided right on this website.



Also explore this website. The blog in particular offers plenty of information on market risk management and value-at-risk.

References

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- Holton, Glyn A. (2014). *Value-at-Risk: Theory and Practice*, 2nd ed. e-book at http://value-at-risk.net.

Coherent risk measure

Coherent risk measure

What mathematical properties a general, appropriate and reasonable risk measurement should satisfy? The third important approach in risk management (Artzner, Delbaen, *et al*. 1997, 1999)

Coherent risk measure： $\rho : X \to R$

A、 Transitional invariance：

$$\rho(x + \alpha r_0) = \rho(x) - \alpha, \quad \forall\, x \in X, \quad \forall\, \alpha \in R$$

here, r_0 is the risk-free return rate

B、 Sub-additivity：

$$\rho(x + y) \leqslant \rho(x) + \rho(y), \quad \forall\, x, y \in X$$

Coherent risk measure

C、 Positive homogeneity:

$$\rho(\lambda x) = \lambda \rho(x), \quad \forall\, \lambda \geqslant 0, \quad \forall\, x \in X$$

D、 Monotonicity:

$$\rho(y) \leqslant \rho(x), \quad \text{if} \;\; x \leqslant y, \quad \forall\, x, y \in X$$

- $A \rightarrow \rho(x + \rho(x) r_0) = 0$ the combination between risk-free asset and risky assets are always efficient in reducing the risk: risk can be controlled
- $B \rightarrow$ Combining risky assets does not bring extra risks

Coherent risk measure

- $C \to$ The size of the holding asset has a direct impact on risk, (eg, large enough to affect the timing of liquidation), the lack of liquidity is considered.
- $D \to$ Exclude variance and all semi-variance measures

Weak Coherent risk measure: B and C is replaced by convexity

Typical Coherent risk measure

$$x_{(\alpha)} = \inf\left\{x \in R : P[X \leqslant x] \geqslant \alpha\right\},$$

$$x^{(\alpha)} = \inf\left\{x \in R : P[X \leqslant x] > \alpha\right\}, \quad E[X^-] < \infty$$

TCE—Tail Conditional Expectation, Tail VaR

$$\mathrm{TCE}_\alpha(X) = -E\left[X \mid X \leqslant x_{(\alpha)}\right]$$

Coherent risk measure

WCE—Worst Conditional Expectation

$$\mathrm{WCE}_\alpha(X) = -\inf\left\{E[X \mid A] : A \in F, P[A] > \alpha\right\}$$

Tail Mean

$$\mathrm{TM}_\alpha(X) = \alpha^{-1}\left(E[X1_{\{X\leqslant x(\alpha)\}}] + x_{(\alpha)}\left(\alpha - P[x \leqslant x_{(\alpha)}]\right)\right)$$

ES—Expected Shortfall

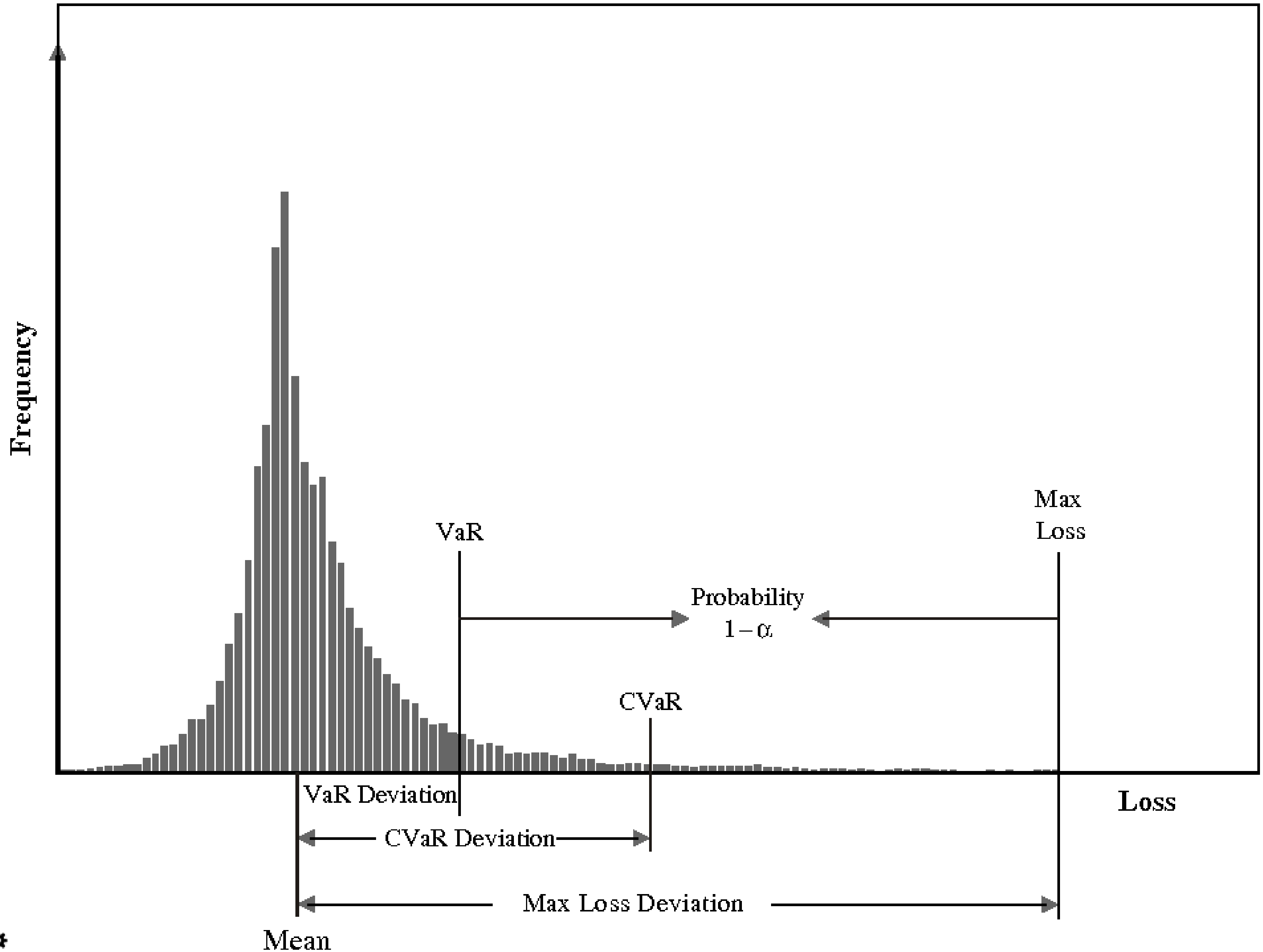
$$ES_\alpha(X) = -\mathrm{TM}_\alpha(X)$$

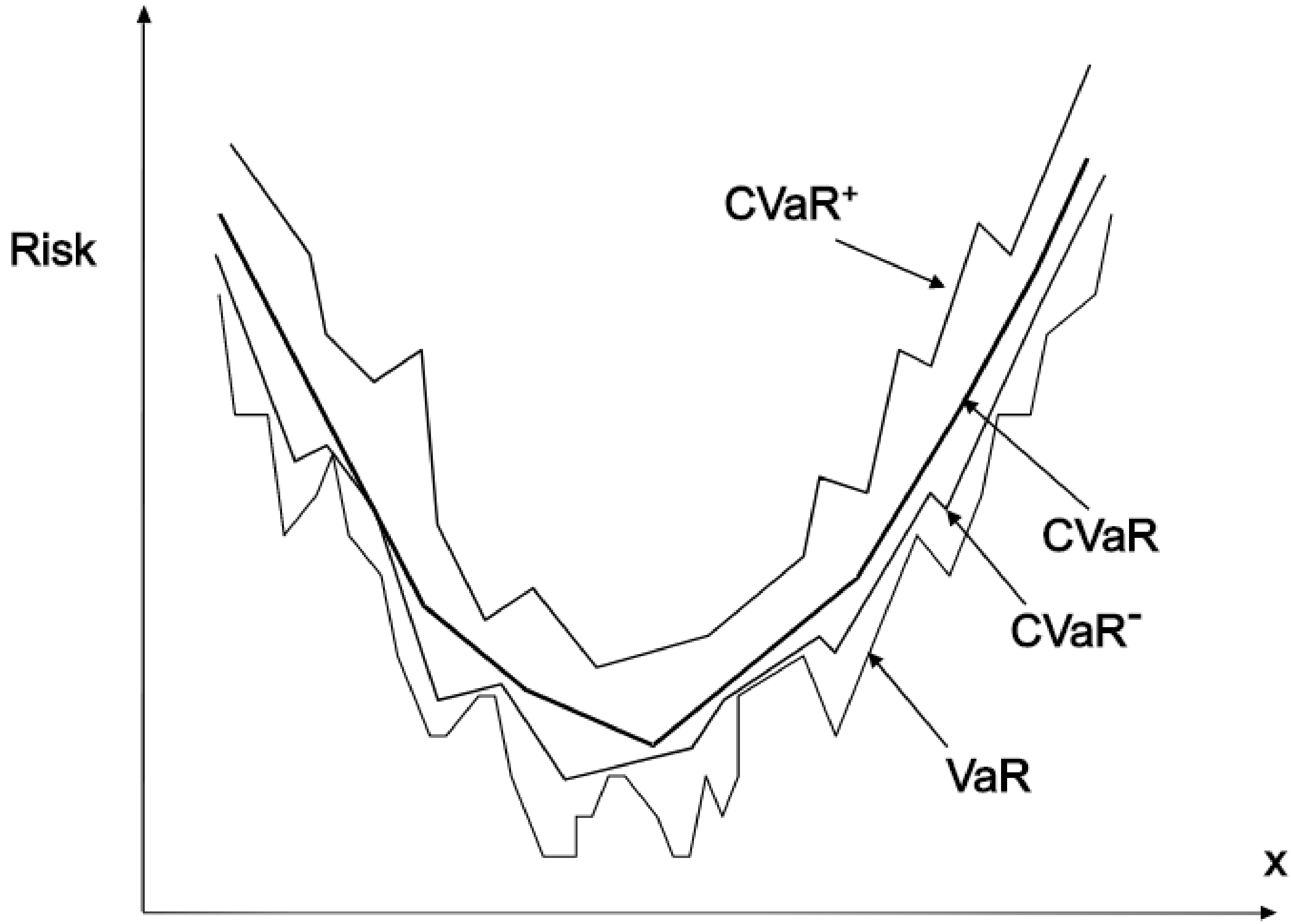
CVaR—Conditional Value-at-risk

The average loss/gain exceeding VaR

α of investment loss – mean value of tail distribution

$$\mathrm{CVaR}_\alpha(X) = E\{X \mid X \geqslant VaR_\alpha(X)\} = \inf\left\{a + \frac{1}{1-\alpha}E[X-a]^+ : a \in R\right\}$$


Frequency
VaR
Max
Loss
Probability
1 – α
CVaR
VaR Deviation
Loss
CVaR Deviation
Max Loss Deviation
Mean



CVaR is convex, but VaR, CVaR $^-$,CVaR$^+$ may be non-convex,
inequalities are valid: VaR $\leq$ CVaR$^-$ $\leq$ CVaR $\leq$ CVaR$^+$

Coherent risk measure

If $X \succ_{\mathrm{SSD}} Y$, then

$$\mathrm{CVaR}_\alpha(Y) \leqslant \mathrm{CVaR}_\alpha(X),\ \mathrm{CVaR}_\alpha(X) \geqslant \mathrm{VaR}_\alpha(X)$$

For continuous X:

$$ES_\alpha(X) = WCE_\alpha(X) = TCE_\alpha(X) = \mathrm{CVaR}(X)$$

Artzner *et al.* (1999), Uryasev (2000), Pflug (2000),
Acerbi & Tasche (2002), Rockafellar & Uryasev (2002)

A simple CVaR based portfolio selection model

$$\begin{aligned} \min\quad & \mathrm{CVaR}_\alpha(-w^T R) \leftrightarrow \max \mathrm{CVaR}_{(1-\alpha)}(w^T R) \\ \text{s.t.}\quad & w^T r \geqslant \bar{r}, \\ & w^T e = 1, \\ & w \geqslant 0, \end{aligned}$$

$\downarrow$

Coherent risk measure

$$
\begin{aligned}
\min_{a,x}\quad & a + \frac{1}{1-\alpha} E[z] \\
\text{s.t.}\quad & z \geqslant -w^{\mathrm{T}} R - a, \\
& w^T r \geqslant \overline{r}, \\
& w^T e = 1, \\
& w \geqslant 0, \\
& a \geqslant 0
\end{aligned}
\qquad
\begin{aligned}
\min_{a,x,z}\quad & a + \frac{1}{(1-\alpha)N} \sum_{i=1}^{N} z^i \\
\text{s.t.}\quad & z^i \geqslant -w^T R^i - a, \quad i = 1, \dots, N, \\
& w^T \widehat{r} \geqslant \overline{r}, \\
& w^T e = 1, \\
& z^i \geqslant 0, \quad i = 1, \dots, N, \\
& x \geqslant 0.
\end{aligned}
$$

Coherent risk measure

CVaR for general loss distribution

$f(x,y)$: loss function of decision variable $x \in X \subseteq R^n$; $y \in R^m$: random vector; $P(y)$: density function of y; $\phi(x,\alpha) = \int_{f(x,y)\leqslant\alpha} p(y)\mathrm{d}y$: cumulated distribution function of x

$$\beta - \mathrm{VaR} : \alpha_\beta(x) = \min\left\{\alpha \in R : \phi(x,\alpha) \geqslant \beta\right\},$$

$$\beta - \mathrm{CVaR} : \phi_\beta(x) = (1-\beta)^{-1} \int_{f(x,y)\geqslant\alpha_\beta(x)} f(x,y)p(y)\mathrm{d}y, \quad \beta \in (0,1)$$

Auxiliary functions

$$F_\beta : X \times R \to R,$$

$$F_\beta(x,\alpha) = \alpha + (1-\beta)^{-1} \int_{y\in R^m} \left[f(x,y) - \alpha\right]^+ p(y)\mathrm{d}y$$

Convex, continuously differentiable

Coherent risk measure

Computation of CVaR for general loss distribution:

$$\phi_\beta(x) = \min_{\alpha \in R} F_\beta(x, \alpha), \quad A_\beta(x) = \arg\min_{\alpha \in R} F_\beta(x, \alpha)$$

Non-empty, bounded closed interval

Computation of VaR: $\alpha_\beta(x) =$ the Left endpoint of horizon $A_\beta(x)$.

$$\alpha_\beta(x) \in \arg\min_{\alpha \in R} F_\beta(x, \alpha), \quad \phi_\beta(x) = F_\beta\big(x, \alpha_\beta(x)\big)$$

Minimizing $\beta - \mathrm{CVaR}$ over $x \in X$: $\min\limits_{x \in X} \ \phi_\beta(x)$

Minimizing $F_\beta(x, \alpha)$ over $(x, \alpha) \in X \times R$: $\min\limits_{(x,\alpha) \in X \times R} \ F_\beta(\alpha)$

(x^*, α^*) minimizes the latter one $\Leftrightarrow$ x^* minimizes the former one, and $\alpha^* \in A_\beta(x^*)$

Coherent risk measure

Discretization:

$$y \to y^1, y^2, \cdots, y^J,$$

$$F_\beta(x,\alpha) \to \overline{F}_\beta(x,\alpha) = \alpha + \upsilon \sum_{j=1}^{J} \left[f(x,y^j) - \alpha\right]^+, \quad \upsilon = J^{-1}(1-\beta)^{-1}$$

If $f(x,y)$ is linear to $x \Leftrightarrow \overline{F}_\beta(x,\alpha)$ is piece-wise linear and convex

Bring Auxiliary variable:

$$z_j, \quad j = 1, 2, \cdots, J,$$

$$\overline{F}_\beta(x,\alpha) = \alpha + \upsilon \sum_{j=1}^{T} z_j,$$

$$z_j \geqslant f(x,y^j) - \alpha, \quad z_j \geqslant 0, \quad j = 1, 2, \cdots, J$$

Coherent risk measure

Single-period portfolio optimization model with transaction costs

- n risky securities: $S_i, \quad i = 1, 2, \cdots, n$
- current hold $x^0 = (x_1^0, x_2^0, \cdots, x_n^0)$
- current price $q = (q_1, q_2, \cdots, q_n) \to q^T x^0$ current wealth
- Optimal portfolio $x = (x_1, x_2, \cdots, x_n) =?$

The prices of securities at the end of the holding period is $y = (y_1, y_2, \cdots, y_n)$ depends on scenarios,
Loss function: $f(x, y, x^0, q) = -y^T x + q^T x^0$
Objective function: expected portfolio return

$$R(x) = \frac{1}{v} \sum_{i=1}^{n} -E[y_i] x_i, \quad v = \sum_{i=1}^{n} q_i x_i^0$$

Coherent risk measure

Transaction costs: linear transaction fees, proportional to the number of trading shares

Balance constraint

$$\sum_{i=1}^{n} q_i x_i^0 = \sum_{i=1}^{n} c_i q_i |x_i^0 - x_i| + \sum_{i=1}^{n} q_i x_i, \quad c_i \text{ Proportional cost factor,}$$

$$\sum_{i=1}^{n} q_i x_i^0 = \sum_{i=1}^{n} c_i q_i (\underline{\delta}_i + \bar{\delta}_i) + \sum_{i=1}^{n} q_i x_i,$$

$$x_i^0 - \underline{\delta}_i + \bar{\delta}_i = x_i, \quad i = 1, 2, \cdots, n,$$

$$\underline{\delta}_i \geqslant 0, \quad \bar{\delta}_i \geqslant 0, \quad i = 1, 2, \cdots, n,$$

One of $\underline{\delta}_i, \bar{\delta}_i$ should be zero!

Coherent risk measure

Value constraint

Not allowing the holding of a certain security i exceeds a certain percentage of the total value of the portfolio

$$q_i x_i \leqslant v_i \sum_{i=1}^{n} q_i x_i$$

CVaR constraints:

Government usually require capital reserves according to a VaR value of the investment bank's portfolio.This can be achieved by adding CVaR constraints. Letting the upper bound of CVaR is w, such as w could be a maximum VaR value, then

$$\alpha + v \sum_{j=1}^{T} z_j \leqslant w, \;\; z_j \geqslant \sum_{i=1}^{n} (-y_i^j x_i + q_i x_i^0) - \alpha, \;\; z_j \geqslant 0, \; j = 1, 2, \cdots, J$$

$\Rightarrow$ Portfolio Optimization Model:

Coherent risk measure

$$
\begin{aligned}
\min_{x,\alpha} \quad & \frac{1}{v}\sum_{j=1}^{n} -E[y_i]x_i \\
\text{s.t.} \quad & \alpha + v\sum_{j=1}^{T} z_j \leqslant w, \\
& z_j \geqslant \sum_{i=1}^{n}(-y_i^j x_i + q_i x_i^0) - \alpha, \quad z_j \geqslant 0, \quad j = 1, 2, \cdots, J, \\
& \sum_{i=1}^{n} q_i x_i^0 = \sum_{i=1}^{n} c_i(\underline{\delta}_i + \bar{\delta}_i) + \sum_{i=1}^{n} q_i x_i, \\
& x_i \leqslant v_i \sum_{k=1}^{n} q_k x_k, \quad i = 1, 2, \cdots, n, \\
& x_i^0 - {}_i + \bar{\delta}_i = x_i, \quad i = 1, 2, \cdots, n, \\
& 0 \leqslant {}_i \leqslant \underline{\delta}_i^{\max}, \quad 0 \leqslant \bar{\delta}_i \leqslant \bar{\delta}_i^{\max}, \quad i = 1, 2, \cdots, n, \\
& \underline{x}_i \leqslant x_i \leqslant \bar{x}_i, \quad i = 1, 2, \cdots, n.
\end{aligned}
$$

LP! Scenario generation?

Coherent risk measure

Remarks:

- For coherent or convex risk measure, there is still much work to do! Better coherent and convex risk measures?

But:

- Existing coherent risk measures usually only consider the first-order change of the tail part, such as the average value
- Excepting CVaR, other coherent risk measure is too complicated to compute and use
- How to modify and improve existing non-coherent risk measure (such as the moment based risk measure), to make it coherent or convexity while keeping its feature and property.
- Evaluation of existing coherent risk measure

Coherent risk measure

Yamai & Yoshiba(2002):

Compares ES and VaR:

- ES (CVaR) is easier to decompose and optimize than VaR
- To reach a good accuracy, ES (CVaR) requires a large number of samples

Based on Evaluation results:

- Improve existing risk measure and design new coherent risk measure: more reasonable and effective
- Investor's behavior? psychology?
- New trend: Different investors: different risk measure