Time Consistent Multi-period Robust Risk Measures and Portfolio Selection Models

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Aug. 25, 2018, Xi'an

Contents

Introduction (Cont'd)

Traditional risk measure

- An aggregation function $\rho : L_p(\Omega, \mathscr{F}, P) \to R$ with respect to the probability P, here $1 \leq p < \infty$
- CVaR can be described as follows:

$$CVaR(x) = \inf_{v} \{v + \epsilon^{-1}\mathbb{E}_P[\, x - v \,]_+\},$$

$\epsilon \in (0, 1]$ is a given loss tolerant probability (say, 5%)

★ The computation of risk measure relies on the underlying distribution P

Introduction (Cont'd)

- Traditional distribution assumptions, such as normal or student's t, does not fit the financial data well
- Fully distributional information is hardly known in practice

Deal with the unknown distribution

- Sample average approximation (Shapiro et al. [2009])
- Parametrical robust optimization (Bertsimas et al. [2011])
- Distributionally robust optimization (El Ghaoui et al. [2003])

Introduction (Cont'd)

Distributionally robust optimization

- First proposed by Scarf (1958) and Žáčková (1966)

Typical uncertainty sets

- Box uncertainty (Natarajan et al., 2010)
- Ellipsoidal uncertainty (Ermoliev et al., 1985)
- Known first two order moments (El Ghaoui et al., 2003; Natarajan, Sim 2011; Chen, He, Zhang, 2010)

Introduction (Cont'd)

- Imprecise first two order moments (Delage and Ye, 2010; Cheng and Lisser, 2014)

- Mixture distribution uncertainty (Zhu and Fukushima, 2009)

- Probabilistic distance based uncertainty (Wasserstein distance, Pflug and Wozabal, 2012,2014; Phi-divergence, Ben-Tal et al. 2013, Guan and Jiang, 2017; K-L distance, Hu and Hong, 2014)

Introduction (Cont'd)

Tractable transformation methods

- Second order cone programming
- Semi-definite programming

Worst-case risk measure

Estimate ρ by assuming P belongs to an uncertainty set $\mathscr{P}$. This gives us the following worst-case risk measure (Zhu and Fukoshima, 2009):

Definition 1 For given risk measure ρ, the worst-case risk measure with respect to $\mathscr{P}$ is defined as $w\rho(x) \triangleq \sup_{P\in\mathscr{P}} \rho(x)$.

★ By constructing different uncertainty sets $\mathscr{P}$, we can derive different versions of worst-case risk measures.

Introduction (Cont'd)

Application of worst-case risk measures

- Lobo and Boyd [1999]: worst-case variance, variance uncertainty, transformed to seme-definite program
- El Ghaoui et al. [2003]: worst-case VaR, mean and variance uncertainty , transformed to SOCP
- Zhu and Fukushima [2009]: worst-case CVaR, mixture distribution uncertainty, transformed to linear or SOCP

Introduction (Cont'd)

- Chen, He, Zhang, 2010: worst-case LPM and worst-case CVaR, known mean and variance uncertainty, transformed to SOCP

- Jonathan Yu-Meng Li [2018]: worst-case law invariant coherent risk measures, known mean and variance uncertainty, transformed to SOCP

★ Above studies are all in static case

Introduction (Cont'd)

Why dynamic risk measure?

- Decisions are made dynamically (at discrete times).
- The information changes frequently over time. The risk measure should adapt to the information flow.
- Static risk measure always leads to myopic decisions, while many investors prefer long-term investment.

Good dynamic risk measure

- Dynamic monotonicity
- Dynamic convexity
- Time consistency

Introduction (Cont'd)

Consider a probability space $(\Omega, \mathcal{F}, P)$, with $\mathcal{F}$ denoting the set of subsets of Ω, and filtration $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, ..., \mathcal{F}_T)$.

Time consistency A conditional risk mapping $\rho_{t,T} : L_p(\mathcal{F}_T) \to L_p(\mathcal{F}_t)$ is time consistent, if for any $0 \le t < \theta \le T-1$, $Z, W \in \mathcal{F}_T$, $\rho_{\theta,T}(Z) \le \rho_{\theta,T}(W)$ implies that $\rho_{t,T}(Z) \le \rho_{t,T}(W)$.

- Relationship between conditional risks
- Leads to the consistency of dynamic decision

Introduction (Cont'd)

Examples of terminal wealth risk measures

$$Var(Z_T \mid \mathcal{F}_t) = E[(Z_T - E(Z_T \mid \mathcal{F}_t))^2 | \mathcal{F}_t]$$

Li and Ng[2000],Cakmak[2004],Celikyurt[2007], Cui et al. [2010]

$$VaR_\alpha(Z_T \mid \mathcal{F}_t) = \inf_{z \in R}\{z | P(Z_T \geq z \mid \mathcal{F}_t) \leq \alpha\}$$

Cheridito and Stadje[2008], Leippold et al. [2006], Cuoco et al.(2007)

$$CVaR_\alpha(Z_T \mid \mathcal{F}_t) = \inf_{z \in R}\{z + \frac{1}{1-\alpha} E[(Z_T - z)^+ \mid \mathcal{F}_t]\}$$

Geman and Ohana [2008], Boda and Filar [2006]

All these are not time consistent !

Introduction (Cont'd)

Additive CVaR: Not time consistent !

$$A\_CVAR_{t,T} = \sum_{s=t}^{T} \beta_s CVaR_{\alpha_s}(Y_s|\mathcal{F}_t)$$

Pflug and Römisch (2007), Fábián (2008) Recursive CVaR: Time consistent !

$$\rho_{t,T}(Z) = \rho_{t,t+1}(\rho_{t+1,T}(Z)),\ t = 0, 1, \ldots, T-1,$$

$$\rho_{t,t+1}(\cdot|\mathcal{F}_t) = CVaR_{\alpha_t}(\cdot|\mathcal{F}_t)$$

Selden [1978], Kreps and Porteus [1978], Pflug and Römisch[2007]

Introduction (Cont'd)

Key issues in:

- Computation of multi-period risk measure
- Modelling of multi-stage portfolio selection problems

Describe the information process

- Period-wise independent
- Time series models: AR, ARMA, ARCH, GARCH
- Markovian process: Regime switching

Known distribution of the random process $\Longrightarrow$ if not known or partially known?

Introduction (Cont'd)

Multi-period robust optimization

- Robust Markov control(Ben-tal et al., 2009): transaction probability matrix uncertainty
- Adjustable robust optimization (ARO): distribution uncertainty
  - ARO can be solved by dynamic programming technique (Shapiro, 2011, Chen, He, Zhang, 2010)
  - ARO make a worst-case estimation at the current period on the basis of the worst-case estimation at the next period $\Longrightarrow$ excessively conservative

Introduction (Cont'd)

- Scenario tree approach : realization value or branching probability uncertainty
  - known distribution structure $\Longrightarrow$ parametric robust (Rustem Gulpinar, 2007)
- Period-wise independent sets : a series of work from Shapiro & Xin, 2011, 2014, 2017
  - loss of dynamic in uncertainty sets $\Longrightarrow$ time inconsistent
  - find time inconsistent bounds

Introduction (Cont'd)

Two important issues

- Time consistency
- Tractability

Our contributions

- A series of works on constructing multi-stage distributionally robust optimization modes
- Discussing the dynamics of the uncertainty set
- Proposing efficient solution method
- Focusing on financial decision making problems

Contribution 1

- Proposing additive form multi-stage worst-case risk measure
- Apply to multi-stage portfolio selection problem
  - Known first two order moments
  - Period-wise independent uncertainty sets
  - Closed-form solution
- Shapiro's additive form worst-case risk measure $\Longrightarrow$ time inconsistent, intractability
- ARO approach $\Longrightarrow$ too conservative
- Our model $\Longrightarrow$ time consistent, closed-form solution, not much conservative

Introduction (Cont'd)

Contribution 2

- Additive form worst-case risk measure with regime switching
  - Partial states (regimes) observable; others uncertain
- Apply to multi-stage portfolio selection problem
  - State dependent moments information
  - State to state: partial dependent!
  - Scenario tree approach & deterministic equivalence method $\Longrightarrow$ SOCP
- Regime-dependent optimal decision

Contents

Basic setting

- There are $T+1$ time points and T periods
- Random loss process $\{x_t,\ t = 0, 1, \cdots T\}$, defined on the probability space $(\Omega, \mathcal{F}, P)$, and adapted to the filtration $\mathcal{F}_t,\ t = 0, 1, \cdots, T$
- $\mathcal{F}_0 = \{0, \Omega\}$, and $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$, for $t = 0, 1, \cdots, T-1$
- $P_t := P|\mathcal{F}_t$
- $x_t \in \mathcal{L}_t = \mathcal{L}_p(\Omega, \mathcal{F}_t, P_t)$
- $\mathcal{L}_{t,T} = \mathcal{L}_t \times \cdots \times \mathcal{L}_T$
- $x_{t,T} = (x_t, \cdots, x_T) \in \mathcal{L}_{t,T}$

Typical multi-period risk measure

- A conditional mapping $\rho_{t,T}(\cdot) : \mathcal{L}_{t+1,T} \to \mathcal{L}_t$
- Separable expected conditional (SEC) mapping:

$$\rho_{t,T}(x_{t+1,T}) = \sum_{i=t+1}^{T} \mathbb{E}_{P_t} \left[\rho_{i|\mathcal{F}_{i-1}}(x_i) \middle| \mathcal{F}_t \right], \; t = 0, 1, \cdots, T-1.$$

Considering the distributional uncertainty

★ At each period t, P_t is required to belong to an uncertainty set $\mathscr{P}_t$ which contains all possible probability distributions of random loss x_t and is observable at time point $t-1$.

★ $\mathscr{P}_1, \mathscr{P}_2, \cdots, \mathscr{P}_T$ are mutually independent.

- We obtain a robust estimation of the one-period conditional risk at period t: $\sup_{P_t \in \mathscr{P}_t} \rho_{t|\mathcal{F}_{t-1}}(x_t)$

- Then all the estimations of risks at different periods are added together with respect to their conditional expectations

$\Longrightarrow$ This gives us the multi-period worst-case risk measure.

Worst case risk measure

For $t = 0, 1, \cdots, T-1$ and $x_{t+1,T} \in \mathcal{L}_{t+1,T}$,

$$w\rho_{t,T}(x_{t+1,T}) = \sum_{i=t+1}^{T} \mathbb{E}_{P_t}\left[\sup_{P_i \in \mathscr{P}_i} \rho_{i|\mathcal{F}_{i-1}}(x_i) \,\middle|\, \mathcal{F}_t\right]$$

is called the conditional worst-case risk mapping. The sequence of the risk mappings $\{w\rho_{t,T}\}_{t=0}^{T-1}$ is called the multi-period worst-case risk measure.

Dynamic formulation

$$w\rho_{t-1,T}(x_{t,T}) = \left(\sup_{P_t \in \mathscr{P}_t} \rho_{t|\mathcal{F}_{t-1}}(x_t)\right) + \mathbb{E}_{P_{t-1}}\left[w\rho_{t,T}(x_{t+1,T})|\mathcal{F}_{t-1}\right],\ t = 1, 2, \cdots, T.$$

Compared with the adjustable robust optimization (ARO)

- $w\rho$: take the worst-case estimation for the first part only
- ARO: take the worst-case estimations for both two parts

$\Longrightarrow$ The worst-case estimation will not be cumulated to the earlier period. Not that conservative than ARO.

Time consistency

- If $\rho_{t|\mathcal{F}_{t-1}}$ associated with the any probability distribution $P_t \in \mathscr{P}_t$ is monotone, $t = 1, 2, \cdots, T$, then the corresponding multi-period worst-case risk measure $\{w\rho_{t,T}\}_{t=0}^{T-1}$ is time consistent.

Coherency

- If $\rho_{t|\mathcal{F}_{t-1}}$ associated with any probability distribution $P_t \in \mathscr{P}_t$ is coherent, the corresponding multi-period worst-case risk measure is dynamic coherent.

Multi-stage portfolio selection problem

Market setting

- There are n risky assets in the security market
- $r_t = [r_t^1, \cdots, r_t^n]^\top$: the random return rates at period t
- $u_{t-1} = [u_{t-1}^1, \cdots, u_{t-1}^n]^\top$: the vector of cash amounts invested in the risky assets at the beginning of period t

$$\mathscr{P}_t = \left\{ P \middle| \mathbb{E}_{P_{t-1}}[r_t] = \mu_t, \mathrm{Cov}_{P_{t-1}}[r_t] = \Gamma_t \right\}.$$

Multi-stage portfolio selection problem

We consider a multi-criteria approach with respect to the expected final wealth and wCVaR measure as follows:

$$\begin{aligned} \max_{u} \quad & \mathbb{E}\left[w_T\right] - \lambda \cdot \sum_{t=1}^{\top} \mathbb{E}\left[\sup_{P_t \in \mathscr{P}_t} CVaR_{t|\mathcal{F}_{t-1}}(-w_t)\right], \\ \text{s.t.} \quad & e^{\top} u_{t-1} = w_{t-1},\ t = 1, \cdots, T. \\ & r_t^{\top} u_{t-1} = w_t,\ t = 1, \cdots, T. \end{aligned}$$

Here, $e = [1, \cdots, 1]^{\top}$. λ is the risk aversion coefficient.

Multi-stage portfolio selection problem

Introduce the following notations

$$a_t = e^\top \Gamma_t^{-1} e, \;\; b_t = e^\top \Gamma_t^{-1} \mu_t, \;\; c_t = \mu_t^\top \Gamma_t^{-1} \mu_t,$$

$$\kappa_t = \sqrt{\frac{1-\epsilon_t}{\epsilon_t}}, \;\; t = 1, \cdots, T, \;\; z_T = 1,$$

$$z_{t-1} = (\lambda + z_t) d_t - \lambda \kappa_t \sqrt{\frac{1}{a_t c_t - b_t^2}(c_2 - 2b_t s_t + a_t s_t^2)}, \;\; t = 2, \cdots, T,$$

$$h_t = \left(\frac{\lambda \kappa_t}{\lambda + z_t}\right)^2 \frac{1}{a_t c_t - b_t^2}, \;\; \Delta_t = 4(h_t a_t - 1)(a_t c_t - b_t^2),$$

$$d_t = \frac{2b(a_t h_t - 1) + \sqrt{\Delta_t}}{2a_t(a_t h_t - 1)}, \; t = 1, \cdots, T.$$

We can solve the mean-wCVaR problem analytically.

Multi-stage portfolio selection problem

Theorem 1 Suppose that the wealth w_t at each period t is non-negative, and the investor is risk averse such that $\lambda + z_t$ is always non-negative. Then, if $a_t h_t - 1 \geq 0$ for all $t = 1, \cdots, T$, the optimal investment policy for problem (4)-(6) is

$$u_{t-1} = \left(\Gamma_t^{-1} e \, \Gamma_t^{-1} \mu_t\right) \frac{1}{a_t c_t - b_t^2} \begin{pmatrix} c_t & -b_t \\ -b_t & a_t \end{pmatrix} \begin{pmatrix} 1 \\ d_t \end{pmatrix} w_{t-1}, \; t = 1, \cdots, T.$$

If $a_t h_t - 1 < 0$ for some t, $1 \leq t \leq T$, the optimal portfolio at period $t-1$ trends to infinity, and the problem (4)-(6) is unbounded.

Empirical results

We compare the following three dynamic portfolio selection models

- wCVaR: mean-wCVaR model
- MV: dynamic MV model in Li et al. (2000)
- LPM2: multistage portfolio selection model with robust second order lower partial moment (LPM2) as the risk measure in Chen et al. (2011)

We simulated the models for 100 times

- Use mean and variance in Example 2 of Li et al. (2000)
- Generate return rate samples by Gussian Distribution
- $T = 4$

Empirical results

Table 1: Characteristics of the terminal wealths among 100 groups of samples

	mean			variance		
	wCVaR	MV	LPM2	wCVaR	MV	LPM2
minimum	1.8387	-1.9208	1.1080	0.1628	160.7812	0.0792
maximum	2.1989	6.0885	1.2659	0.2793	1143.9345	0.3350
average value	2.0184	1.8296	1.1875	0.2162	504.9351	0.1466

Empirical results

- MV model gains high wealth under best cases, and suffers extreme large loss under worst cases
- When the actual distribution has bias from Guassian (extreme cases), MV model performs badly
- Robust technique can efficiently reduce the expected wealth loss and investment risk under extreme cases

Empirical results

- wCVaR model is not that extremely conservative as the LPM2 model, and it makes a good balance between providing a high terminal wealth and controlling the extreme risk
- We propose in this paper a multi-period worst-case risk measure, which measures the dynamic risk period-wise from a distributionally robust perspective.
- We apply CVaR to construct multi-stage robust portfolio selection models and show that they can be solved analytically.

Contents

Regime switching

- Regime switching can reflect dynamic correlations of return rates in different economic cycles.
- The regime process is $s_1, \cdots, s_T$.
- Possible regimes are $s^1, s^2, \cdots, s^J$.
- Stationary Markovian chain with the following transition probability matrix:

$$Q = \begin{pmatrix} Q_{s^1 s^1} & Q_{s^1 s^2} & \cdots & Q_{s^1 s^J} \\ Q_{s^2 s^1} & Q_{s^2 s^2} & \cdots & Q_{s^2 s^J} \\ \cdots & \cdots & \cdots & \cdots \\ Q_{s^J s^1} & Q_{s^J s^2} & \cdots & Q_{s^J s^J} \end{pmatrix}.$$

Product space

- Regime process belongs to $(S, \mathcal{S}, Q)$, and the corresponding filtration it generates is $\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \cdots \subseteq \mathcal{S}_T$.
- Consider $\{x_t,\ t = 0, 1, \cdots, T\}$ on the product space $(\Omega \times S, \mathcal{F} \times \mathcal{S}, P \times Q)$.
- At each period t, $t = 0, 1, \cdots, T$, x_t is adapted to the filtration $\mathcal{F}_t \times \mathcal{S}_t$.
- From the stationary assumption for s_t, we know that $Q|\mathcal{S}_\tau \equiv Q|\mathcal{S}_t$.

$\Longrightarrow$ $x_t \in L_p(\Omega \times S, \mathcal{F}_t \times \mathcal{S}_t, P_t \times Q), p \geq 2.$

To distinguish the influence of $\mathcal{F}_t$ and that of $\mathcal{S}_t$.

Conditional risk mapping

$\rho_{t-1,t}(\cdot) : L_p(\Omega \times S, \mathcal{F}_t \times \mathcal{S}_t, P_t \times Q) \to L_p(\Omega \times S, \mathcal{F}_{t-1} \times \mathcal{S}_{t-1}, P_{t-1} \times Q)$. We separate $\rho_{t-1,t}(\cdot)$ into two levels:

- The conditional risk mapping under given regime s_t, $\rho_{t|\mathcal{F}_{t-1}}(\cdot) : L_p(\Omega \times S, \mathcal{F}_t \times \mathcal{S}_t, P_t \times Q) \to L_p(\Omega \times S, \mathcal{F}_{t-1} \times \mathcal{S}_t, P_{t-1} \times Q)$
- The regime-dependent risks are combined by $g_t(\cdot) : L_p(\Omega \times S, \mathcal{F}_{t-1} \times \mathcal{S}_t, P_{t-1} \times Q) \to L_p(\Omega \times S, \mathcal{F}_{t-1} \times \mathcal{S}_{t-1}, P_{t-1} \times Q)$

Distributionally robust counterpart

- The uncertainty set $\mathscr{P}_t(s_t)$ at period t is associated with the regime $s_t \in S_t$.
- With respect to the regime based uncertainty set, the worst-case estimation of the one-period risk at period t is $w\rho_{s_t}(x_t) = \sup_{P_t \in \mathscr{P}_t(s_t)} \rho_{t|\mathcal{F}_{t-1}}(x_t)$,

Multi-period worst-regime risk measure: find the worst-regime, and the multi-period robust risk measures are formulated in a SEC way.

Multi-period worst-regime risk measure

For $t = 0, 1, \cdots, T-1$ and $x_{t+1,T} \in \mathcal{L}_{t+1,T}$,

$$wr\rho_{t,T}(x_{t+1,T}; s_t) = \sum_{i=t+1}^{T} \mathbb{E}\left[\sup_{s_i \in S_i} \sup_{P_i \in \mathscr{P}_i(s_i)} \rho_{i|\mathcal{F}_{i-1}}(x_i) \middle| \mathcal{F}_t \times \mathcal{S}_t\right]$$

is called the conditional worst-regime risk mapping. And the sequence of the conditional worst-regime risk mappings $\{wr\rho_{t,T}\}_{t=0}^{T-1}$ is called the multi-period worst-regime risk measure.

$wr\rho$ cares about the worst regime and ignores other regimes, a very conservative risk evaluation.

$\Longrightarrow$ Weight all sub worst-case risk measures under different regimes.

Multi-period mixed worst-case risk measure

For $t = 0, 1, \cdots, T-1$ and $x_{t+1,T} \in \mathcal{L}_{t+1,T}$,

$$mw\rho_{t,T}(x_{t+1,T}; s_t) \quad = \sum_{i=t+1}^{T} \mathbb{E}\left[\mathbb{E}\left[\sup_{P_i \in \mathscr{P}_i(s_i)} \rho_{i|\mathcal{F}_{i-1}}(x_i) \middle| \mathcal{S}_{i-1} \right] \middle| \mathcal{F}_t \times \mathcal{S}_t \right]$$

is called the conditional mixed worst-case risk mapping. And the sequence of the conditional mixed worst-case risk mappings $\{mw\rho_{t,T}\}_{t=0}^{T-1}$ is called the multi-period mixed worst-case risk measure.

$mw\rho$ takes the information under all regimes into consideration.

Dynamic formulations

$$wr\rho_{t-1,T}(x_{t,T};s_{t-1}) = \left(\sup_{s_t\in S_t}\left(\sup_{P_t\in\mathscr{P}_t(s_t)}\rho_{t|\mathcal{F}_{t-1}}(x_t)\right)\right) + \mathbb{E}\left[wr\rho_{t,T}(x_{t+1,T};s_t)\middle|\mathcal{F}_{t-1}\times\mathcal{S}_{t-1}\right], t=1,2,\cdots,T.$$

$$mw\rho_{t-1,T}(x_{t,T};s_{t-1}) = \left(\mathbb{E}\left[\sup_{P_t\in\mathscr{P}_t(s_t)}\rho_{t|\mathcal{F}_{t-1}}(x_t)\middle|\mathcal{S}_{t-1}\right]\right) + \mathbb{E}\left[mw\rho_{t,T}(x_{t+1,T};s_t)\middle|\mathcal{F}_{t-1}\times\mathcal{S}_{t-1}\right], t=1,2,\cdots,T.$$

$\Longrightarrow$ time consistency of the two multi-period robust risk measures.

Multi-period portfolio selection models

The mean-mwCVaR model with transaction costs and market restriction constraints.

$$
\begin{aligned}
\max_{u} \quad & \left\{ \mathbb{E}[w_T; s_0] - \lambda \cdot mwCVaR_{0,T}(-w_{1,T}; s_0) \right\}, \\
\text{s.t.} \quad & w_0 = u_0^\top e + \alpha^\top (u_0)^+ + \beta^\top (u_0)^-, \\
& w_t = u_t^\top e + \alpha^\top (u_t - u_{t-1})^+ \\
& \qquad + \beta^\top (u_t - u_{t-1})^-, t = 1, \cdots, T-1, \\
& w_{t+1} = u_t^\top r_{t+1},\ t = 0, \cdots, T-1, \\
& \underline{u} \le u_t \le \overline{u},\ t = 0, \cdots, T-1,
\end{aligned}
$$

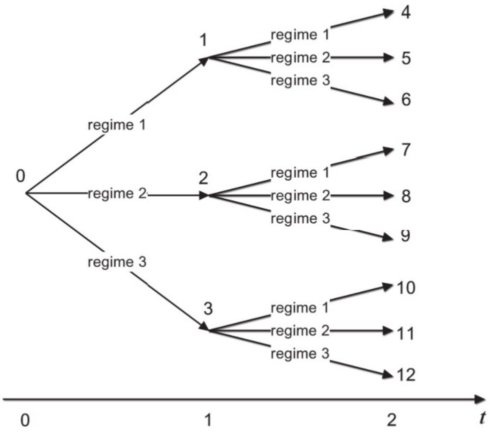
Multi-period portfolio selection models

The mean-wrCVaR model with transaction costs and market restriction constraints.

$$
\begin{aligned}
\max_{u} \quad & \left\{ \mathbb{E}[w_T; s_0] - \lambda \cdot wrCVaR_{0,T}(-w_{1,T}; s_0) \right\}, \\
\text{s.t.} \quad & w_0 = u_0^\top e + \alpha^\top (u_0)^+ + \beta^\top (u_0)^-, \\
& w_t = u_t^\top e + \alpha^\top (u_t - u_{t-1})^+ \\
& \qquad + \beta^\top (u_t - u_{t-1})^-, t = 1, \cdots, T-1, \\
& w_{t+1} = u_t^\top r_{t+1},\ t = 0, \cdots, T-1, \\
& \underline{u} \leq u_t \leq \overline{u},\ t = 0, \cdots, T-1,
\end{aligned}
$$

Model transformation

We adopt a scenario tree to transform the mean-mwCVaR and mean-wrCVaR models



Model transformation

Some notations

- K^+: the set of all nodes at periods $1, 2, \cdots, T$;
- $N(K^+)$: the number of nodes in K^+;
- K^-: the set of all nodes at periods $0, 1, \cdots, T-1$;
- $N(K^-)$: the number of nodes in K^-;
- $t(k)$: the number of period of node k;
- $s(k)$: the regime of node k;
- $Q(k; s_0)$: node k's appearing probability in the tree.

Model transformation

- For a node $k \in K^+$, the unique predecessor is denoted as k^-;
- $\mu(k)$: the estimated expectation value of r_t at node k;
- $\Gamma(k)$: the estimation value of the conditional covariance matrix;
- The uncertainty set with respect to the regime $s(k)$

$$\mathscr{P}(k) = \Big\{ P \Big| \mathbb{E}_{P_{t-1}}[r_t | \mathcal{F}_{t-1}, s_t = s(k)] = \mu(k), \\ \Gamma_{P_{t-1}}[r_t | \mathcal{F}_{t-1}, s_t = s(k)] = \Gamma(k) \Big\}.$$

Model transformation

Under the scenario tree setting, the mean-mwCVaR model is equivalent to the following cone programming problem:

Object:

$$\max_{u,y,z,g,u^+,u^-} \Big\{(1+\lambda)w_0 + \sum_{k\in K^+}(1+(T-t(k^-)-1)\lambda)Q(k;s_0)(\mu(k)-e)^\top u(k^-)$$

$$-\lambda\sum_{k\in K^+}Q(k;s_0)y(k) - (1+T\lambda)\big(\alpha^\top u^+(0)+\beta^\top u^-(0)\big)$$

$$-\sum_{k\in K^-\backslash\{0\}}(1+(T-t(k))\lambda)\left[\alpha^\top u^+(k)+\beta^\top u^-(k)\right]\Big\}$$

Model transformation

Constraints:

$$
\begin{aligned}
\text{s.t.}\quad & \Gamma^{1/2}(k)u(k^-) = z(k),\ k \in K^+, \\
& (\mu(k) - e)^\top u(k^-) + y(k) = \kappa(k)g(k),\ k \in K^+, \\
& ||z(k)||_2 \le g(k),\ k \in K^+, \\
& u(0) = u^+(0) - u^-(0), \\
& w_0 = u(0)^\top e + \alpha^\top u^+(0) + \beta^\top u^-(0), \\
& u(k) - u(k^-) = u^+(k) - u^-(k),\ k \in K^- \backslash \{0\}, \\
& u(k^-)^\top \mu(k) = u(k)^\top e + \alpha^\top u^+(k) + \beta^\top u^-(k),\ k \in K^- \backslash \{0\}, \\
& u^+(k), u^-(k) \ge 0,\ k \in K^-, \\
& \underline{u} \le u(k) \le \overline{u},\ k \in K^-,
\end{aligned}
$$

The above SOCP has $(n+2)N(K^+) + 3nN(K^-)$ variables, $(n+1)N(K^+) + (n+1)N(K^-)$ linear constraints and $N(K^+)$ standard second order cone constraints.

Model transformation

Under the scenario tree setting, the mean-wrCVaR model is equivalent to the following cone programming problem:

Object:

$$\min_{u,y,z,g,u^+,u^-} \Big\{ (1+\lambda)w_0 + \sum_{k\in K^+} (1+(T-t(k)-1)\lambda)Q(k;s_0)(\mu(k)-e)^\top u(k^-)$$

$$-\lambda \sum_{k\in K^-} Q(k;s_0)y(k) - (1+T\lambda)\big(\alpha^\top u^+(0)+\beta^\top u^-(0)\big)$$

$$+\sum_{k\in K^-\setminus\{0\}} (1+(T-t(k))\lambda)\left[\alpha^\top u^+(k)+\beta^\top u^-(k)\right]\Big\}$$

Model transformation

Constraints:

$$
\begin{aligned}
\text{s.t.}\quad & \Gamma^{1/2}(k)u(k^-) = z(k),\ k \in K^+, \\
& (\mu(k) - e)^\top u(k^-) + y(k^-) = \kappa(k)g(k),\ k \in K^+, \\
& ||z(k)||_2 \le g(k),\ k \in K^+, \\
& u(0) = u^+(0) - u^-(0), \\
& w_0 = u(0)^\top e + \alpha^\top u^+(0) + \beta^\top u^-(0), \\
& u(k) - u(k^-) = u^+(k) - u^-(k),\ k \in K^- \backslash \{0\}, \\
& u(k^-)^\top \mu(k) = u(k)^\top e + \alpha^\top u^+(k) + \beta^\top u^-(k),\ k \in K^- \backslash \{0\}, \\
& u^+(k), u^-(k) \ge 0,\ k \in K^-, \\
& \underline{u} \le u(k) \le \overline{u},\ k \in K^-,
\end{aligned}
$$

The above SOCP has $(n+1)N(K^+) + (3n+1)N(K^-)$ variables, $(n+1)N(K^+) + (n+1)N(K^-)$ linear constraints and $N(K^+)$ standard second order cone constraints.

Empirical results

Market setting (Dow Jones, S & P500)

- 10 stocks from different industries in American stock markets
- We use adjusted daily close-prices of these stocks on every Monday to compute their weekly logarithmic return rates rom February 14, 1977 to January 30, 2012
- We divide the market into three regimes: the bull regime; the consolidation regime and the bear regime

Empirical results

Determining regime (NYSF, AMEX, NASDAQ)

- Use MKT-RF (Fama and French, 1993) to determine regime
- Effective time window with 28 weeks, centered on the examining week
- Add all MKT-RF in the effective time window and compare with pre-set benchmark
- Sum larger than 1.0 $\Rightarrow$ bull regime
- Sum smaller than -1.0 $\Rightarrow$ bear regime
- Sum between -1.0 and 1.0 $\Rightarrow$ consolidation regime

Empirical results

Estimating regime transition probability

Counting the relevant historical transition times

$$Q = \begin{bmatrix} 0.9475 & 0.0336 & 0.0189 \\ 0.3333 & 0.3148 & 0.3519 \\ 0.0471 & 0.0634 & 0.8895 \end{bmatrix}.$$

- Stable to stay in the bull or bear regime
- High possibility to switch from the consolidation regime into the bull or bear regime

Empirical results

Table 10: Expected return rates (%) under different regimes

	DIS	DOW	ED	GE	IBM	MRK	MRO	MSI	PEP	JNJ
$\mu(s^1)$	0.2486	0.1845	0.1165	0.2260	0.1290	0.1884	0.1639	0.2291	0.1825	0.1511
$\mu(s^2)$	0.0206	-0.0116	0.1413	0.0110	-0.1879	0.1027	0.2251	0.0817	0.1653	0.1273
$\mu(s^3)$	-0.1921	-0.1583	0.0897	-0.1545	0.0035	-0.0691	-0.0274	-0.2706	-0.0199	0.0366
μ	0.1004	0.0681	0.1098	0.0970	0.0718	0.1046	0.1090	0.0676	0.1196	0.1147

- Both first and second order moments have significant difference among different regimes.
- The estimated covariance matrices have the same feature.

Empirical results

Find the optimal portfolios of mean-wCVaR mean-wrCVaR, mean-mwCVaR models by solving the SOCPs

Table 11: Root optimal portfolios

	DIS	DOW	ED	GE	IBM	MRK	MRO	MSI	PEP	JNJ
$u^*_{wCVaR}(s_0)$	0.0000	0.0000	0.3000	0.0000	0.3000	0.0000	0.0000	0.0000	0.2995	0.1005
$u^*_{wrCVaR}(s_0)$	0.0000	0.0000	0.3000	0.0000	0.3000	0.0000	0.0000	0.0000	0.1367	0.2633
$u^*_{mwCVaR}(s_0 = s^1)$	0.0000	0.0000	0.3000	0.0000	0.3000	0.0000	0.1385	0.0000	0.2615	0.0000
$u^*_{mwCVaR}(s_0 = s^2)$	0.0000	0.0000	0.3000	0.0000	0.3000	0.0000	0.0550	0.0000	0.3000	0.0450
$u^*_{mwCVaR}(s_0 = s^3)$	0.0000	0.0000	0.3000	0.0000	0.3000	0.0000	0.0000	0.0000	0.1492	0.2508

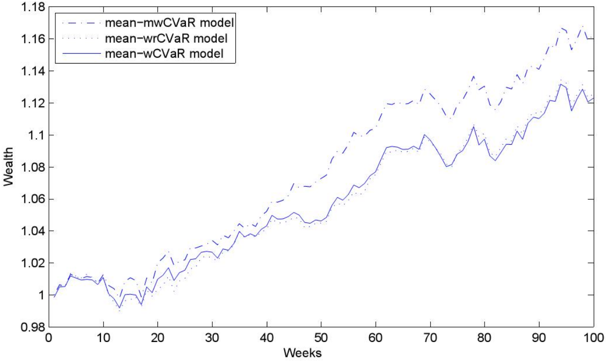
$\epsilon_t(s_t) = 0.05,\ \lambda = 20,\ \underline{u} = 0,\ \overline{u} = 0.3.$

Empirical results

- Both the optimal portfolios of mean-wVaR model and mean-wrVaR model do not rely on the current regime.
- The mean-mwVaR model provides us with three optimal portfolios under three different regimes.
- That is because the estimation of mwVaR relies on the regime appearing probability in the future.
- The strategy derived under regime-dependent robust models reveals more information about market regimes than the traditional worst-case risk measures.

Empirical results

Out the out-of-sample test by rolling forward for 100 weeks, this provides us three out-of-sample accumulated wealth series



Empirical results

Table 12: Statistics of out-of-sample performances

model	mean-wCVaR	mean-wrCVaR	mean-mwCVaR
maximum (%)	1.1020	1.0683	1.2713
minimum (%)	-1.4588	-1.4586	-1.2030
mean (%)	0.1229	0.1234	0.1627
variance ($\times$1.0e-4)	0.2639	0.2688	0.2957
skewness	-0.4449	-0.4343	-0.1873

- Mean-wCVaR and mean-wrCVaR models have similar performance
- Mean-mwCVaR model provides much higher return rate than the other two in terms of the maximum and mean

Empirical results

Table 13: Out-of-sample performances under different regimes

model	regime	bull	consolidation	bear
	weight (weeks)	69	6	25
mean-wCVaR	mean (%)	0.1421	0.2729	0.0339
	variance ($\times$1.0e-4)	0.2455	0.3133	0.3129
mean-wrCVaR	mean (%)	0.1370	0.2401	0.0579
	variance ($\times$1.0e-4)	0.2542	0.3230	0.3129
mean-mwCVaR	mean (%)	0.1938	0.2588	0.0535
	variance ($\times$1.0e-4)	0.2902	0.3421	0.3087

- Under consolidation market: All three are similar
- Under bear market: mean-wrCVaR is best
- Under bull market: mean-mwCVaR is best

Empirical results

Different sites of stock pools:

- 10 stocks from Dow Jones IA, S & P 500
- 50 stocks from S & P 500 $\supset$ “10 stocks"
- 100 stocks from S & P 500 $\supset$ “50 stocks"
- Adjusted daily close-prices to compute their daily logarithmic return rates from March 20, 2011 to March 3, 2015

Empirical results

Separate the historical daily data into

- The in-sample period: March 20,2011 to October 7, 2014
- The out-of-sample period: October 8, 2014 to March 3, 2015

Divide the marhet into three regimes

- Using the effective time window method stated abore
- In the out-of-sample period:
  - Bull regime: 68 days
  - Consolidation regime: 15 days
  - Bear regime: 17 days

Empirical results

Table 14: Statistics of out-of-sample return series got under three models with different stocks pools

mean-wCVaR		10 stocks	50 stocks	100 stocks
total	mean (%)	0.0331	0.0473	0.0771
	variance ($\times$10e-4)	0.608	0.639	0.728
bull	mean (%)	0.001	-0.0483	-0.0494
	variance ($\times$10e-4)	0.5415	0.7933	1.2447
consolidation	mean (%)	0.5026	0.528	0.5006
	variance ($\times$10e-4)	0.4668	0.3368	0.4225
bear	mean (%)	-0.2565	0.0006	0.1164
	variance ($\times$10e-4)	0.8118	0.7361	1.0421

Empirical results

Table 15: Statistics of out-of-sample return series got under three models with different stocks pools

mean-wrCVaR		10 stocks	50 stocks	100 stocks
total	mean (%)	0.0324	0.0465	0.0613
	variance ($\times$10e-4)	0.612	0.745	1.109
bull	mean (%)	0.0001	-0.0321	0.0585
	variance ($\times$10e-4)	0.5227	0.6859	0.742
consolidation	mean (%)	0.5029	0.5256	0.5306
	variance ($\times$10e-4)	0.5068	0.3517	0.6414
bear	mean (%)	-0.2492	-0.0572	-0.2489
	variance ($\times$10e-4)	0.8339	0.5223	0.5315

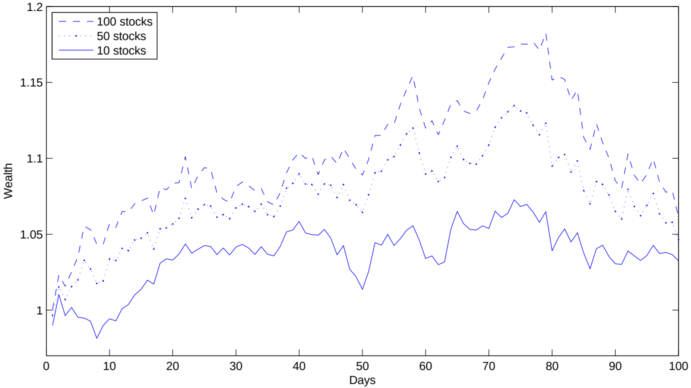
Table 16: Statistics of out-of-sample return series got under three models with different stocks pools

mean-mwCVaR		10 stocks	50 stocks	100 stocks
total	mean (%)	0.0370	0.0817	0.0855
	variance ($\times$10e-4)	0.621	0.739	1.072
bull	mean (%)	0.0078	0.006	-0.0143
	variance ($\times$10e-4)	0.5522	0.7751	1.1805
consolidation	mean (%)	0.4995	0.535	0.5345
	variance ($\times$10e-4)	0.4839	0.4224	0.4313
bear	mean (%)	-0.2545	-0.0154	0.0885
	variance ($\times$10e-4)	0.8125	0.7317	1.0806

Empirical results

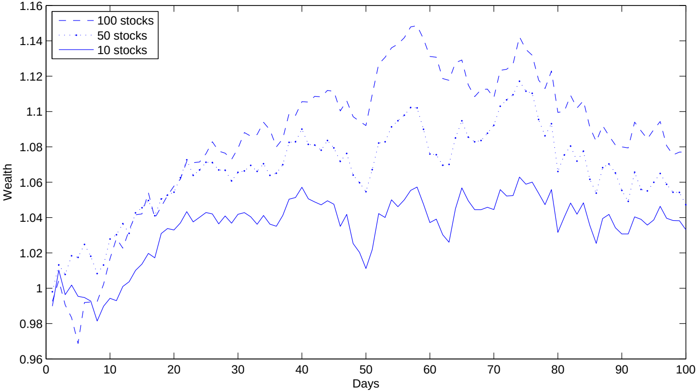
- The solution times for the encountered SOCP problems with 10 stocks are between 0.42 seconds and 0.55 seconds;
- The solution times for the encountered SOCP problems with 50 stocks are between 0.45 seconds and 1.59 seconds;
- The solution times for the encountered SOCP problems with 100 stocks are between 0.55 seconds and 7.60 seconds.

Empirical results



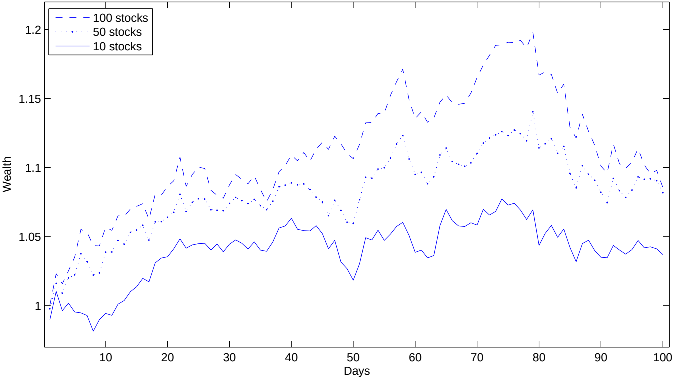
The out-of-sample accumulative wealth series got under the mean-wCVaR model

Empirical results



The out-of-sample accumulative wealth series got under the mean-wrCVaR model

Empirical results



The out-of-sample accumulative wealth series got under the mean-mwCVaR model

Empirical results

- The mean-mwCVaR model constantly provides much greater return rate than the other two models, independently of the three stock pools.
- The mean-wrCVaR model always makes the most powerful control of risk under the worst regime.
- As the size of the stock pool becomes larger and larger, the out-of-sample return rates got under the three models generally become greater too.

Empirical results

When the market is:

- Under the bull regime, the portfolio selection models with a smaller stock pool perform better;
- Under the consolidation regime, the performance of the portfolio selection models with a smaller stock pool is similar to that of the portfolio selection models with a larger stock pool;
- Under the bear regime, the portfolio selection models with a larger stock pool significantly perform better.

Empirical results

During a medium-term or long-term real investment process

- When the investor finds that the market is constantly going high, he/she can focus on the best performing stocks and balance his/her investment among them;
- When he/she finds that the market is turning down, the investor should diversify his/her investment in more assets even if the performance of some assets is not so good as the best performing stocks temporarily;
- Enlarging the stock pool and adopting the multi-period robust portfolio selection model can efficiently avoid the large risks which the investor may suffer under bad market regimes.

Contents

Conclusions

- We propose three multi-period robust risk measures.
- Closed-form solution for multi-period robust portfolio selection problem with multi-period worst-case CVaR.
- With scenario tree technique, we solve the multi-period robust portfolio selection problem with regime switching by SOCP.
- Numerical results demonstrate the efficiency and flexibility of the proposed models.

- Jia Liu, Zhiping Chen, Yongchang Hui. Time consistent multi-period worst-case risk measure in robust portfolio selection. *Journal of the Operations Research Society of China*, 2018, 6: 139-158.
- Jia Liu, Zhiping Chen, Time Consistent Multi-period Robust Risk Measures and Portfolio Selection Models with Regime-switching, *European Journal of Operational Research*, 2018, 268: 373-385.

Thank You Very Much for Your Attention!