Time Consistent Recursive Risk Measures Under Regime Switching and Factor Models and Their Application in Dynamic Portfolio Selection

Jia Liu
(with Zhiping Chen)

Xi'an Jiaotong University

TEL:0086-13572886780, E-mail: liu.jia@stu.xjtu.edu.cn

Outline

- Introduction
- Two-level information structure
- Regime-based recursive risk measure
- Dynamic portfolio selection problem
- Numerical experiments
- Conclusions

Introduction

Why dynamic risk measure?

- Decisions are made dynamically (at discrete times).
- The information changes frequently over time. The risk measure should adapt to the information flow.
- Static risk measure always leads to myopic decisions, while many investors prefer long-term investment.

Good dynamic risk measure

- Dynamic monotonicity
- Dynamic convexity
- Time consistency

Time consistency

Consider a probability space $(\Omega, \mathcal{F}, P)$, with $\mathcal{F}$ denoting the set of subsets of Ω, and filtration $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, ..., \mathcal{F}_T)$. Accumulated return $Z_t = Y_1 + Y_2 + \cdots Y_t,\ t = 1, 2, \cdots, T$. Z_t, Y_t are adapt to $\mathcal{F}_t$.

Time consistency

A conditional risk mapping $\rho_{t,T} : L_p(\mathcal{F}_T) \to L_p(\mathcal{F}_t)$ is time consistent, if for any $0 \le t < \theta \le T - 1$, $Z, W \in \mathcal{F}_T$, $\rho_{\theta,T}(Z) \le \rho_{\theta,T}(W)$ implies that $\rho_{t,T}(Z) \le \rho_{t,T}(W)$.

Time consistency describes the consistent relationship of risks among different stages, which ensures the rationality of the risk measure and the resulting dynamic portfolio selection model.

Wang [1999], Cheridito, Delbaen, Kupper[2006], Detlefsen, Scandolo [2005], Roorda, Schumacher, Engwerda[2005], Artzner,Delbaen,Eber,Heath,Ku [2007], Kovacevic, Pflug[2009], Ruszczyński[2010], Acciaio, Penner[2011]

The final stage risk measure

The final stage risk measure

$$Var(Z_T|\mathcal{F}_t) = E[(Z_T - E(Z_T|\mathcal{F}_t))^2|\mathcal{F}_t]$$

Li and Ng[2000],Cakmak[2004],Celikyurt[2007], Cui et al. [2010]

$$VaR_\alpha(Z_T|\mathcal{F}_t) = inf_{z\in R}\{z|P(Z_T \geq z|\mathcal{F}_t) \leq \alpha\}$$

Cheridito and Stadje[2008], Basak and Shapiro [2001], Berkelaar et al. [2005], Leippold et al. [2006], Cuoco et al.(2007)

$$CVaR_\alpha(Z_T|\mathcal{F}_t) = inf_{z\in R}\{z + \frac{1}{1-\alpha}E[(Z_T - z)^+|\mathcal{F}_t]\}$$

Geman and Ohana [2008], Boda and Filar [2006]

All these are not time consistent!

Other multi-period risk measures

Separable average CVaR

$$SA\_CVAR_{t,T} = \sum_{s=t}^{T} \beta_s CVaR_{\alpha_s}(Y_s|\mathcal{F}_t)$$

Not time consistent!

Recursive CVaR

$$R\_CVAR_{t,T} = CVaR_{\alpha_t}(R\_CVAR_{t+1,T}|\mathcal{F}_t)$$

Time consistent!

Selden [1978], Kreps and Porteus [1978], and Duffie and Epstein[1992], Pflug and Römisch[2007]

Recursive risk measure

The recursive risk measure is time consistent when the one-step conditional risk measure is monotonicity.

If a conditional risk mapping is monotonicity, time consistent, regularity and translation invariance, then for all t it can be represented recursively, $\rho_{t,T}(Z) = \rho_{t,t+1}(\rho_{t+1,T}(Z))$. (Theorem 1, Ruszczyński[2010])

But, the multi-stage portfolio selection problem based on recursive risk measure is hard to solve!

In our paper, we propose a two-level information structure, and define the regime-based recursive risk measure on it.

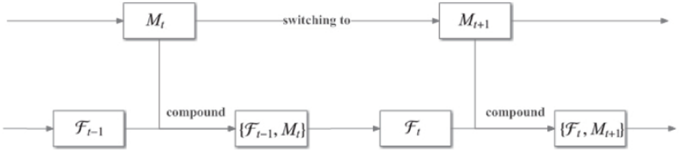
Our feature: more realistic, tractable, and reasonable.

Two-level information structure

The outer level reflects the endogenous information of macro market states such as bear market, bull market, consolidation market, we call them market regimes M_t.

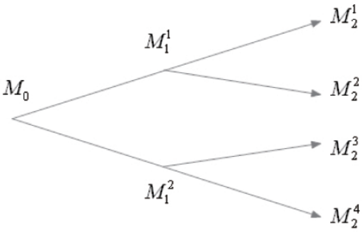
The endogenous assumption means which regime to appear only depends on historical regimes. and translates with the probability $P(M_t^k, M_{t+1}^j)$.

The inner level reflects the exogenous information driven by M_t, such as the return rates of assets R_t and the value of indexes F_t.



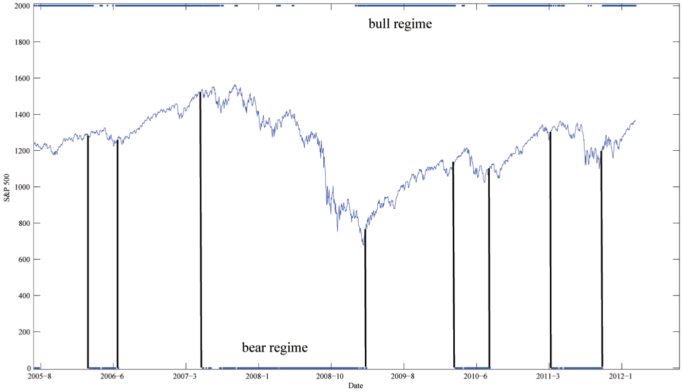
The regime switching

We suppose that there are K_t regimes, $M_t^k, k = 1, 2, ..., K_t$, at period t, $t = 1, ..., T$. The regime switching only relies on historical regimes. Hence, the switching structure can be represented as a scenario tree.



Generally, such process is supposed to be Markovian.

Regime switching



Regime switching of S&P 500 from Aug. 2005 to Jan. 2012

Factor model

Modern finance theory tells us that stock prices are determined by economic factors and financial market risk factors.

Chen, Ross and Roll: five-factor model [1986]

- growth rate of industrial production (IP)
- unexpected inflation (UI)
- change of expected inflation (DEI)
- yield spread (YS)
- credit spread (CS)

Fama and French: three-factor model [1993]

- the excess market portfolio return over the risk free asset (MKT)
- difference of returns on the small and large stock portfolios (SML)
- difference of returns on high and low book-to-market ratio portfolio returns (HML)

Factor model (Cont'd)

We assume the following linear relationship holds between the return rates of risky assets and the market factors,

$$R_t = A + B \cdot F_t + c_t,$$

where, $A = (a_1, a_2, ..., a_n)' \in R^n$ denotes the inherent expected return vector of risky assets, $B = (b_{ij}) \in R^{n \times L}$ is the factor loading coefficient matrix, and $c_t = (c_t^1, c_t^2, ..., c_t^n)' \in R^n$ is the residual term.

- Factor can reflect the macro market regime properly.
- Factor model can efficiently reduce the dimension of the large-scale problem.

Factor forecasting model

We use the q-order autoregression model $AR(q)$ to forecast the value of market factors. Concretely we assume, under particular regime M_t,

$$F_t(M_t) = s_0(M_t) + \sum_{i=1}^{q} s_i(M_t)F_{t-i}(M_t) + e_t(M_t),$$

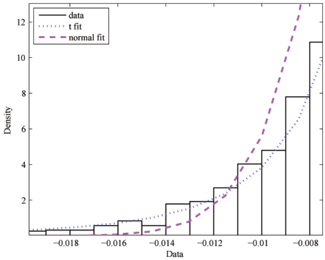
where $s_0(M_t), s_i(M_t), i = 1, 2, ..., q$, and $e_t(M_t)$ are the autoregression parameters and residual term, respectively.

The residual term $e_t(M_t)$ and c_t are assumed following to the joint student t distribution with the same degree of freedom.

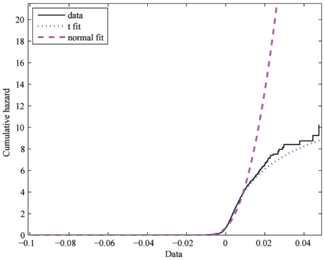
Distribution fitting performance

Comparison between normal fitting and student t fitting of S&P 500 index from Aug. 2005 to Jan. 2012

	ν	μ	β/σ	$ln(L)$	AIC	BIC
t fit	3.11037	0.000199297	0.00265426	65229.5	−130453.0	−130430.0
normal fit	$+\infty$	0.000122412	0.00426178	63164.4	−126324.8	−126309.5



Probability density function



Cumulative hazard funcion

Conditional mean and variance

The conditional mean and variance of factors are:

$$E[F_t^j|\mathcal{F}_{t-1}, M_t] = s_0(M_t) + \sum_{i=1}^{q} s_i(M_t)F_{t-i}^j, \; j = 1, 2, ..., L,$$

$$\sigma^2[F_t^j|\mathcal{F}_{t-1}, M_t] = \sigma^2_{F_t^j}(M_t) \; j = 1, 2, ..., L.$$

The conditional mean and variance of stock return rates are:

$$E[R_t^i|\mathcal{F}_{t-1}, M_t] = a_i + \sum_j b_{i,j}(M_t)(s_0(M_t) + \sum_{i=1}^{q} s_i(M_t)F_{t-i}^j), \; i = 1, 2, ..., n,$$

$$\sigma^2[R_t^i|\mathcal{F}_{t-1}, M_t] = \sum_j b_{i,j}^2(M_t)\sigma^2_{F_t^j}(M_t) + \sigma^2_{c_t^i}(M_t), \; i = 1, 2, ..., n,$$

$$\sigma[R_t^i, R_t^k|\mathcal{F}_{t-1}, M_t] = \sum_j b_{i,j}(M_t)b_{k,j}(M_t)\sigma^2_{F_t^j}(M_t), \; i, k = 1, 2, ..., n, \; i \neq k.$$

Regime-based risk measure

We construct the multi-period risk measure in two-level.

The inner level measures the one-step conditional investment risk mapping under individual regimes M_t,

$$\rho_{M_t}(\cdot) : L_p(\mathcal{F}_t) \to L_p(\mathcal{F}_{t-1}(M_t)) \tag{1.1}$$

We require $\rho_{M_t}(\cdot)$ satisfy translation invariance, monotonicity, and convexity.
It is called the weak coherent risk measure (Artzner et al.[1997], Carr et al.[2001]), or the convex risk measure (Fölmer and Scchied[2002], Frittelli and Rosazza[2002]).

Properties

The outer level combines all the "sub-risks" together through proper regime switching probabilities.

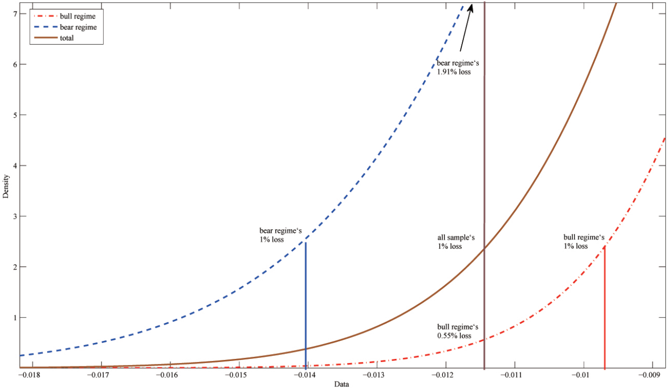
$$\rho_{t-1,T}(Z) = E_{M_{t-1}}[\rho_{M_t}(\rho_{t,T}(Z))], \tag{1.2}$$

Theorem

If the one-step conditional risk mapping ρ_{M_t} satisfies translation invariance, monotonicity, and convexity, then the recursive risk measure (1.2) satisfies the following properties:
(1) For all $Z, W \in \mathcal{F}_T$ such that $Z \leq W$, $\rho_{t,T}(Z) \leq \rho_{t,T}(W)$,
(2) For any $0 \leq t < \theta \leq T-1$, $Z, W \in \mathcal{F}_T$, $\rho_{t,T}(Z) \leq \rho_{t,T}(W)$ holds if $\rho_{\theta,T}(Z) \leq \rho_{\theta,T}(W)$.
(3)$\rho_{t,T}(\lambda Z + (1-\lambda)W) \leq \lambda\rho_{t,T}(Z) + (1-\lambda)\rho_{t,T}(W)$ holds for all $Z, W \in \mathcal{F}_T$, $t \leq T$.

properties (1)-(3) are so-called dynamic monotonicity, time consistency, and dynamic convexity, respectively (Wang[1999], Ruszczyński[2010], Cheridito et al[2006]).

VaR(CVaR) under different regimes



Tail density curves under different regimes

Portfolio selection model

The investor joins the market at time 0 with an initial wealth x_0 and plans to invest his/her wealth in the stock market for T consecutive periods in a self-financing way. Let x_t be the total wealth at the beginning of the tth period, and let $u_t^i, i = 1, 2, ..., n$, be the cash amount invested in the ith risky asset at stage t. Here we assume u_t is M_t adapted.

$$x_{t+1} = e_{t+1}^0 x_t + R'_{t+1} u_t,\ \ t = 0, 1, ..., T-1.$$

We maximize the expected final wealth, while controlling the multi-period investment risk not exceeding a given thredhold δ, which forms the optimization problem $P1(\delta)$;

$$\begin{aligned} P1(\delta): \quad \text{Max} \quad & E(x_T) \\ \text{s.t.} \quad & \rho_{0,T}(-x_T) \leq \delta, \\ & x_{t+1} = e_{t+1}^0 x_t + R'_{t+1} u_t,\ \ t = 0, 1, ..., T-1; \end{aligned}$$

Portfolio selection model(Cont'd)

We minimize the multi-period investment risk, while requiring the expected final wealth to be at least ϵ, which forms the optimization problem $P2(\epsilon)$;

$$
\begin{aligned}
P2(\epsilon): \quad \text{Min} \quad & \rho_{0,T}(-x_T) \\
\text{s.t.} \quad & E(x_T) \geq \epsilon, \\
& x_{t+1} = e^0_{t+1} x_t + R'_{t+1} u_t, \ t = 0, 1, ..., T-1;
\end{aligned}
$$

We maximize the linear combination of the expected terminal wealth and the investment risk by introducing a risk averse factor ω, which forms the optimization problem $P3(\omega)$.

$$
\begin{aligned}
P3(\omega): \quad \text{Max} \quad & E(x_T) - \omega \rho_{0,T}(-x_T), \\
\text{s.t.} \quad & x_{t+1} = e^0_{t+1} x_t + R'_{t+1} u_t, \ t = 0, 1, ..., T-1.
\end{aligned}
$$

Transformation

Let $P(M_t, k)$ denote the conditional probability of the kth scenario generated by node M_t, and $d(M_t, k)$ the accumulated autocorrelation coefficients along the sub-path $M_t, M_{t+1}, ..., M_T$, beginning from node M_t, of the corresponding kth scenario. Then we have the following recursive formula about components of $d(M_t, k)$.

$$d_i^0(M_t, k) = s_i(M_t, k),\ i = 0, 1, ..., q,$$
$$d_0^{k+1}(M_t, k) = d_0^k(M_t, k) + d_1^k(M_t, k)s_0(M_t, k),\ k = 1, 2, ..., T,$$
$$d_i^{k+1}(M_t, k) = d_{i+1}^k(M_t, k) + d_1^k(M_t, k)s_i(M_t, k),\ i = 1, 2, ..., q-1,\ k = 1, ...$$
$$d_q^{k+1}(M_t, k) = d_1^k(M_t, k)s_q(M_t, k),\ k = 1, 2, ..., T,$$

Transformation(Cont'd)

At any stage t, the conditional recursive risk measure can be expressed as

$$\rho_{t,T}(-x_T) = -E[x_T|\mathcal{F}_t] + \sum_{\tau=t+1}^{T} E_{M_t}[E_{M_{t+1}}[....E_{M_{\tau-1}}[C_{\sigma_\tau(M_\tau)}\sqrt{\sigma^2(E[x_T|\mathcal{F}_\tau]|\mathcal{F}_{\tau-1}, M_\tau)}]]],$$

where

$$E[x_T|\mathcal{F}_t] = \sum_{k\in\aleph^{T-t}M_t} P(M_t,k)(R^0_{t,T}x_t + \sum_{s=t}^{T-1} R^0_{s+1,T}(A + B\cdot(d_0^{s-t}(M_t,k) + \sum_{i=1}^{q} d_i^{s-t}(M_t,k)F_{t+1-i}))'u_s,$$

$$C_{\sigma_s(M_\tau)} = \frac{\nu}{\nu-1}(1+\frac{F_0^{-1}(\alpha(M_\tau))}{\nu})\frac{f_0(F_0^{-1}(\alpha(M_\tau)))}{\alpha(M_\tau)}\sqrt{\frac{\nu-2}{\nu}},$$

$$\sigma^2(-E[x_T|\mathcal{F}_\tau]|\mathcal{F}_{\tau-1}, M_\tau) = \sigma^2(E[x_T|\mathcal{F}_\tau]|\mathcal{F}_{\tau-1}, M_\tau),$$

$$= \sum_j(\sigma_{F_\tau^j}(M_\tau))^2(\sum_{k\in\alpha^{T-\tau}M_\tau} P(M_\tau,k)(R^0_{\tau,T}(b_j(M_\tau)'u_{\tau-1}) + \sum_{s=\tau}^{T-1} R^0_{s+1,T}d_1^{s-\tau}(M_\tau,k,j)(b_j(M_t)'u_s)))^2$$

$$+\sum_i(\sigma_{c_\tau^i})^2(\sum_{k\in\alpha^{T-\tau}M_\tau}(R^0_{\tau,T}u^i_{\tau-1}))^2,\ \tau = t+1,...,T-1.$$

Transformed problem

By introducing auxiliary variables, $P1(\delta)$ can be transformed into the following standard second-order cone program (SOCP):

$$\text{Max} \quad \sum_{k\in\aleph^T M_0} P(M_0,k)(R^0_{0,T}x_t + \sum_{s=0}^{T-1} R^0_{s+1,T}(A + B\cdot(d^s_0(M_0,k) + \sum_{i=1}^{q} d^s_i(M_0,k)F_{1-i}))'u_s)$$

$$\text{s.t.} \quad \sum_{t=1}^{T}(\sum_{M_t\in\mathcal{F}_{M_t}} P(M_t)(C_{\sigma_t(M_t)}V_t(M_t))) \le \delta,$$

$$\sqrt{\sum_{k\in\aleph^{T-t}M_t} y_t(M_t,k)'y_t(M_t,k) + z_t(M_t)'z_t(M_t)} \le V_t(M_t),\ M_t\in\Xi(M)\setminus\{M_0\},$$

$$P(M_t,k)\sigma_{F_j}(M_t)(R^0_{t,T}(b_j(M_t)'u_{t-1}) + \sum_{s=t}^{T-1} R^0_{s+1,T}d_1^{s-t}(M_t,k,j)(b_j(M_t)'u_s)) = y^j_t(M_t,k),$$

$$M_t\in\Xi(M)\setminus\{M_0\},\ k\in\aleph^{T-t}M_t,\ j=1,2,...,L,$$

$$R^0_{t,T}\sigma_{c,i}u^i_{t-1}\sqrt{\sum_{k\in\aleph^{T-t}M_t} P(M_t,k)^2} = z^i_t(M_t),\ M_t\in\Xi(M)\setminus\{M_0\},\ i=1,2,...,n.$$

Transformed problem(Cont'd)

Similarly, $P2(\epsilon)$ and $P3(\omega)$ can be also transformed into SOCP.

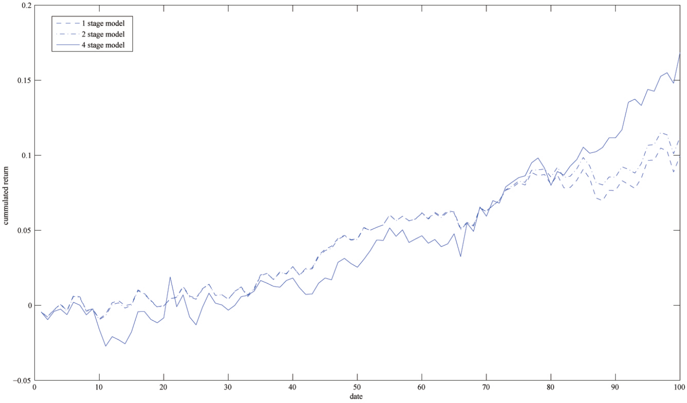
SOCP can be efficiently solved by some commercial optimization softwares, such as MOSEK.

Compared with current method for multi-stage portfolio selection under dynamic risk measure, our method is easier to implement and more efficient computationally.

Numerical experience

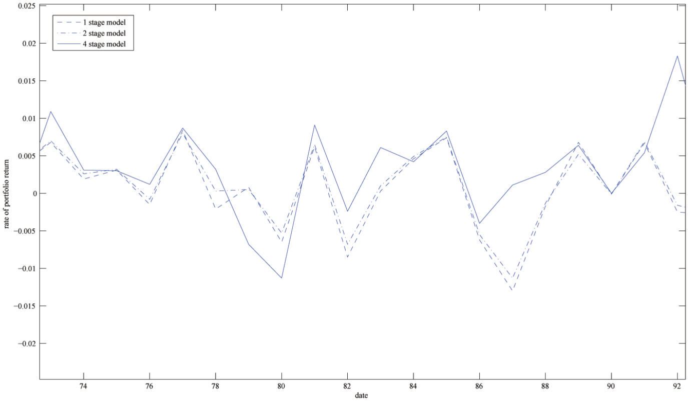
- We randomly choose 13 stocks from different industries in both Dow Jones Industrial Average and S&P 500 Indexes.
- We use 7 factors to explain the return rates.
- We use weekly data from February 14, 1977 to January 30, 2012, and take the last 100 weeks to be the out-of-sample horizon.
- We demonstrate out-of-sample test in a rolling way.

Out-of-sample performance



Out-of-sample cumulated wealth performance

Out-of-sample performance (Cont'd)



Out-of-sample rates of return performance between Week 73 and Week 92

Out-of-sample performance (Cont'd)

Portfolios performance of single and multi stages models

	1 stage		3 stages			
p^1	0.5	0.999	0.5	1	2	5
mean of return rate	0.0012	0.0014	0.0012	0.0016	0.0016	0.0018
std. of return rate	0.0063	0.0141	0.0063	0.0114	0.0127	0.0131
final wealth	1.1240	1.1366	1.1232	1.1593	1.1559	1.1823

portfolio performance of different stages models

	1 stages	2 stages	3 stages	4 stages	5 stages
mean of return rate	9.95E-04	0.0011	0.0013	0.0017	0.0015
std. of return rate	0.0054	0.0052	0.0051	0.0078	0.0135
final wealth	1.0998	1.1124	1.1346	1.1686	1.1460

Conclusions

- The "two-level" information structure can capture the high-kurtosis, fat-tail, and left-skewed properties of financial market.
- The regime-based recursive risk measure is time consistent. And it can reflect the different measure under different market states.
- The corresponding dynamic portfolio selection problem is tractable and performance well in numerical experience .

Thank you!