Stochastic geometric programming with joint probabilistic constraints

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- Part I: Stochastic geometric programs
  - joint chance constraint
  - independent random variables
  - normal distribution
- Part II: Stochastic rectangular geometric programs
  - joint chance constraint
  - independent rows
  - elliptical distribution

Part I: Stochastic geometric programs

Geometric programs

A geometric program can be formulated as

$$(GP) \quad \min_t g_0(t) \text{ s.t. } g_k(t) \le 1,\ k = 1, \cdots, K,\ t \in \mathbb{R}^M_{++}$$

with

$$g_k(t) = \sum_{i \in I_k} c_i \prod_{j=1}^{M} t_j^{a_{ij}},\ k = 0, \cdots, K.$$

$\{I_k,\ k = 0, \cdots, K\}$ is the disjoint index sets of $\{1, \cdots, Q\}$.

We call $c_i \prod_{j=1}^{M} t_j^{a_{ij}}$ a monomial and $g_k(t)$ a posynomial.

Geometric programs (Cont'd)

Geometric programs have a number of practical problems, such as

- shape optimization problems (Boyd et al., 2007)
- electrical circuit design problems (Boyd et al., 2007)
- mechanical engineering problems (Wiebking, 1977)
- economic and managerial problems (Luptáčik, 1981)
- nonlinear network problems (Kim et al., 2007)

Shape optimization problem

Example: A shape optimization problem,

$$
\begin{aligned}
\max_{h,w,\zeta} \quad & hw\zeta \\
\text{s.t.} \quad & 2hw + 2h\zeta \le A_{wall}, \\
& w\zeta \le A_{flr}, \\
& \alpha w \le h, \;\; h \le \beta w, \\
& \gamma w \le \zeta, \;\; \zeta \le \delta w.
\end{aligned}
$$

- maximize the volume of a box-shaped structure with height h, width w and depth ζ
- with constraint on total wall area $2(hw + h\zeta)$, and floor area $w\zeta$

Stochastic geometric programs

Usually, c_i are preset non-negative coefficients.

In practice use, c_i is not known deterministically but randomly.

Considering the randomness of c_i, one can formulate stochastic geometric programs.

Probabilistic constraints are frequently used to control the uncertainty of posynomial constraints.

Stochastic geometric programs (Cont'd)

Stochastic geometric programs with individual probabilistic constraints:

$$
\begin{aligned}
(SGPIPC) \quad & \min_{t \in \mathbb{R}^M_{++}} \quad E\left[\sum_{i \in I_0} c_i \prod_{j=1}^{M} t_j^{a_{ij}}\right] \\
& \text{s.t.} \quad P\left(\sum_{i \in I_k} c_i \prod_{j=1}^{M} t_j^{a_{ij}} \leq 1\right) \geq 1 - \epsilon_k, \ k = 1, \cdots, K.
\end{aligned}
$$

where $\epsilon_k \in (0, 0.5]$ is the tolerance probability for the k-th posynomial constraint.

Stochastic geometric programs (Cont'd)

Stochastic geometric programs with joint probabilistic constraints:

$$(SGPJPC) \quad \min_{t \in \mathbb{R}^M_{++}} \quad E\left[\sum_{i \in I_0} c_i \prod_{j=1}^{M} t_j^{a_{ij}}\right]$$

$$\text{s.t.} \quad P\left(\sum_{i \in I_k} c_i \prod_{j=1}^{M} t_j^{a_{ij}} \leq 1, \ k = 1, \cdots, K\right) \geq 1 - \epsilon.$$

where $\epsilon \in (0, 0.5]$ is the tolerance probability for all the posynomial constraints.

Shape optimization problem

A joint probabilistic constrained shape optimization problem,

$$\begin{aligned} \min_{h,w,\zeta} \quad & h^{-1}w^{-1}\zeta^{-1} \\ \text{s.t.} \quad & P\Big((2/A_{wall})hw + (2/A_{wall})h\zeta \leq 1,\ (1/A_{flr})w\zeta \leq 1\Big) \geq 1-\epsilon, \\ & \alpha h^{-1}w \leq 1,\ (1/\beta)hw^{-1} \leq 1, \\ & \gamma w\zeta^{-1} \leq 1,\ (1/\delta)w^{-1}\zeta \leq 1. \end{aligned}$$

- maximize the volume of a box-shaped structure with height h, width w and depth ζ
- with constraint on total wall area $2(hw + h\zeta)$, and floor area $w\zeta$

Literature reviews

- Dupačová (2009) discussed the (SGPIPC) problem.
- They find a deterministic formulation of the probabilistic constraint when c_i are normally distributed and independent of each other
- However, as far as we know, there is no in-depth research works on the (SGPJPC) problem. Dupačová (2009): non-convex?

Literature reviews (Cont'd)

(SGPJPC) problem are a generalization of stochastic linear program with joint probabilistic constraints

- Earlier research: Miller and Wagner (1965), Prékopa (1995): Separable case, right hand random
- Convexity, sub-differentiability: Prékopa, Henrion, Van Ackooij, 1990-2018.
- $\Rightarrow$ Might be nonconvex, non-differential! Bi-linear term! Indicator function!
- Convex Approximation: CVaR approximation (Zymler, Kuhn, Rustem,2011), Bernstein approximation (Nemirovski, Shapiro, 2006), SOCP approximation (Cheng, Lisser, 2012), D-C approximation (Hong, Yang, Zhang, 2011)
- Sample approaches: SAA, SA, Luetdke, Shapiro, Nemirovski, Campi, Xu, 2008-2018.

Our work

We work on stochastic geometric program with joint probabilistic constraints.

- We suppose that a_{ij} is deterministic and c_i is normally distributed and independent of each other, i.e., $c_i \sim N(E_{c_i}, \sigma_i^2)$.

The following techniques are used:

- standard variable transformation from geometric programming
- piecewise linear approximation
- Sequential convex approximation

Equivalent formulation

As c_i are independent of each other, we have

$$P\left(\sum_{i\in I_k} c_i \prod_{j=1}^{M} t_j^{a_{ij}} \leq 1,\ k=1,\cdots,K\right) \geq 1-\epsilon$$

is equivalent to

$$\prod_{k=1}^{K} P\left(\sum_{i\in I_k} c_i \prod_{j=1}^{M} t_j^{a_{ij}} \leq 1\right) \geq 1-\epsilon.$$

Equivalent formulation (Cont'd)

By introducing auxiliary variables $y_k \in \mathbb{R},\ k = 1, \cdots, K$, (SGPJPC) problem can be equivalently transformed into

$$\begin{aligned}
\min_{t \in \mathbb{R}^M_{++}, y \in \mathbb{R}^K} \quad & E\left[\sum_{i \in I_0} c_i \prod_{j=1}^{M} t_j^{a_{ij}}\right] \\
\text{s.t.} \quad & P\left(\sum_{i \in I_k} c_i \prod_{j=1}^{M} t_j^{a_{ij}} \leq 1\right) \geq y_k,\ k = 1, \cdots, K, \\
& \prod_{k=1}^{K} y_k \geq 1 - \epsilon,\ y_k \geq 0.
\end{aligned}$$

Equivalent formulation (Cont'd)

As $c_i \sim N(E_{c_i}, \sigma_i^2)$, (SGPJPC) problem is further equivalent to

$$\begin{aligned}
\min_{t \in \mathbb{R}^M_{++}, y \in \mathbb{R}^K} \quad & \sum_{i \in I_0} E_{c_i} \prod_{j=1}^{M} t_j^{a_{ij}} \\
\text{s.t.} \quad & \sum_{i \in I_k} E_{c_i} \prod_{j=1}^{M} t_j^{a_{ij}} + \Phi^{-1}(y_k) \sqrt{\sum_{i \in I_k} \sigma_i^2 \prod_{j=1}^{M} t_j^{2a_{ij}}} \le 1, \; k = 1, \cdots, K, \\
& \prod_{k=1}^{K} y_k \ge 1 - \epsilon, \; y_k \ge 0.
\end{aligned}$$

$\Phi^{-1}(y_k)$ is the quantile of standard normal distribution $N(0, 1)$.

Equivalent formulation (Cont'd)

The standard variable transformation $r_j = \log(t_j),\ j = 1, \cdots, M$ and $x_k = \log(y_k),\ k = 1, \cdots, K$ leads to the equivalent formulation:

$$\min_{r \in \mathbb{R}^M, x \in \mathbb{R}^K} \quad \sum_{i \in I_0} E_{c_i} \exp\left\{\sum_{j=1}^{M} a_{ij} r_j\right\}$$

$$\text{s.t.} \quad \sum_{i \in I_k} E_{c_i} \exp\left\{\sum_{j=1}^{M} a_{ij} r_j\right\} + \sqrt{\sum_{i \in I_k} \sigma_i^2 \exp\left\{\sum_{j=1}^{M} (2a_{ij} r_j + \log(\Phi^{-1}(e^{x_k})^2))\right\}} \le 1,\ k = 1, \cdots, K,$$

$$\sum_{k=1}^{K} x_k \ge \log(1 - \epsilon),\ x_k \le 0,\ k = 1, \cdots, K.$$

Convex optimization problem!

Property of $\Phi^{-1}(\cdot)$

$\Phi^{-1}(\cdot)$ is also called the probit function:

$$\Phi^{-1}(z) = \sqrt{2}\,\mathrm{erf}^{-1}(2z-1), \quad z \in (0,1).$$

The inverse error function is a nonelementary function which can be represented by the Maclaurin series:

$$\mathrm{erf}^{-1}(z) = \sum_{p=0}^{\infty} \frac{\lambda_p}{2p+1} \left(\frac{\sqrt{\pi}}{2} z \right)^{2p+1},$$

where $\lambda_0 = 1$ and

$$\lambda_p = \sum_{i=0}^{p-1} \frac{\lambda_i \lambda_{p-1-i}}{(i+1)(2i+1)} > 0, \; p = 1, 2, \cdots.$$

Property of $\log(\Phi^{-1}(e^{x_k})^2)$

- $\log(\Phi^{-1}(e^{x_k})^2)$ is convex for $1 > y_k \geq 1 - \epsilon \geq 0.5$.
- Moreover, $\log(\Phi^{-1}(e^{x_k})^2)$ is always monotonic increasing.
- nonelementary $\log(\Phi^{-1}(e^{x_k})^2) \Rightarrow$ approximate by a piecewise linear function from below:

$$F_s(x_k) = d_s x_k + b_s, \;\; s = 1, \cdots, S,$$

such that

$$F_s(x_k) \leq \log(\Phi^{-1}(e^{x_k})^2), \;\; \forall x_k \in [\log(1-\epsilon), 0), \;\; s = 1, \cdots, S.$$

Piecewise linear approximation

- For a practical use, we can choose the tangent lines of $\log(\Phi^{-1}(e^{x_k})^2)$ at different points in $[\log(1-\epsilon), 0)$, say $\xi_1, \xi_2, \cdots, \xi_S$.
- Then, we have

$$d_s = \frac{2e^{\xi_s}(\Phi^{-1})^{(1)}(e^{\xi_s})}{\Phi^{-1}(e^{\xi_s})}$$

and

$$b_s = -d_s\xi_s + \log(\Phi^{-1}(e^{\xi_s})^2), \ s = 1, \cdots, S.$$

Theorem

Using the piecewise linear function $F(x_k)$, we can found an approximation of (SGPJPC) problem:

$$(SGP_A)$$

$$\begin{aligned}
\min_{r\in\mathbb{R}^M, x\in\mathbb{R}^K} \quad & \sum_{i\in I_0} E_{c_i} \exp\left\{\sum_{j=1}^{M} a_{ij} r_j\right\} \\
& \sum_{i\in I_k} E_{c_i} \exp\left\{\sum_{j=1}^{M} a_{ij} r_j\right\} + \sqrt{\sum_{i\in I_k} \sigma_i^2 \exp\left\{\sum_{j=1}^{M} (2a_{ij} r_j + d_s x_k + b_s)\right\}} \\
& \qquad \leq 1,\ s = 1, \cdots, S,\ k = 1, \cdots, K, \\
& \sum_{k=1}^{K} x_k \geq \log(1-\epsilon),\ x_k \leq 0,\ k = 1, \cdots, K.
\end{aligned}$$

The optimal value is a lower bound of the (SGPJPC) problem. When S goes to infinity, the approximation is tight.

Sequential convex approximation

- Sequential convex approximation $\Rightarrow$ upper bound
- Basic idea: decomposing into subproblems where a subset of variables is fixed alternatively.
- We first fix $y = y^n$ and update t by solving

$$
\begin{aligned}
(SQ_1) \quad & \min_{t \in \mathbb{R}^M_{++}} \quad \sum_{i \in I_0} E_{c_i} \prod_{j=1}^{M} t_j^{a_{ij}} \\
& \text{s.t.} \quad \sum_{i \in I_k} E_{c_i} \prod_{j=1}^{M} t_j^{a_{ij}} + \Phi^{-1}(y_k^n) \sqrt{\sum_{i \in I_k} \sigma_i^2 \prod_{j=1}^{M} t_j^{2a_{ij}}} \leq 1, \\
& \qquad k = 1, \cdots, K
\end{aligned}
$$

Sequential convex approximation (Cont'd)

- and then fix $t = t^n$ and update y by solving

$$
\begin{aligned}
(SQ_2) \quad \min_{y \in \mathbb{R}^K_+} \quad & \sum_{k=1}^{K} \phi_k \, y_k \\
\text{s.t.} \quad & y_k \le \Phi\left(\frac{1 - \sum_{i \in I_k} E_{c_i} \prod_{j=1}^{M} (t_j^n)^{a_{ij}}}{\sqrt{\sum_{i \in I_k} \sigma_i^2 \prod_{j=1}^{M} (t_j^n)^{2a_{ij}}}} \right), \ k = 1, \cdots, K. \\
& \prod_{k=1}^{K} y_k \ge 1 - \epsilon, \ y_k \ge 0, \ k = 1, \cdots, K.
\end{aligned}
$$

- ϕ_k is a chosen searching direction.

Sequential convex approximation (Cont'd)

Algorithm 1 Sequential convex approximation

Initialization:

Choose an initial point y^0 of y feasible for (8). Set $n = 0$.

Iteration:

while $n \geq 1$ and $||y^{n-1} - y^n||$ is small enough **do**

- Solve problem (SQ_1); let t^n, θ^n and v^n denote an optimal solution of t, an optimal solution of the Lagrangian dual variable θ and the optimal value, respectively.
- Solve problem (SQ_2) with $\phi_k = \theta_k^n \cdot (\Phi^{-1})'(y_k^n)\sqrt{\sum_{i \in I_k} \sigma_i^2 \prod_{j=1}^{M} (t_j^n)^{2a_{ij}}}$; let $\tilde{y}$ denote an optimal solution.
- $y^{n+1} \leftarrow y^n + \tau(\tilde{y} - y^n)$, $n \leftarrow n + 1$. Here, $\tau \in (0, 1)$ is the step length.

end while

Output: t^n, v^n

Sequential convex approximation (Cont'd)

Theorem

Algorithm 1 converges in a finite number of iterations and the returned value v^n is a upper bound for problem (SGP).

- Problems (SQ_1) and (SQ_2) are both geometric programs, hence they can be transformed into a convex programming problem, and solved by interior point methods.

Settings

- Set $\alpha = \gamma = 0.5$, $\beta = \delta = 2$, $\epsilon = 5\%$,
- Assume $1/A_{wall} \sim N(0.005, 0.01)$ and $1/A_{flr} \sim N(0.01, 0.01)$.
- By using CVX software, we solve the approximation problems with Matlab R2012b, on a PC with a 2.6 Ghz Intel Core i7-5600U CPU and 12.0 GB RAM.
- We solve five groups of approximation problems with different number of segments, S.

Computational results

Table 1: Computational results

S	Var. Num.	Con. Num.	Low. bound	CPU(s)	Upp. bound	CPU(s)	Gap(%)
1	133	60	0.232	0.5955	0.256	5.5274	9.655
2	184	91	0.234	0.6272	0.256	5.5274	8.789
5	283	153	0.241	0.9480	0.256	5.5274	6.044
10	513	273	0.252	1.3554	0.256	5.5274	1.713
20	973	513	0.256	1.9986	0.256	5.5274	0

Sequential convex approximation algorithm converges within 7 outer iterations

II. Stochastic rectangular geometric programs

Stochastic rectangular geometric programs

Stochastic rectangular geometric programs with joint probabilistic constraints:

$$
\begin{aligned}
(SRGP) \quad & \min_{t \in \mathbb{R}^M_{++}} \quad \mathbb{E}\left[\sum_{i=1}^{I_0} c_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0} \right] \\
& \text{s.t.} \quad \mathbb{P}\left(\alpha_k \leq \sum_{i=1}^{I_k} c_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} \leq \beta_k, \ k = 1, \ldots, K \right) \geq 1 - \epsilon.
\end{aligned}
$$

$1 - \epsilon$ is a prespecified probability with $\epsilon < 0.5$, a_{ij}^k, $k = 1, \ldots, K$, $i = 1, \ldots, I_k$, $j = 1, \ldots, M$, are given parameters and c_i^k, $k = 1, \ldots, K$, $i = 1, \ldots, I_k$, are random parameters with non-negative mean values.

Elliptical distribution

We suppose $c^k = [c_1^k, c_2^k, \ldots, c_{I_k}^k]$ follows a multivariate elliptical distribution $Ellip_{I_k}(\mu^k, \Gamma^k, \varphi_k)$ with $\mu^k = [\mu_1^k, \mu_2^k, \ldots, \mu_{I_k}^k]^\top \geq 0$, and $\Gamma^k = \{\sigma_{i,p}^k,\ i, p = 1, \ldots, I_k\}$ positive definite, $k = 1, \ldots, K$.

Elliptical distribution

A L-dimensional random vector ξ follows an elliptical distribution $Ellip_L(\mu, \Gamma, \varphi)$ if its characteristic function is given by $\mathbb{E}e^{iz^\top c} = e^{iz^\top \mu}\varphi(z^\top \Gamma z)$ where φ is the characteristic generator function, μ is the location parameter, and Γ is the scale matrix.

Moreover, we assume that c^k, $k = 1, \ldots, K$ are pairwise independent.

Elliptical distribution (Cont'd)

Elliptical distributions includes

- normal distribution with $\varphi(t) = \exp\{-\frac{1}{2}t\}$
- student's t distribution with $\varphi(t)$ varying with its degree of freedom
- Cauchy distribution with $\varphi(t) = \exp\{-\sqrt{t}\}$
- Laplace distribution with $\varphi(t) = (1 + \frac{1}{2}t)^{-1}$
- logistic distribution with $\varphi(t) = \frac{2\pi\sqrt{t}}{e^{\pi\sqrt{t}} - e^{-\pi\sqrt{t}}}$

Elliptical distribution (Cont'd)

$Ellip_L(\mu, \Gamma, \varphi)$

- mean value is μ,
- covariance matrix is $\frac{E(r^2)}{\text{rank}(\Gamma)}\Gamma$, where r is the random radius.

Proposition[Embrechts et al., 2005]

If a L-dimensional random vector ξ follows an elliptical distribution $Ellip_L(\mu, \Gamma, \varphi)$, then for any $(L \times N)$-matrix A and any N-vector b, $A\xi + b$ follows an N-dimensional elliptical distribution $Ellip_N(A\mu + b, A\Gamma A^\top, \varphi)$.

Assumptions on the parameters

Assumption 1

We assume that

- $\frac{\phi'_{\varphi_k}(\Phi^{-1}_{\varphi_k}(1-\epsilon))}{\phi_{\varphi_k}(\Phi^{-1}_{\varphi_k}(1-\epsilon))}\Phi^{-1}_{\varphi_k}(1-\epsilon) < -1,\ k = 1,\dots,K,$
- $(\Phi^{-1}_{\varphi_k}(1-\epsilon))^2\sigma^k_{i,p} - \mu^k_i\mu^k_p > 0,\ i,p = 1,\dots,I_k,\ k = 1,\dots,K,$
- $2\sigma^k_{i,p}\left(1 - \frac{\phi'_{\varphi_k}(\Phi^{-1}_{\varphi_k}(z))}{\phi_{\varphi_k}(\Phi^{-1}_{\varphi_k}(z))}\Phi^{-1}_{\varphi_k}(z)\right)\left((\Phi^{-1}_{\varphi_k}(z))^2\sigma^k_{i,p} - \mu^k_i\mu^k_p\right) - (2\sigma^k_{i,p}\Phi^{-1}_{\varphi_k}(z))^2 \geq 0,\ 1-\epsilon \leq z \leq 1,\ i,p = 1,\dots,I_k,\ k = 1,\dots,K.$

Equivalent formulation (Cont'd)

Given Assumption 1, the joint rectangular geometric chance constrained programs can be equivalently reformulated as

$$\min_{t\in\mathbb{R}^M_{++}} \quad \sum_{i=1}^{I_0} \mu_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0} \tag{1}$$

$$\text{s.t.} \quad \Phi_{\varphi_k}^{-1}(z_k^+)\sqrt{\sum_{i=1}^{I_k}\sum_{p=1}^{I_k} \sigma_{i,p}^k \prod_{j=1}^{M} t_j^{a_{ij}^k + a_{pj}^k}} - \sum_{i=1}^{I_k} \mu_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} \le -\alpha_k,\ k = 1,\dots,K. \tag{2}$$

$$\Phi_{\varphi_k}^{-1}(z_k^-)\sqrt{\sum_{i=1}^{I_k}\sum_{p=1}^{I_k} \sigma_{i,p}^k \prod_{j=1}^{M} t_j^{a_{ij}^k + a_{pj}^k}} + \sum_{i=1}^{I_k} \mu_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} \le \beta_k,\ k = 1,\dots,K, \tag{3}$$

$$z_k^+ + z_k^- - 1 \ge y_k,\ 0 \le z_k^+, z_k^- \le 1,\ k = 1,\dots,K, \tag{4}$$

$$\prod_{k=1}^{K} y_k \ge 1-\epsilon,\ 0 \le y_k \le 1,\ k = 1,\dots,K. \tag{5}$$

Convex approximations of constraint (2)

Constraint (2) can be reformulated as

$$2\alpha_k \sum_{i=1}^{I_k} \mu_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} + \sum_{i=1}^{I_k} \sum_{p=1}^{I_k} ((\Phi_{\varphi_k}^{-1}(z_k^+))^2 \sigma_{i,p}^k - \mu_i^k \mu_p^k) \prod_{j=1}^{M} t_j^{a_{ij}^k + a_{pj}^k} \le \alpha_k^2, \ k = 1, ...K.$$

standard variable transformation: $r_j = \log(t_j), \ j = 1, \dots, M$

$$2\alpha_k \sum_{i=1}^{I_k} \mu_i^k \exp\left\{ \sum_{j=1}^{M} a_{ij}^k r_j \right\} + \sum_{i=1}^{I_k} \sum_{p=1}^{I_k} \exp\left\{ \sum_{j=1}^{M} (a_{ij}^k r_j + a_{pj}^k r_j) \right.$$
$$\left. + \log((\Phi_{\varphi_k}^{-1}(z_k^+))^2 \sigma_{i,p}^k - \mu_i^k \mu_p^k) \right\} \le \alpha_k^2, \ k = 1, \dots, K.$$

Hard to deal with: $\log((\Phi_{\varphi_k}^{-1}(z_k^+))^2 \sigma_{i,p}^k - \mu_i^k \mu_p^k)$

Property of $\log((\Phi_{\varphi_k}^{-1}(z_k^+))^2\sigma_{i,p}^k - \mu_i^k\mu_p^k)$

Proposition

Given Assumption 1, $f_{i,p,k}(z_k^+) = \log((\Phi_{\varphi_k}^{-1}(z_k^+))^2\sigma_{i,p}^k - \mu_i^k\mu_p^k)$ is monotone increasing and convex for $z_k^+ \in [1-\epsilon, 1)$, $i, p = 1, \ldots, I_k$, $k = 1, \ldots, K$.

Piece-wise linear approximations

Choose points $\xi_1, \xi_2, \ldots, \xi_S$ in $[1-\epsilon, 1)$

- tangent lines at the points $\rightarrow$ lower approximation
- segments between the points $\rightarrow$ uppser approximation

Approximation of $\log((\Phi_{\varphi_k}^{-1}(z_k^+))^2\sigma_{i,p}^k - \mu_i^k\mu_p^k)$

convex approximation of constraint (2):

$$\begin{cases} 2\alpha_k \sum_{i=1}^{I_k} \mu_i^k \exp\left\{ \sum_{j=1}^{M} a_{ij}^k r_j \right\} + \sum_{i=1}^{I_k}\sum_{p=1}^{I_k} \exp\left\{ \sum_{j=1}^{M} (a_{ij}^k r_j + a_{pj}^k r_j) \right. \\ \qquad \left. + \, \omega_{i,p,k}^L \right\} \leq \alpha_k^2, \; k = 1, \dots, K, \\ d_{s,i,p,k} z_k^+ + b_{s,i,p,k} \leq \omega_{i,p,k}^L, \; s = 1, \dots, S, \; i, p = 1, \dots, I_k, \; k = 1, \dots, K. \end{cases}$$

an inner approximation of the feasible set of $(SRGP)$.

Approximation of $\log((\Phi^{-1}_{\varphi_k}(z_k^+))^2\sigma^k_{i,p} - \mu^k_i\mu^k_p)$

convex approximation of constraint (2):

$$\begin{cases} 2\alpha_k \sum_{i=1}^{I_k} \mu_i^k \exp\left\{ \sum_{j=1}^{M} a_{ij}^k r_j \right\} + \sum_{i=1}^{I_k}\sum_{p=1}^{I_k} \exp\left\{ \sum_{j=1}^{M} (a_{ij}^k r_j + a_{pj}^k r_j) \right. \\ \qquad \left. + \, \omega^U_{i,p,k} \right\} \le \alpha_k^2, \ k = 1, \dots, K, \\ \tilde{d}_{s,i,p,k} z_k^+ + \tilde{b}_{s,i,p,k} \le \omega^U_{i,p,k}, \ s = 1, \dots, S, \ i, p = 1, \dots, I_k, \ k = 1, \dots, K. \end{cases}$$

an outer approximation of the feasible set of $(SRGP)$.

Approximation for (3)

variable transformation $r_j = \log(t_j),\ j = 1, \ldots, M$ to (3)

$$\sqrt{\sum_{i=1}^{I_k}\sum_{p=1}^{I_k}\exp\left\{\sum_{j=1}^{M}(a_{ij}^k r_j + a_{pj}^k r_j) + 2\log(\Phi_{\varphi_k}^{-1}(z_k^-))\right\}} + \sum_{i=1}^{I_k}\mu_i^k \exp\left\{\sum_{j=1}^{M} a_{ij}^k r_j\right\} \le \beta_k,\ k = 1, \ldots, K.$$

To deal with : $\log(\Phi_{\varphi_k}^{-1}(z_k^-))$

Approximation of $\log(\Phi_{\varphi_k}^{-1}(z_k^-))$

Property of $\log(\Phi_{\varphi_k}^{-1}(z_k^-))$

Given Assumption 1, $\log(\Phi_{\varphi_k}^{-1}(z_k^-))$ is monotone increasing and convex on $[1-\epsilon, 1)$.

Choose points $\xi_1, \xi_2, \ldots, \xi_S$ in $[1-\epsilon, 1)$

- tangent lines at the points → lower approximation
- segments between the points → upper approximation

Main result

Approximation $(SRGP_L)$: lower bound of $(SRGP)$

$$\min_{r,z^+,z^-,x,\omega^L,\tilde{\omega}^L} \quad \sum_{i=1}^{I_0} \mu_i^0 \exp\left\{\sum_{j=1}^{M} a_{ij}^0 r_j\right\}$$

$$\text{s.t.} \quad 2\alpha_k \sum_{i=1}^{I_k} \mu_i^k \exp\left\{\sum_{j=1}^{M} a_{ij}^k r_j\right\} + \sum_{i=1}^{I_k}\sum_{p=1}^{I_k} \exp\left\{\sum_{j=1}^{M}(a_{ij}^k r_j + a_{pj}^k r_j) + \omega_{i,p,k}^L\right\} \le \alpha_k^2,\ k = 1,\dots,K$$

$$d_{s,i,p,k} z_k^+ + b_{s,i,p,k} \le \omega_{i,p,k},\ s = 1,\dots,S,\ i,p = 1,\dots,I_k,\ k = 1,\dots,K,$$

$$\sum_{i=1}^{I_k} \mu_i^k \exp\left\{\sum_{j=1}^{M} a_{ij}^k r_j\right\} + \sqrt{\sum_{i=1}^{I_k}\sum_{p=1}^{I_k} \exp\left\{\sum_{j=1}^{M}(a_{ij}^k r_j + a_{pj}^k r_j) + 2\widetilde{\omega}_k^L\right\}} \le \beta_k,\ k = 1,\dots,K,$$

$$l_{s,k} z_k^- + q_{s,k} \le \widetilde{\omega}_k^L,\ s = 1,\dots,S,\ k = 1,\dots,K,$$

$$z_k^+ + z_k^- - 1 \ge e^{x_k},\ 0 \le z_k^+, z_k^- \le 1,\ k = 1,\dots,K,$$

$$\sum_{k=1}^{K} x_k \ge \log(1-\epsilon),\ x_k \le 0,\ k = 1,\dots,K.$$

The bound is tight!

Main result

Approximation $(SRGP_U)$: upper bound of $(SRGP)$

$$\min_{r,z^+,z^-,x,\omega^U,\tilde{\omega}^U} \quad \sum_{i=1}^{I_0} \mu_i^0 \exp\left\{\sum_{j=1}^{M} a_{ij}^0 r_j\right\}$$

$$\text{s.t.} \quad 2\alpha_k \sum_{i=1}^{I_k} \mu_i^k \exp\left\{\sum_{j=1}^{M} a_{ij}^k r_j\right\} + \sum_{i=1}^{I_k}\sum_{p=1}^{I_k} \exp\left\{\sum_{j=1}^{M} (a_{ij}^k r_j + a_{pj}^k r_j) + \omega_{i,p,k}^U\right\} \le \alpha_k^2, \ k = 1, \ldots, I$$

$$\tilde{d}_{s,i,p,k} z_k^+ + \tilde{b}_{s,i,p,k} \le \omega_{i,p,k}^U, \ s = 1, \ldots, S, \ i,p = 1, \ldots, I_k, \ k = 1, \ldots, K,$$

$$\sum_{i=1}^{I_k} \mu_i^k \exp\left\{\sum_{j=1}^{M} a_{ij}^k r_j\right\} + \sqrt{\sum_{i=1}^{I_k}\sum_{p=1}^{I_k} \exp\left\{\sum_{j=1}^{M} (a_{ij}^k r_j + a_{pj}^k r_j) + 2\tilde{\omega}_k^U\right\}} \le \beta_k, \ k = 1, \ldots, K,$$

$$\tilde{l}_{s,k} z_k^- + \tilde{q}_{s,k} \le \tilde{\omega}_k^U, \ s = 1, \ldots, S, \ k = 1, \ldots, K,$$

$$z_k^+ + z_k^- - 1 \ge e^{x_k}, \ 0 \le z_k^+, z_k^- \le 1, \ k = 1, \ldots, K,$$

$$\sum_{k=1}^{K} x_k \ge \log(1-\epsilon), \ x_k \le 0, \ k = 1, \ldots, K.$$

The bound is tight!

Shape optimization problem

Consider a joint probabilistic constrained rectangular shape optimization problem,

$$
\begin{aligned}
(SCP) \quad \min_{h,w,\zeta} \quad & h^{-1}w^{-1}\zeta^{-1} \\
\text{s.t.} \quad & \mathbb{P}\left(\begin{array}{c} \alpha_{wall} \leq (2/A_{wall})hw + (2/A_{wall})h\zeta \leq \beta_{wall} \\ \alpha_{flr} \leq (1/A_{flr})w\zeta \leq \beta_{flr} \end{array} \right) \geq 1-\epsilon, \\
& \gamma_{wh}h^{-1}w \leq 1, \ (1/\gamma_{hw})hw^{-1} \leq 1, \\
& \gamma_{w\zeta}w\zeta^{-1} \leq 1, \ (1/\gamma_{\zeta w})w^{-1}\zeta \leq 1.
\end{aligned}
$$

Example 1

- Set $\alpha = \gamma = 0.5$, $\beta = \delta = 2$, $\epsilon = 5\%$,
- Assume $1/A_{wall} \sim N(0.005, 0.01)$ and $1/A_{flr} \sim N(0.01, 0.01)$.
- By using CVX software, we solve the approximation problems with Matlab R2012b, on a PC with a 2.6 Ghz Intel Core i7-5600U CPU and 12.0 GB RAM.
- We solve 6 groups of approximation problems with different number of segments, S.

Computational results

Table 1: Computational results of approximations for normal distribution

S	Var.	Con.	UB	CPU(s)	Var.	Con.	LB	CPU(s)	Gap(%)
1	16	18	5.9119	1.2795	19	19	5.7587	1.2102	2.66
2	16	25	5.8188	0.9406	19	26	5.7587	1.0237	1.04
5	16	46	5.7644	0.9360	19	47	5.7639	1.0676	0.01
10	16	81	5.7645	1.1247	19	82	5.7643	0.9494	0.00
20	16	151	5.7644	1.2374	19	152	5.7643	1.2228	0.00
100	16	711	5.7644	2.0650	19	712	5.7643	1.9789	0.00

The bound is tight!

Example 1

Set $2/A_{wall}$ follows a Student's t distribution

- location parameter $\mu_{2/A_{wall}} = 0.01$
- the scale parameter $\Gamma_{2/A_{wall}} = 0.01$
- the degree of freedom $v_{2/A_{wall}} = 4$
- $2/A_{wall}$ and $1/A_{flr}$ are pairwise independent.

- Solved in CVX software
- We solve 7 groups of approximation problems with different number of segments, S.

Computational results

Table 2: Computational results of approximations for Student's t distribution

S	Var.	Con.	UB	CPU(s)	Var.	Con.	LB	CPU(s)	Gap(%)
1	16	18	13.8794	1.1772	19	19	5.8498	1.2815	137.26
2	16	25	8.8903	1.0984	19	26	5.8498	1.1373	51.98
5	16	46	6.0468	0.9796	19	47	5.8699	0.8857	3.01
10	16	81	5.9111	1.0510	19	82	5.8716	1.0800	0.67
20	16	152	5.8915	1.2446	19	152	5.8717	1.0739	0.34
100	16	711	5.8760	2.1234	19	712	5.8717	1.8112	0.07
500	16	3511	5.8725	5.9124	19	3512	5.8717	5.7727	0.01

The bound is also tight! Not that good than the normal case.

Conclusions

- Convex reformulation of joint chance constrained (rectangular) geometric problems
- Asymptotic tight upper and lower bounds

Further work

- Mixture-normal distributions, hyperbolic distribution et al.
- Random a_{ij}.
- Ambiguity of the distribution: distributional robust cases.

Thank you!