Multi-stage portfolio selection problem with dynamic stochastic dominance constraints

Jia Liu

Xi'an Jiaotong University

TEL:086-82663661, E-mail: jialiu@xjtu.edu.cn

Joint work with Yu Mei, Zhiping Chen and Bingbing Ji (Xi'an Jiaotong University)

(July, 25, 2021, Chengdu)

Outline

- Introduction
- Model description
- Scenario tree representation
- Approximations
  - Upper approximation
  - Lower approximation
- Numerical results
- Conclusion

Introduction

Portfolio selection

- utility preference $\max_x \mathbb{E}[u(r^\top x)]$
- risk preference $\max_x \mathbb{E}(r^\top x) - \lambda\rho(r^\top x)$
- probabilistic preference $\max_x \mathbb{E}[r^\top x] \quad \text{s.t.} \mathbb{P}(r^\top x \geq y) \geq 1 - \epsilon$

Introduction

Portfolio selection

- utility preference $\max_x \mathbb{E}[u(r^\top x)]$
- risk preference $\max_x \mathbb{E}(r^\top x) - \lambda\rho(r^\top x)$
- probabilistic preference $\max_x \mathbb{E}[r^\top x] \quad \text{s.t.} \mathbb{P}(r^\top x \geq y) \geq 1 - \epsilon$

Investment under preference ambiguity.

- Stochastic dominance (with a benchmark)
  - Dominance test (Levy, Post, Kuosmanen)
  - optimization, (Dentcheva, Ruszczyński, Luedtke, Schultz)
- Preference robust optimization
  - pairwise, moments, nominal,
  - Armbruster, Delage, Xu H.F., Homen-de-Mello,T., Hu J., Haskell
- State-dependent risk-aversion parameter

Introduction

Basic definitions of stochastic dominance:

Definition 1 (FSD)

$X \in \mathscr{L}_p$ dominates $Y \in \mathscr{L}_p$ in the first order, denoted $X \succeq_{(1)} Y$, if

$$P\{X \leq \eta\} \leq P\{Y \leq \eta\}, \quad \forall \eta \in R$$

We define expected shortfall function
$F_2(X;\eta) = \int_{-\infty}^{\eta} F(X;\alpha)d\alpha = \mathbb{E}[(\eta - X)_+]$.

Definition 2 (SSD)

$X \in \mathscr{L}_p$ dominates $Y \in \mathscr{L}_p$ in the second order, denoted $X \succeq_{(2)} Y$, if

$$F_2(X;\eta) \leq F_2(Y;\eta), \ \forall \eta \in R$$

Second-order stochastic dominance is particularly popular in industry since it models risk-averse preferences.

Introduction

Proposition 1

- $X \geq_{(1)} Y$ *iff* $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$ *for all* $u \in \mathscr{U}_1$*, here* $\mathscr{U}_1$ *denotes the set of all nondecreasing functions* $u : R \rightarrow R$.
- $X \geq_{(2)} Y$ *iff* $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$ *for all* $u \in \mathscr{U}_2$*, here* $\mathscr{U}_2$ *denotes the set of all concave and nondecreasing functions* $u : R \rightarrow R$.

- Dentcheva and Ruszczyński (2003) first considered optimization problem with SSD and derived the optimality conditions.
- Dentcheva and Ruszczyński (2006) developed duality relations and solved the dual problem by utilizing the piecewise linear structure of the dual functional
- Luedtke (2008) get new linear formulations for SSD with finite distributed benchmark
- Drapkin, Gollmer, Gotzes, Schultz, et al. (2011a,2011b) study cases where the random variables are induced by mixed-integer linear recourse

Introduction

Solution methods

- Sampling approaches are the most popular solution method (see, Dentcheva and Ruszczyński, 2003, Liu, Sun and Xu, 2016)
- Cut plane methods are the most efficient solution algorithm (see, e.g., Rudolf and Ruszczyński, 2003; Homem-de-Mello and Mehrotra, 2009; Sun, Xu, et al., 2013).

Strong application background in finance

- e.g., portfolio selection, index tracking applications (Dentcheva and Ruszczyński, 2006, Meskarian, Fliege and Xu 2014; Chen, Zhuang, L., 2019)

Our focus:

- Dynamic extension: compare random sequences
- Application: portfolio selection

Related works

Multi-stage portfolio selection + Stochastic dominance

- Introduce one univariate SD constraint on a certain random variable, such as the terminal wealth (Moriggia et al., 2019), the final cost (Singh and Djarmaraja, 2020) or the expected shortfall (Haskell and Jain, 2013)
  → The risks at intermediate stages cannot be controlled.
- Consider several univariate SD constraints (Yang et al., 2010; Kopa et al., 2018)
  → The risks at intermediate and final stages are handled separately and independently; Cannot reflect the dynamics of the random sequences.
- Adopt multivariate SD to characterize the risk in portfolio selection problems (Petrová, 2019)
  → Treat components in the wealth sequence equally; A discount rate sequence is needed.

What they loss: intertemporal preference ambiguity

Definition

Definition 3 (Dynamic SSD, Dentcheva and Ruszczyński, 2008)

Random sequence $(x_1, \cdots, x_T)$ dynamically dominates $(y_1, \cdots, y_T)$ in the second order with respect to a discount rate sequence set D, if

$$\sum_{t=1}^{T} \rho_t x_t \succeq_{(2)} \sum_{t=1}^{T} \rho_t y_t, \ \forall \rho \in D. \quad (1)$$

Choices of set D:

- Finite set: $D_1 = \{\rho^1, \cdots, \rho^k\}$
- Decreasing discount rate sequence set:
  $D_2 = \{\rho \in [0,1]^T \,|\, \rho_t \geq \rho_{t+1}, t = 1, \ldots, T-1\}$
- Product discount rate sequence set:
  $D_3 = \{[\rho_1, \rho_1\rho_2, \cdots, \prod_{t=1}^{T} \rho_t]^\top \mid \rho_t \in R_t \subset [0,1], \ t = 1, \cdots, T\}$
- Discount rate sequence set based on a reference:
  $D_4 = \{\rho \in [0,1]^T \,|\, \rho \geq \hat{\rho}\} \cap D_2$

Introduction

Our motivations

- Dynamic extension: compare random sequences to control risks
- Application: portfolio selection problems

Our contributions

- Adopt the dynamic SSD constraints to better control intermediate and final risks
- Derive an upper bound approximation and a lower bound approximation based on scenario tree representation
- Establish the convergence of the upper bound approximation

Model description

Dynamic settings

- n risky assets and one risk-free asset
- joins the market at time 0 with a positive initial wealth x_0
- invest for T periods
- at the beginning of each period, the current wealth can be reallocated
- the whole investment process is self-financing
- there exist transaction costs when buying or selling risky assets
- consider all the random processes on a probability space $(\Omega, \mathscr{F}, P)$, $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \cdots \subset \mathscr{F}_T \subset \mathscr{F}$
- u_t (cash amounts invested), b_t (buy), s_t (sell) $\lhd \mathscr{F}_t, t = 1, \cdots, T-1$

Dynamic SSD constrained portfolio selection model

Make our investment wealth process $\{x_t\}_{t=0,\cdots,T}$ preferable over the benchmark wealth process $\{y_t\}_{t=0,\cdots,T}$. Assume that $y_0 = x_0$ and y_t is also $\mathscr{F}_t$-measurable.

Typical benchmarks:

- a market index
- the equally weighted portfolio
- a portfolio suggested by a fund manager

Dynamic SSD constrained portfolio selection model

Our model:

$$\max_{u,b,s,x} \quad \mathbb{E}[x_T] \tag{2}$$

$$\text{s.t.} \quad x_{t+1} = r_{t+1}^{\top} u_t + r_{t+1}^{rf}[x_t - c_b\|b_t\|_1 - c_s\|s_t\|_1], \; t = 0, 1, \cdots, T-1, \tag{3}$$

$$u_0 = b_0, \; s_0 = 0, \; u_t = u_{t-1} + b_t - s_t, \; t = 1, \cdots, T-1, \tag{4}$$

$$\sum_{t=1}^{T} \rho_t x_t \succeq_{(2)} \sum_{t=1}^{T} \rho_t y_t, \; \forall \rho \in D, \tag{5}$$

$$\|u_t\|_1 \leq x_t - c_b\|b_t\|_1 - c_s\|s_t\|_1, \; t = 0, 1, \cdots, T-1, \tag{6}$$

$$u_t, b_t, s_t \in \mathbb{R}_+^n, \; u_t, b_t, s_t \lhd \mathscr{F}_t, \; t = 0, 1, \cdots, T-1, \tag{7}$$

$$x_t \in \mathbb{R}_+, t = 1, \cdots, T. \tag{8}$$

Model description

Polyhedral discount rate sequence set:

Assumption 1

D is a polyhedral set with m constraints, that is, $D := \{\rho \in \mathbb{R}^T | C\rho \leq d\}$, $C \in \mathbb{R}^{m\times T}$, $d \in \mathbb{R}^m$.

- Such D is a convex set
- Covers D_2, D_3, and D_4 summarized aforementioned

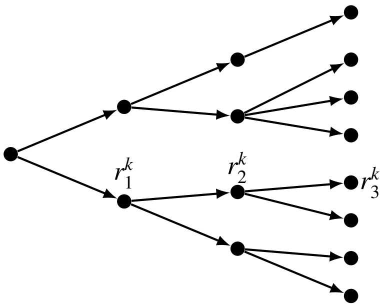
Difficulties in solving the model:

- stochastic: the randomness of multi-stage return rates (Sec. 3)
- semi-infinite: infinitely many constraints (Sec. 4)
- non-smooth: $(\cdot)_+$ (linearization)

Scenario tree approach

Scenario tree:

- Represents a discretised estimate of the random data process and associated appearing probabilities at future stages (Gülpınar and Rustem, 2007)
- Can be generated by different approaches without relying on any distribution assumption (Topaloglou et al., 2008)

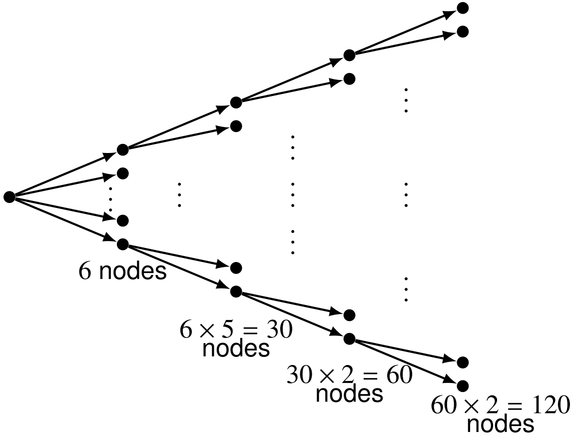


Scenario generation methodology

- Use the mixed normal distribution composed of two Gaussian components to characterize the residual item in the ARMA model.
- The Gaussian distributions of the two components are specified in advance.
- The time-varying weights are dynamically adjusted by a autoregression type model.
- Estimate the parameters by expectation maximization algorithm and the maximum likelihood estimation method.
- Generate the scenario tree sequentially by Monte Carlo sampling and K-means clustering algorithm.

Scenario tree

A 4-stage scenario tree with the branching structure 6-5-2-2:



Reformulation

The dynamic SSD constraints can be equivalently described as:

- for all nondecreasing and concave functions $u : \mathbb{R} \to \mathbb{R}$ and all $\rho \in D$, it holds true that

$$\mathbb{E}\Big[u\Big(\sum_{t=1}^{T} \rho_t x_t\Big)\Big] \geq \mathbb{E}\Big[u\Big(\sum_{t=1}^{T} \rho_t y_t\Big)\Big].$$

- for any $\eta \in \mathbb{R}$ and $\rho \in D$, it holds true that

$$\mathbb{E}\Big[\Big(\eta - \sum_{t=1}^{T} \rho_t x_t\Big)_+\Big] \leq \mathbb{E}\Big[\Big(\eta - \sum_{t=1}^{T} \rho_t y_t\Big)_+\Big], \tag{9}$$

where $(\cdot)_+ = \max(0, \cdot)$.

Reformulation of dynamic SSD

According to reformulation in Luedtke (2008), the dynamic SSD constraints hold if and only if for any $\rho \in D$, there is a $\pi \in \mathbb{R}_+^{K\times K}$ satisfying

$$\sum_{j=1}^{K} \sum_{t=1}^{T} \rho_t y_t^j \pi_{kj} \leq \sum_{t=1}^{T} \rho_t x_t^k, \ k = 1, \cdots, K, \tag{10}$$

$$\sum_{j=1}^{K} \pi_{kj} = 1, \ k = 1, \cdots, K, \tag{11}$$

$$\sum_{k=1}^{K} p^k \sum_{j=1}^{s-1} \pi_{kj} \leq \sum_{j=1}^{s-1} p^j, \ s = 2, \cdots, K. \tag{12}$$

Reformulation of dynamic SSD

According to reformulation in Luedtke (2008), the dynamic SSD constraints hold if and only if for any $\rho \in D$, there is a $\pi \in \mathbb{R}_+^{K\times K}$ satisfying

$$\sum_{j=1}^{K}\sum_{t=1}^{T}\rho_t y_t^j \pi_{kj} \le \sum_{t=1}^{T}\rho_t x_t^k,\ k=1,\cdots,K, \tag{10}$$

$$\sum_{j=1}^{K}\pi_{kj} = 1,\ k=1,\cdots,K, \tag{11}$$

$$\sum_{k=1}^{K}p^k\sum_{j=1}^{s-1}\pi_{kj} \le \sum_{j=1}^{s-1}p^j,\ s=2,\cdots,K. \tag{12}$$

(10)-(12) can be written in a compact form as

$$\max_{\rho\in D}\min_{\pi\in\Pi}\left\{\max_{k=1,\cdots,K} f^k(\rho,\pi,x)\right\} \le 0, \tag{13}$$

where $f^k(\rho,\pi,x) = \sum_{j=1}^K\sum_{t=1}^T\rho_t y_t^j\pi_{kj} - \sum_{t=1}^T\rho_t x_t^k$, and

$$\Pi = \left\{\pi\in\mathbb{R}_+^{K\times K}\middle|\ \begin{array}{l}\sum_{j=1}^K\pi_{kj}=1,\ k=1,\cdots,K,\\ \sum_{k=1}^K p^k\sum_{j=1}^{s-1}\pi_{kj} \le \sum_{j=1}^{s-1}p^j,\ s=2,\cdots,K\end{array}\right\}.$$

Scenario tree representation

Scenario tree representation:

$$
\begin{aligned}
\max_{u,b,s,x} \quad & \sum_{k=1}^{K} p^k x_T^k \\
\text{s.t.} \quad & x_{t+1}^k = (r_{t+1}^k)^\top u_t^k + r_{t+1}^{rf}[x_t^k - c_b\|b_t^k\|_1 - c_s\|s_t^k\|_1],\ t = 0, 1, \cdots, T-1,\ k = 1, \cdots, K, \\
& u_0^k = b_0^k,\ s_0^k = 0,\ u_t^k = u_{t-1}^k + b_t^k - s_t^k,\ t = 1, \cdots, T-1,\ k = 1, \cdots, K, \\
& \|u_t^k\|_1 \le x_t^k - c_b\|b_t^k\|_1 - c_s\|s_t^k\|_1,\ t = 0, 1, \cdots, T-1,\ k = 1, \cdots, K, \\
& u_0^j = u_0^k,\ j, k = 1, \cdots, K, \\
& u_t^j = u_t^k,\ b_t^k = b_t^j,\ s_t^k = s_t^j,\ j \in \mathcal{A}(k,t),\ k = 1, \cdots, K,\ t = 1, \cdots, T-1, \\
& \max_{\rho \in D} \min_{\pi \in \Pi} \left\{ \max_{k=1,\cdots,K} f^k(\rho, \pi, x) \right\} \le 0, \\
& u \in \mathbb{R}_+^{n\times T\times K}, b \in \mathbb{R}_+^{n\times T\times K}, s \in \mathbb{R}_+^{n\times T\times K}, x \in \mathbb{R}^{T\times K}.
\end{aligned} \tag{14}
$$

Upper approximation

Upper approximation:

We properly select L samples to form a subset $D^L \subset D$. Then $\max_{\rho \in D^L} \min_{\pi \in \Pi} \left\{ \max_{k=1,\cdots,K} f^k(\rho, \pi, x) \right\}$ provides a lower bound to the left-hand side of (13) and an upper bound to the original optimization problem.

Proposition 2

We have

$$\max_{\rho \in D} \min_{\pi \in \Pi} \left\{ \max_{k=1,\cdots,K} f^k(\rho, \pi, x) \right\} \geq \max_{\rho \in D^L} \min_{\pi \in \Pi} \left\{ \max_{k=1,\cdots,K} f^k(\rho, \pi, x) \right\}. \tag{15}$$

Upper approximation

An upper bound formulation for problem (14):

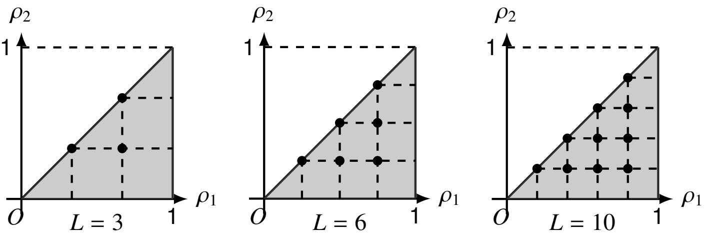
$$
\begin{aligned}
\max_{u,b,s,x,\pi,\tau} \quad & \sum_{k=1}^{K} p^k x_T^k \\
\text{s.t.} \quad & x_{t+1}^k = (r_{t+1}^k)^\top u_t^k + r_{t+1}^{rf}[x_t^k - c_b\|b_t^k\|_1 - c_s\|s_t^k\|_1],\ t = 0,\cdots,T-1,\ k = 1,\cdots,K, \\
& u_0^k = b_0^k,\ s_0^k = 0,\ u_t^k = u_{t-1}^k + b_t^k - s_t^k,\ t = 1,\cdots,T-1,\ k = 1,\cdots,K, \\
& \|u_t^k\|_1 \le x_t^k - c_b\|b_t^k\|_1 - c_s\|s_t^k\|_1,\ t = 0,1,\cdots,T-1,\ k = 1,\cdots,K, \\
& u_0^j = u_0^k,\ k,j = 1,\cdots,K, \\
& u_t^j = u_t^k,\ b_t^k = b_t^j,\ s_t^k = s_t^j,\ j \in \mathcal{A}(k,t),\ k = 1,\cdots,K,\ t = 1,\cdots,T-1, \\
& \tau^l \le 0,\ l = 1,\cdots,L, \\
& \tau^l \ge \sum_{j=1}^{K}\sum_{t=1}^{T} \rho_t^l y_t^j \pi_{kj}^l - \sum_{t=1}^{T} \rho_t^l x_t^k,\ k = 1,\cdots,K,\ l = 1,\cdots,L, \\
& \sum_{j=1}^{K} \pi_{kj}^l = 1,\ k = 1,\cdots,K,\ l = 1,\cdots,L, \\
& \sum_{k=1}^{K} p^k \sum_{j=1}^{s-1} \pi_{kj}^l \le \sum_{j=1}^{s-1} p^j,\ s = 2,\cdots,K,\ l = 1,\cdots,L, \\
& u \in \mathbb{R}_+^{n\times T\times K}, b \in \mathbb{R}_+^{n\times T\times K}, s \in \mathbb{R}_+^{n\times T\times K}, x \in \mathbb{R}^{T\times K}, \pi \in \mathbb{R}_+^{K\times K\times L}, \tau \in \mathbb{R}^L.
\end{aligned}
\tag{16}
$$

Upper approximation: Convergence

Assumption 2

There exist positive numbers A_1 and A_2 such that for any positive integer L and vector $\rho \in D$, there exists a $\rho^L \in D^L$ with $\|\rho - \rho^L\|_2 \leq \frac{1}{A_1 L^{A_2}}$.

Example of D^L:



$$A_1 = \frac{1}{\sqrt{2}} \text{ and } A_2 = \frac{1}{2}$$

Upper approximation: Convergence

Convergence:

Proposition 3

Under Assumption 2, we have

$$\lim_{L\to\infty}\max_{\rho\in D^L}\min_{\pi\in\Pi}\left\{\max_{k=1,\cdots,K} f^k(\rho,\pi,x)\right\} = \max_{\rho\in D}\min_{\pi\in\Pi}\left\{\max_{k=1,\cdots,K} f^k(\rho,\pi,x)\right\}. \qquad (17)$$

Denote the feasible solution sets of problem (14) and upper approximation problem by $\mathcal{F}$ and $\mathcal{F}_L$, the optimal solution sets by $\mathcal{S}$ and $\mathcal{S}_L$, and the optimal values by v and v_L, respectively. Write the decision variable as $z = (u, b, s, x)$.

Theorem 4

We have $\mathcal{F} = \lim_{L\to\infty}\mathcal{F}_L$, $\limsup_{L\to\infty}\mathcal{S}_L \subset \mathcal{S}$, $v = \lim_{L\to\infty} v_L$.

Lower approximation

Lower approximation and error estimate:

Proposition 4

We have

$$\max_{\rho\in D}\min_{\pi\in\Pi}\left\{\max_{k=1,\cdots,K} f^k(\rho,\pi,x)\right\} \le \min_{\pi\in\Pi}\max_{\rho\in D}\left\{\max_{k=1,\cdots,K} f^k(\rho,\pi,x)\right\}. \quad (18)$$

Proposition 5

There exists a positive constant $C_1 < \infty$ *such that*

$$\min_{\pi\in\Pi}\max_{\rho\in D}\left\{\max_{k=1,\cdots,K} f^k(\rho,\pi,x)\right\} - \max_{\rho\in D}\min_{\pi\in\Pi}\left\{\max_{k=1,\cdots,K} f^k(\rho,\pi,x)\right\} \le C_1\mu(\Pi), \quad (19)$$

where $\mu(\Pi) = \max_{a,b\in\Pi} ||a-b||_\infty$ *is the diameter of* Π *under the* ∞*-norm for matrix.*

Lower approximation

A lower bound formulation for problem (14) using the duality theory:

$$
\begin{aligned}
\max_{u,b,s,x,\pi,\alpha} \quad & \sum_{k=1}^{K} p^k x_T^k \\
\text{s.t.} \quad & x_{t+1}^k = (r_{t+1}^k)^\top u_t^k + r_{t+1}^{rf}[x_t^k - c_b\|b_t^k\|_1 - c_s\|s_t^k\|_1], \ t = 0,1,\cdots,T-1, \ k = 1,\cdots,K, \\
& u_0^k = b_0^k, \ s_0^k = 0, \ u_t^k = u_{t-1}^k + b_t^k - s_t^k, \ t = 1,\cdots,T-1, \ k = 1,\cdots,K, \\
& \|u_t^k\|_1 \le x_t^k - c_b\|b_t^k\|_1 - c_s\|s_t^k\|_1 \ t = 0,1,\cdots,T-1, \ k = 1,\cdots,K, \\
& u_0^j = u_0^k, \ k,j = 1,\cdots,K, \\
& u_t^j = u_t^k, \ b_t^k = b_t^j, \ s_t^k = s_t^j, \ j \in \mathcal{A}(k,t), \ k = 1,\cdots,K, \ t = 1,\cdots,T-1, \\
& d^\top \alpha^k \le 0, \ k = 1,\cdots,K, \qquad (20) \\
& \sum_{j=1}^{K} \pi_{kj} y_t^j - x_t^k - C_t^\top \alpha^k \le 0, \ t = 1,\cdots,T, \ k = 1,\cdots,K, \\
& \sum_{j=1}^{K} \pi_{kj} = 1, \ k = 1,\cdots,K, \\
& \sum_{k=1}^{K} p^k \sum_{j=1}^{s-1} \pi_{kj} \le \sum_{j=1}^{s-1} p^j, \ s = 2,\cdots,K, \\
& u \in \mathbb{R}_+^{n\times T\times K}, b \in \mathbb{R}_+^{n\times T\times K}, s \in \mathbb{R}_+^{n\times T\times K}, x \in \mathbb{R}^{T\times K}, \pi \in \mathbb{R}_+^{K\times K}, \alpha \in \mathbb{R}_+^{m\times K}.
\end{aligned}
$$

In-sample tests

Convergence of the gap:

Table: Computation results of the upper bound formulation (16) and the lower bound formulation (20).

	upper bound formulation (16)				lower bound formulation (20)				
L	Var.#	Con.#	Upper B.	CPU(s)	Var.#	Con.#	Lower B.	CPU(s)	Gap(%)
1	27841	2399	1.0712	186.35	28320	2878	1.0709	529.34	0.0289
5	85445	3839	1.0711	200.19	28320	2878	1.0709	529.34	0.0205
20	301460	9239	1.0711	1275.91	28320	2878	1.0709	529.34	0.0159
50	733490	20039	1.0710	6582.56	28320	2878	1.0709	529.34	0.0075

Performance of rolling window test

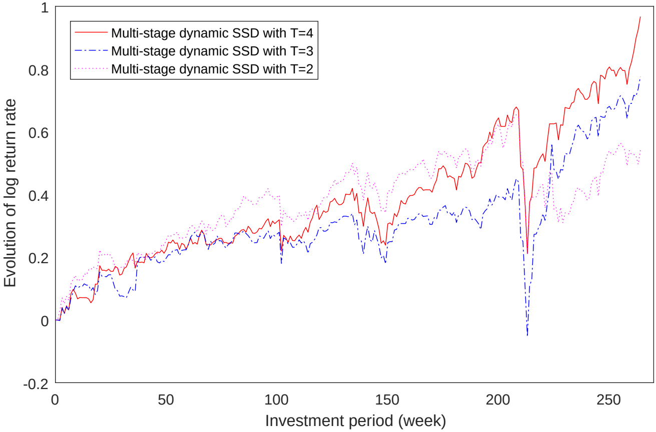


Figure: Evolution of logarithmic return rates w.r.t the investment period for the multi-stage dynamic SSD constrained model.

Performance of rolling window test

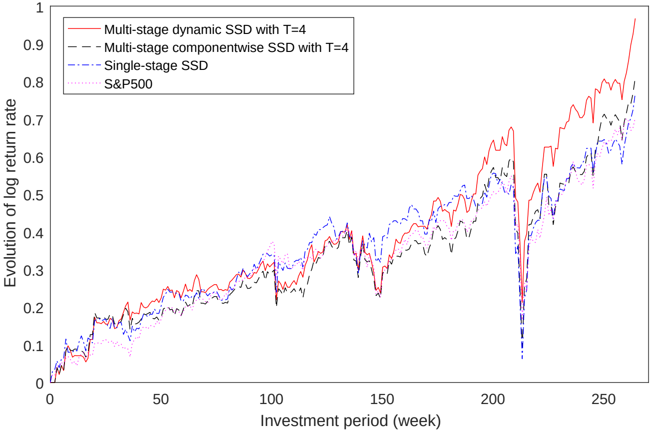


Figure: Comparison of logarithmic return rates w.r.t the investment period of different models.

Out-of-sample tests

Table: Out-of-sample performance statistics of the optimal portfolios' excess return rates got under different models.

Model		Stage	Mean(%)	Std.	Sharpe ratio	Proportion below S&P500	Mean below S&P 500(%)	$\text{CVaR}_{0.1}$(%)
Multi-stage	Dynamic SSD (Lower B.) (20)	$T = 4$	0.3664	0.0302	0.1214	48.5%	-0.8513	-6.7194
		$T = 3$	0.2939	0.0326	0.0903	52.7%	-0.8864	-6.7557
		$T = 2$	0.2051	0.0303	0.0677	48.1%	-1.1322	-7.3125
	Componentwise SSD	$T = 4$	0.3070	0.0316	0.0971	49.6%	-0.9344	-7.1828
		$T = 3$	0.2662	0.0317	0.0841	52.7%	-0.8317	-6.7427
		$T = 2$	0.2176	0.0301	0.0724	48.5%	-1.0393	-7.2125
Single-stage	SSD	$T = 1$	0.2911	0.0308	0.0945	46.2%	-1.0621	-6.7657
	S&P500	$T = 1$	0.2664	0.0250	0.1064	-	-	-5.7020

Observations

- The dynamic SSD with $T = 4$ provides the largest mean value and the largest Sharpe ratio among all the models.
- Compared with the single-stage SSD constrained model, the multi-stage models have a larger conditional mean of return rates below that of S&P500.
- For dynamic SSD, the mean value of return rates, the Sharpe ratio, the conditional mean value of return rates below that of S&P500, and $\text{CVaR}_{0.1}$ increase with the number of stages.

Conclusions

Summary:

- Study multi-stage dynamic SSD constrained portfolio selection model
- Derive an upper approximation and a lower approximation, by solving linear programming problems
- The upper approximation is convergent
- The numerical results verifies the validity of the proposed model

Further works:

- Dynamic preference robust optimization model
- How to design the discount rate sequence set?

Thank you!