Stochastic geometric programming with joint probabilistic constraints

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Outline

Geometric programs

A geometric program can be formulated as

$$(GP) \quad \min_t g_0(t) \text{ s.t. } g_k(t) \leq 1,\ k = 1, \cdots, K,\ t \in \mathbb{R}^M_{++}$$

with

$$g_k(t) = \sum_{i \in I_k} c_i \prod_{j=1}^{M} t_j^{a_{ij}},\ k = 0, \cdots, K.$$

$\{I_k,\ k = 0, \cdots, K\}$ is the disjoint index sets of $\{1, \cdots, Q\}$.

We call $c_i \prod_{j=1}^{M} t_j^{a_{ij}}$ a monomial and $g_k(t)$ a posynomial.

Geometric programs (Cont'd)

Geometric programs have a number of practical problems, such as

- shape optimization problems (Boyd et al., 2007)
- electrical circuit design problems (Boyd et al., 2007)
- mechanical engineering problems (Wiebking, 1977)
- economic and managerial problems (Luptáčik, 1981)
- nonlinear network problems (Kim et al., 2007)

Stochastic geometric programs

Usually, c_i are preset non-negative coefficients.

In practice use, c_i is not known deterministically but randomly.

Considering the randomness of c_i, one can formulate stochastic geometric programs.

Probabilistic constraints are frequently used to control the uncertainty of posynomial constraints.

Stochastic geometric programs (Cont'd)

Stochastic geometric programs with individual probabilistic constraints:

$$
\begin{aligned}
(SGPIPC) \quad & \min_{t \in \mathbb{R}^M_{++}} \quad E\left[\sum_{i \in I_0} c_i \prod_{j=1}^{M} t_j^{a_{ij}}\right] \\
& \text{s.t.} \quad P\left(\sum_{i \in I_k} c_i \prod_{j=1}^{M} t_j^{a_{ij}} \leq 1\right) \geq 1 - \epsilon_k, \; k = 1, \cdots, K.
\end{aligned}
$$

where $\epsilon_k \in (0, 0.5]$ is the tolerance probability for the k-th posynomial constraint.

Stochastic geometric programs (Cont'd)

Stochastic geometric programs with joint probabilistic constraints:

$$
\begin{aligned}
(SGPJPC) \quad & \min_{t \in \mathbb{R}^M_{++}} \quad E\left[\sum_{i \in I_0} c_i \prod_{j=1}^{M} t_j^{a_{ij}}\right] \\
& \text{s.t.} \quad P\left(\sum_{i \in I_k} c_i \prod_{j=1}^{M} t_j^{a_{ij}} \leq 1, \; k = 1, \cdots, K\right) \geq 1 - \epsilon.
\end{aligned}
$$

where $\epsilon \in (0, 0.5]$ is the tolerance probability for all the posynomial constraints.

Literature reviews

- Dupačová (2009) discussed the (SGPIPC) problem.
- They find a deterministic formulation of the probabilistic constraint when c_i are normally distributed and independent of each other
- However, as far as we know, there is no in-depth research works on the (SGPJPC) problem.

Literature reviews (Cont'd)

(SGPJPC) problem are a generalization of stochastic linear program with joint probabilistic constraints

- Miller and Wagner (1965) showed that joint probabilistic constrained problems are equivalent to concave deterministic problems under some independence assumptions
- For some specific cases, such as the right hand side random vector being multivariate normally distributed, Prékopa (1995) showed that the joint probabilistic constraint problems are convex.
- Cheng and Lisser (2012) propose some approximations for linear programs with joint probabilistic constraints.

Our work

We work on stochastic geometric program with joint probabilistic constraints.

- We suppose that a_{ij} is deterministic and c_i is normally distributed and independent of each other, i.e., $c_i \sim N(E_{c_i}, \sigma_i^2)$.

The following techniques are used:

- standard variable transformation from geometric programming
- piecewise linear approximation
- Sequential convex approximation

Equivalent formulation

As c_i are independent of each other, we have

$$P\left(\sum_{i \in I_k} c_i \prod_{j=1}^{M} t_j^{a_{ij}} \leq 1,\ k = 1, \cdots, K\right) \geq 1 - \epsilon$$

is equivalent to

$$\prod_{k=1}^{K} P\left(\sum_{i \in I_k} c_i \prod_{j=1}^{M} t_j^{a_{ij}} \leq 1\right) \geq 1 - \epsilon.$$

Equivalent formulation (Cont'd)

By introducing auxiliary variables $y_k \in \mathbb{R},\ k = 1, \cdots, K$, (SGPJPC) problem can be equivalently transformed into

$$\min_{t \in \mathbb{R}^M_{++}, y \in \mathbb{R}^K} \quad E\left[\sum_{i \in I_0} c_i \prod_{j=1}^{M} t_j^{a_{ij}}\right]$$

$$\text{s.t.} \quad P\Big(\sum_{i \in I_k} c_i \prod_{j=1}^{M} t_j^{a_{ij}} \leq 1\Big) \geq y_k,\ k = 1, \cdots, K,$$

$$\prod_{k=1}^{K} y_k \geq 1 - \epsilon,\ y_k \geq 0.$$

Equivalent formulation (Cont'd)

As $c_i \sim N(E_{c_i}, \sigma_i^2)$, (SGPJPC) problem is further equivalent to

$$\begin{aligned} \min_{t\in\mathbb{R}^M_{++}, y\in\mathbb{R}^K} \quad & \sum_{i\in I_0} E_{c_i} \prod_{j=1}^{M} t_j^{a_{ij}} \\ \text{s.t.} \quad & \sum_{i\in I_k} E_{c_i} \prod_{j=1}^{M} t_j^{a_{ij}} + \Phi^{-1}(y_k) \sqrt{\sum_{i\in I_k} \sigma_i^2 \prod_{j=1}^{M} t_j^{2a_{ij}}} \leq 1, \ k = 1, \cdots, K, \\ & \prod_{k=1}^{K} y_k \geq 1 - \epsilon, \ y_k \geq 0. \end{aligned}$$

$\Phi^{-1}(y_k)$ is the quantile of standard normal distribution $N(0, 1)$.

Equivalent formulation (Cont'd)

The standard variable transformation $r_j = \log(t_j),\ j = 1, \cdots, M$ and $x_k = \log(y_k),\ k = 1, \cdots, K$ leads to the equivalent formulation:

$$\min_{r\in\mathbb{R}^M, x\in\mathbb{R}^K} \quad \sum_{i\in I_0} E_{c_i} \exp\left\{\sum_{j=1}^{M} a_{ij} r_j\right\}$$

$$\text{s.t.} \quad \sum_{i\in I_k} E_{c_i} \exp\left\{\sum_{j=1}^{M} a_{ij} r_j\right\} + \sqrt{\sum_{i\in I_k} \sigma_i^2 \exp\left\{\sum_{j=1}^{M} (2a_{ij} r_j + \log(\Phi^{-1}(e^{x_k})^2))\right\}} \leq 1,\ k = 1, \cdots, K,$$

$$\sum_{k=1}^{K} x_k \geq \log(1-\epsilon),\ x_k \leq 0,\ k = 1, \cdots, K.$$

Property of $\Phi^{-1}(\cdot)$

$\Phi^{-1}(\cdot)$ is also called the probit function:

$$\Phi^{-1}(z) = \sqrt{2}\,\mathrm{erf}^{-1}(2z-1), \quad z \in (0,1).$$

The inverse error function is a nonelementary function which can be represented by the Maclaurin series:

$$\mathrm{erf}^{-1}(z) = \sum_{p=0}^{\infty} \frac{\lambda_p}{2p+1}\left(\frac{\sqrt{\pi}}{2}z\right)^{2p+1},$$

where $\lambda_0 = 1$ and

$$\lambda_p = \sum_{i=0}^{p-1} \frac{\lambda_i \lambda_{p-1-i}}{(i+1)(2i+1)} > 0, \; p = 1, 2, \cdots.$$

Property of $\log(\Phi^{-1}(e^{x_k})^2)$

- $\log(\Phi^{-1}(e^{x_k})^2)$ is convex for $1 > y_k \geq 1 - \epsilon \geq 0.5$.
- Moreover, $\log(\Phi^{-1}(e^{x_k})^2)$ is always monotonic increasing.
- We can approximate $\log(\Phi^{-1}(e^{x_k})^2)$ by a piecewise linear function from below:

$$F_s(x_k) = d_s x_k + b_s,\ \ s = 1, \cdots, S,$$

such that

$$F_s(x_k) \leq \log(\Phi^{-1}(e^{x_k})^2),\ \ \forall x_k \in [\log(1-\epsilon), 0),\ \ s = 1, \cdots, S.$$

Piecewise linear approximation

- For a practical use, we can choose the tangent lines of $\log(\Phi^{-1}(e^{x_k})^2)$ at different points in $[\log(1-\epsilon), 0)$, say $\xi_1, \xi_2, \cdots, \xi_S$.
- Then, we have

$$d_s = \frac{2e^{\xi_s}(\Phi^{-1})^{(1)}(e^{\xi_s})}{\Phi^{-1}(e^{\xi_s})}$$

and

$$b_s = -d_s\xi_s + \log(\Phi^{-1}(e^{\xi_s})^2), \;\; s = 1, \cdots, S.$$

Theorem

Using the piecewise linear function $F(x_k)$, we can found an approximation of (SGPJPC) problem:

$$(SGP_A)$$

$$\min_{r\in\mathbb{R}^M, x\in\mathbb{R}^K} \quad \sum_{i\in I_0} E_{c_i} \exp\left\{\sum_{j=1}^{M} a_{ij} r_j\right\}$$

$$\sum_{i\in I_k} E_{c_i} \exp\left\{\sum_{j=1}^{M} a_{ij} r_j\right\} + \sqrt{\sum_{i\in I_k} \sigma_i^2 \exp\left\{\sum_{j=1}^{M} (2a_{ij} r_j + d_s x_k + b_s)\right\}} \leq 1,\ s = 1, \cdots, S,\ k = 1, \cdots, K,$$

$$\sum_{k=1}^{K} x_k \geq \log(1-\epsilon),\ x_k \leq 0,\ k = 1, \cdots, K.$$

The optimal value is a lower bound of the (SGPJPC) problem. When S goes to infinity, the approximation is tight.

Sequential convex approximation

- Sequential convex approximation ⇒ upper bound
- Basic idea: decomposing into subproblems where a subset of variables is fixed alternatively.
- We first fix $y = y^n$ and update t by solving

$$
\begin{aligned}
(SQ_1) \quad & \min_{t \in \mathbb{R}^M_{++}} \quad \sum_{i \in I_0} E_{c_i} \prod_{j=1}^{M} t_j^{a_{ij}} \\
& \text{s.t.} \quad \sum_{i \in I_k} E_{c_i} \prod_{j=1}^{M} t_j^{a_{ij}} + \Phi^{-1}(y_k^n) \sqrt{\sum_{i \in I_k} \sigma_i^2 \prod_{j=1}^{M} t_j^{2a_{ij}}} \leq 1, \\
& \qquad k = 1, \cdots, K
\end{aligned}
$$

Sequential convex approximation (Cont'd)

- and then fix $t = t^n$ and update y by solving

$$(SQ_2) \quad \min_{y \in \mathbb{R}^K_+} \sum_{k=1}^{K} \phi_k \, y_k$$

$$\text{s.t.} \quad y_k \leq \Phi\left(\frac{1 - \sum_{i \in I_k} E_{c_i} \prod_{j=1}^{M} (t_j^n)^{a_{ij}}}{\sqrt{\sum_{i \in I_k} \sigma_i^2 \prod_{j=1}^{M} (t_j^n)^{2a_{ij}}}} \right), \; k = 1, \cdots, K.$$

$$\prod_{k=1}^{K} y_k \geq 1 - \epsilon, \; y_k \geq 0, \; k = 1, \cdots, K.$$

- ϕ_k is a chosen searching direction.

Sequential convex approximation (Cont'd)

Algorithm 1 Sequential convex approximation

Initialization:

Choose an initial point y^0 of y feasible for (8). Set $n = 0$.

Iteration:

while $n \geq 1$ and $||y^{n-1} - y^n||$ is small enough **do**

- Solve problem (SQ_1); let t^n, θ^n and v^n denote an optimal solution of t, an optimal solution of the Lagrangian dual variable θ and the optimal value, respectively.
- Solve problem (SQ_2) with $\phi_k = \theta_k^n \cdot (\Phi^{-1})'(y_k^n)\sqrt{\sum_{i\in I_k} \sigma_i^2 \prod_{j=1}^M (t_j^n)^{2a_{ij}}}$; let $\tilde{y}$ denote an optimal solution.
- $y^{n+1} \leftarrow y^n + \tau(\tilde{y} - y^n)$, $n \leftarrow n+1$. Here, $\tau \in (0,1)$ is the step length.

end while

Output: t^n, v^n

Sequential convex approximation (Cont'd)

Theorem

Algorithm 1 converges in a finite number of iterations and the returned value v^n is a upper bound for problem (SGP).

- Problems (SQ_1) and (SQ_2) are both geometric programs, hence they can be transformed into a convex programming problem, and solved by interior point methods.

Shape optimization problem

Consider a joint probabilistic constrained shape optimization problem,

$$\begin{aligned} \min_{h,w,\zeta} \quad & h^{-1}w^{-1}\zeta^{-1} \\ \text{s.t.} \quad & P\Big((2/A_{wall})hw + (2/A_{wall})h\zeta \leq 1,\ (1/A_{flr})w\zeta \leq 1\Big) \geq 1-\epsilon, \\ & \alpha h^{-1}w \leq 1,\ (1/\beta)hw^{-1} \leq 1, \\ & \gamma w\zeta^{-1} \leq 1,\ (1/\delta)w^{-1}\zeta \leq 1. \end{aligned}$$

- maximize the volume of a box-shaped structure with height h, width w and depth ζ
- with constraint on total wall area $2(hw + h\zeta)$, and floor area $w\zeta$

Settings

- Set $\alpha = \gamma = 0.5$, $\beta = \delta = 2$, $\epsilon = 5\%$,
- Assume $1/A_{wall} \sim N(0.005, 0.01)$ and $1/A_{flr} \sim N(0.01, 0.01)$.
- By using CVX software, we solve the approximation problems with Matlab R2012b, on a PC with a 2.6 Ghz Intel Core i7-5600U CPU and 12.0 GB RAM.
- We solve five groups of approximation problems with different number of segments, S.

Computational results

Table 1: Computational results

S	Var. Num.	Con. Num.	Low. bound	CPU(s)	Upp. bound	CPU(s)	Gap(%)
1	133	60	0.232	0.5955	0.256	5.5274	9.655
2	184	91	0.234	0.6272	0.256	5.5274	8.789
5	283	153	0.241	0.9480	0.256	5.5274	6.044
10	513	273	0.252	1.3554	0.256	5.5274	1.713
20	973	513	0.256	1.9986	0.256	5.5274	0

Sequential convex approximation algorithm converges within 7 outer iterations

Conclusions

- We discussed (SGPJPC) problem under normal distribution
- We find an upper bound and a lower bound for (SGPJPC) problem

Further work

- (SGPJPC) problem under elliptical distributions, log-normal distribution et al.
- (SGPIPC) and (SGPJPC) problems with random a_{ij}.
- Distributional robust (SGPIPC) and (SGPJPC) problems with distribution uncertainty.

Thank you!