

Distributionally robust chance constrained geometric optimization

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Outline of the talk

- 1 **Introduction**
- 2 **Individual** robust geometric chance constraints
- 3 **Joint** robust geometric chance constraints
- 4 Joint uncertainty of a_{ij}^k and c_i^k
- 5 **Numerical** results

Distributionally robust geometric optimization

A geometric program can be formulated as:

$$\min_t g_0(t) \text{ s.t. } g_k(t) \leq 1, \quad k = 1, \dots, K, \quad t \in \mathbb{R}_{++}^M$$

with

$$g_k(t) = \sum_{i=1}^{l_k} c_i^k \prod_{j=1}^M t_j^{a_{ij}^k}, \quad k = 0, \dots, K.$$

- $c_i^k \prod_{j=1}^M t_j^{a_{ij}^k}$ is called a monomial, and $g_k(t)$ is called posynomial.
- The posynomials might have different parameters $c_i^k \geq 0$ and a_{ij}^k .
- Geometric programs are not convex with respect to t
- They are convex with respect to $\{r : r_j = \log t_j, j = 1, \dots, M\}$.

Applications (cf. S. Boyd, 2007, R. Wiebking, 1977, M. Luptáčík, 1981, S. Kim et al., 2007)

- Wireless communications
- Semiconductor device engineering
- Floor planning
- Digital circuit gate sizing
- Economic and managerial problems
- Wire sizing
- ...

Distributionally robust geometric optimization

Individual chance constraints are studied in S. Rao (1996) and J. Dupačová (2009):

$$(ISGP) \quad \min_{t \in \mathbb{R}_{++}^M} \mathbb{E}_{F_0} \left[\sum_{i=1}^{I_0} c_i^0 \prod_{j=1}^M t_j^{a_{ij}^0} \right]$$
$$\text{s.t.} \quad \mathbb{P}_{F_k} \left(\sum_{i=1}^{I_k} c_i^k \prod_{j=1}^M t_j^{a_{ij}^k} \leq 1 \right) \geq 1 - \epsilon_k, \quad k = 1, \dots, K$$

- c_i^k is random (normally distributed), a_{ij}^k is constant.
- Reformulated as geometric programs.

Dupačová J (2009) Stochastic geometric programming: approaches and applications. In Brožová V, Kvasnicka R, eds. Proceedings of MME09, 63-66.

Recently, L. and A. Lasser (2016) studied stochastic geometric problems with joint chance constraints; i.e.;

$$\begin{aligned} (JSGP) \quad & \min_{t \in \mathbb{R}_{++}^M} \mathbb{E}_{F_0} \left[\sum_{i=1}^{I_0} c_i^0 \prod_{j=1}^M t_j^{a_{ij}^0} \right] \\ & \text{s.t.} \quad \mathbb{P}_F \left(\sum_{i=1}^{I_k} c_i^k \prod_{j=1}^M t_j^{a_{ij}^k} \leq 1, k = 1, \dots, K \right) \geq 1 - \epsilon, \end{aligned}$$

- c_i^k is random (normally distributed), a_{ij}^k is constant.
- Reformulated as convex programming problems when $\epsilon \leq 0.5$.

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- 5 Numerical results

Distributionally robust geometric optimization

We consider the distributionally robust geometric programs with individual chance constraints and the ambiguity of F_0 , F_k or F :

$$(IRGP) \quad \min_{t \in \mathbb{R}_{++}^M} \quad \sup_{F \in \mathcal{F}_0} \mathbb{E}_{F_0} \left[\sum_{i=1}^{I_0} c_i^0 \prod_{j=1}^M t_j^{a_{ij}^0} \right]$$
$$\text{s.t.} \quad \inf_{F_k \in \mathcal{F}_k} \mathbb{P}_{F_k} \left(\sum_{i=1}^{I_k} c_i^k \prod_{j=1}^M t_j^{a_{ij}^k} \leq 1 \right) \geq 1 - \epsilon_k, \quad k = 1, \dots, K.$$

where \mathcal{F}_0 , \mathcal{F}_k , $k = 1, \dots, K$ are the uncertainty sets, which contain all the distributions of F_0 , F_k , $k = 1, \dots, K$.

- c_i^k is random, a_{ij}^k is constant.

Uncertainty sets with known moments

- We consider the uncertainty sets \mathcal{F}_k , $k = 0, \dots, K$, with **known two first order moments** information (L. El Ghaoui et al, 2003, L. Chen et al, 2011).

Assumption 1

The uncertainty sets are

$$\mathcal{F}_k = \{F_k \mid \mathbb{E}_{F_k}[c^k] = \mu^k, \text{Cov}_{F_k}[c^k] = \Gamma^k\}, \quad k = 0, \dots, K.$$

where $\text{Cov}_F[c^k] = \mathbb{E}_{F_k} \left[(c^k - \mathbb{E}_{F_k}[c^k]) (c^k - \mathbb{E}_{F_k}[c^k])^\top \right]$,

- $\mu^k = [\mu_1^k, \mu_2^k, \dots, \mu_{l_k}^k]^\top$ and $\Gamma^k = \{\sigma_{i,j}^k\}$.
- μ_i^k is the reference value of the **expected value** of c_i^k .
- $\sigma_{i,j}^k$ is the reference value of the **covariance** between c_i^k and c_j^k ,
- We assume that $\mu^k \geq 0$, Γ^k is positive definite matrix, and $\sigma_{i,j}^k \geq 0$.

Theorem 1

Suppose that Assumption 1 holds. Then (IRGP) is equivalent to

$$\begin{aligned} (\text{IRGP}_1) \quad & \min_{t \in \mathbb{R}_{++}^M} \sum_{i=1}^{l_0} \mu_i^0 \prod_{j=1}^M t_j^{a_{ij}^0} \\ & \text{s.t.} \quad \sum_{i=1}^{l_k} \mu_i^k \prod_{j=1}^M t_j^{a_{ij}^k} + \sqrt{\frac{1 - \epsilon_k}{\epsilon_k}} \sqrt{\sum_{i=1}^{l_k} \sum_{p=1}^{l_k} \sigma_{i,p}^k \prod_{j=1}^M t_j^{a_{ij}^k + a_{pj}^k}} \leq 1, \\ & \quad k = 1, \dots, K. \end{aligned}$$

Sketch of the proof:

- Lemma 2.1 (L. Chen et al., 2011)
- Chebychev inequality (N. Rujeerapaiboon et al., 2016)

Distributionally robust geometric optimization

Problem $(IRGP_1)$ is not convex w.r.t t , it can be transformed into a convex problem using $r_j = \log(t_j)$, $j = 1, \dots, M$

$$\begin{aligned} (IRGP_{1s}) \quad & \min_{r \in \mathbb{R}^M} \sum_{i=1}^{I_0} \mu_i^0 \exp \left\{ \sum_{j=1}^M a_{ij}^0 r_j \right\} \\ \text{s.t.} \quad & \sqrt{\frac{1 - \epsilon_k}{\epsilon_k}} \sqrt{\sum_{i=1}^{I_k} \sum_{p=1}^{I_k} \sigma_{i,p}^k \exp \left\{ \sum_{j=1}^M (a_{ij}^k + a_{pj}^k) r_j \right\}} \\ & + \sum_{i=1}^{I_k} \mu_i^k \exp \left\{ \sum_{j=1}^M a_{ij}^k r_j \right\} \leq 1, \quad k = 1, \dots, K, \end{aligned}$$

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Distributionally robust geometric optimization

Uncertainty sets with unknown moments

We consider the uncertainty sets \mathcal{F}_k , $k = 0, \dots, K$, for (IRGP). (E. Delage and Y. Ye, 2010; J. Cheng et al. 2003; N. Rujeerapaiboon et al., 2015).

Assumption 2

- The uncertainty sets are

$$\mathcal{F}_k = \left\{ F_k \left| \begin{array}{l} (\mathbb{E}_{F_k}[c^k] - \mu^k)^\top (\Gamma^k)^{-1} (\mathbb{E}_{F_k}[c^k] - \mu^k) \leq \pi_1^k, \\ \text{COV}_{F_k}[c^k] \preceq_D \pi_2^k \Gamma^k. \end{array} \right. \right\},$$

- We assume that $\mu^k \geq 0$, Γ^k is positive definite matrix, and $\sigma_{i,j}^k \geq 0$, $k = 0, 1, \dots, K$.
- $\pi_1^k, \pi_2^k \in \mathbb{R}$ are two *scale parameters* controlling the size of the uncertainty sets.
- $A \preceq_D B$ means that for any $x \in \mathbb{R}^n$, we have $x^\top A x \leq x^\top B x$.

Distributionally robust geometric optimization

Uncertainty sets with unknown moments

Theorem 2

Given Assumption 2, (IRGP) is equivalent to

$$\begin{aligned} (\text{IRGP}_2) \quad & \min_{t \in \mathbb{R}_{++}^M} \sum_{i=1}^{l_0} \mu_i^0 \prod_{j=1}^M t_j^{a_{ij}^0} + \sqrt{\pi_1^0} \sqrt{\sum_{i=1}^{l_0} \sum_{p=1}^{l_0} \sigma_{i,p}^0 \prod_{j=1}^M t_j^{a_{ij}^0 + a_{pj}^0}} \\ \text{s. t.} \quad & \sum_{i=1}^{l_k} \mu_i^k \prod_{j=1}^M t_j^{a_{ij}^k} + \\ & \left(\sqrt{\frac{1-\epsilon_k}{\epsilon_k}} \sqrt{\pi_2^k} + \sqrt{\pi_1^k} \right) \sqrt{\sum_{i=1}^{l_k} \sum_{p=1}^{l_k} \sigma_{i,p}^k \prod_{j=1}^M t_j^{a_{ij}^k + a_{pj}^k}} \leq 1, \quad k = 1, \dots, K. \end{aligned}$$

Sketch of the proof:

- Same ingredients as the previous Theorem extended to this case.

Distributionally robust geometric optimization

Uncertainty sets with unknown moments

With the standard variable transformation $r_j = \log(t_j)$, $j = 1, \dots, M$, we can transform (IRGP₂) into

$$\begin{aligned} \min_{r \in \mathbb{R}^M} \quad & \sum_{i=1}^{l_0} \mu_i^0 \exp \left\{ \sum_{j=1}^M a_{ij}^0 r_j \right\} + \sqrt{\pi_1^0} \sqrt{\sum_{i=1}^{l_0} \sum_{p=1}^{l_0} \sigma_{i,p}^0 \exp \left\{ \sum_{j=1}^M (a_{ij}^0 + a_{pj}^0) r_j \right\}} \\ \text{s. t.} \quad & \left(\sqrt{\frac{1-\epsilon_k}{\epsilon_k}} \sqrt{\pi_2^k} + \sqrt{\pi_1^k} \right) \sqrt{\sum_{i=1}^{l_k} \sum_{p=1}^{l_k} \sigma_{i,p}^k \exp \left\{ \sum_{j=1}^M (a_{ij}^k + a_{pj}^k) r_j \right\}} \\ & + \sum_{i=1}^{l_k} \mu_i^k \exp \left\{ \sum_{j=1}^M a_{ij}^k r_j \right\} \leq 1, \quad k = 1, \dots, K. \end{aligned}$$

which is a convex optimization problem.

- Avoid SDP reformulation (Delage and Ye 2010; Cheng, Delage and Lisser 2014)

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Uncertainty in the density

Distributionally robust geometric optimization

Uncertainty sets controlled by the **distance between the true distribution and the reference distribution** of c^k (A. Ben-Tal, 2013; Z. Hu and J. Hong, 2013; R. Jiang and Y. Guan, 2016).

Assumption 3

The uncertainty sets are

$$\mathcal{F}_k = \{F_k \mid D_{DL}(F_k \parallel F_k^0) \leq \kappa_k\}, \quad k = 0, \dots, K.$$

where D_{DL} is the **Kullback-Leibler divergence distance**

$$D_{DL}(F_k \parallel F_k^0) = \int_{\Omega} \phi \left(\frac{f_{F_k}(c^k)}{f_{F_k^0}(c^k)} \right) f_{F_k^0}(c^k) dc^k,$$

- F_k^0 is the **reference distribution** of c^k , $f_{F_k}(c^k)$ and $f_{F_k^0}(c^k)$ are the density functions of the true distribution and the reference distribution of c^k on Ω ,
- κ_k is a parameter controlling the **size of the uncertainty set**, $k = 0, \dots, K$.
 $\phi(t) = t \log t - t + 1$, for $t \geq 0$, and $\phi(t) = \infty$, ow.

Uncertainty in the density

Distributionally robust geometric optimization

We use Theorem 1 in Z. Hu and J. Hong (2013) for the following proposition:

Proposition 3

Given Assumption 3, the objective function is equivalent to

$$\inf_{\alpha \in (0, \infty)} \alpha \log \mathbb{E}_{F_0^0} \left[\exp \left\{ \left(\sum_{i=1}^{I_0} c_i^0 \prod_{j=1}^M t_j^{a_{ij}^0} \right) / \alpha \right\} \right] + \alpha \kappa_0.$$

and Theorem 1 and Proposition 4 in R. Jiang and Y. Guan (2016),

Proposition 4

Given Assumption 3, the constraint is equivalent to

$$\mathbb{P}_{F_k^0} \left(\sum_{i=1}^{I_k} c_i^k \prod_{j=1}^M t_j^{a_{ij}^k} \leq 1 \right) \geq 1 - \epsilon'_k, \quad k = 1, \dots, K,$$

where $\epsilon'_k = 1 - \inf_{x \in (0, 1)} \left\{ \frac{e^{-\kappa_k x^{1-\epsilon_k}} - 1}{x-1} \right\}, \quad k = 1, \dots, K.$

Uncertainty in the density

(IRGP) with normal reference distribution

Assume that the reference distribution F_k^0 follows a normal distribution:

- mean vector $\mu^k = [\mu_1^k, \mu_2^k, \dots, \mu_{l_k}^k]^\top \geq 0$
- positive definite covariance matrix $\Gamma^k = \{\sigma_{i,j}^k, i, j = 1, \dots, l_k\}, \forall k$.

Theorem 5

Given Assumption 3 and normal distribution assumption for $F_k^0, k = 0, 1, \dots, K$, (IRGP) is equivalent to

$$\begin{aligned} (\text{IRGP}_{3N}) \quad & \min_{t \in \mathbb{R}_{++}^M} \sum_{i=1}^{l_0} \mu_i^0 \prod_{j=1}^M t_j^{a_{ij}^0} + \sqrt{2\kappa_0 \sum_{i=1}^{l_0} \sum_{p=1}^{l_0} \sigma_{i,p}^0 \prod_{j=1}^M t_j^{a_{ij}^0 + a_{pj}^0}} \\ \text{s.t.} \quad & \sum_{i=1}^{l_k} \mu_i^k \prod_{j=1}^M t_j^{a_{ij}^k} + \Phi^{-1}(1 - \epsilon'_k) \sqrt{\sum_{i=1}^{l_k} \sum_{p=1}^{l_k} \sigma_{i,p}^k \prod_{j=1}^M t_j^{a_{ij}^k + a_{pj}^k}} \leq 1, \quad k = 1, \dots, K. \end{aligned}$$

Here, $\Phi^{-1}(\cdot)$ is the quantile of the standard normal distribution $N(0, 1)$.

Proof: Based on previous theorems

Uncertainty in the density

Data-driven (IRGP)

Historical data based discrete reference distribution F_k^0 .

Theorem 6

Given Assumption 3, we further assume F_k^0 follows a discrete distribution with H possible scenarios $\tilde{c}^k(1), \tilde{c}^k(2), \tilde{c}^k(H)$, associated with their probabilities $\frac{1}{H}$, $k = 0, 1, \dots, K$. Then, problem (IRGP) is equivalent to (IRGP_{3D})

$$\begin{aligned} \min_{r \in \mathbb{R}^M, \alpha \in (0, \infty), \varsigma} \quad & \alpha \log \left(\frac{1}{H} \sum_{h=1}^H \exp \left\{ \left(\sum_{i=1}^{I_0} \tilde{c}_i^0(h) \exp \left\{ \sum_{j=1}^M a_{ij}^0 r_j \right\} \right) / \alpha \right\} \right) + \alpha \kappa_0, \\ \text{s.t.} \quad & \frac{1}{H} \sum_{h=1}^H (1 - \varsigma_h^k) \geq 1 - \epsilon'_k, \quad k = 1, 2, \dots, K, \\ & \sum_{i=1}^{I_k} \tilde{c}_i^k(h) \exp \left\{ \sum_{j=1}^M a_{ij}^k r_j \right\} \leq M \varsigma_h^k + 1, \quad h = 1, \dots, H, \quad k = 1, 2, \dots, K, \\ & \varsigma_h^k \in \{0, 1\}, \quad h = 1, \dots, H, \quad k = 1, 2, \dots, K. \end{aligned}$$

Proof: Propositions 4 and 5 and previous theorems.

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Distributionally robust geometric optimization

We consider the distributionally robust geometric programs with **joint** chance constraints and the ambiguity of F_0 or F :

$$(JRGP) \quad \min_{t \in \mathbb{R}_{++}^M} \quad \sup_{F_0 \in \mathcal{F}_0} \mathbb{E}_{F_0} \left[\sum_{i=1}^{I_0} c_i^0 \prod_{j=1}^M t_j^{a_{ij}^0} \right]$$
$$\text{s.t.} \quad \inf_{F \in \mathcal{F}} \mathbb{P}_F \left(\sum_{i=1}^{I_k} c_i^k \prod_{j=1}^M t_j^{a_{ij}^k} \leq 1, \quad k = 1, \dots, K \right) \geq 1 - \epsilon,$$

where \mathcal{F}_0 and \mathcal{F} are the uncertainty sets, which contain all the distributions of F_0 and F .

- c_i^k is random, a_{ij}^k is constant.

Uncertainty with known moments

Joint chance constraints case

- We consider an uncertainty sets \mathcal{F} for (JRGP), with known two first order moments, and with **pairwise independent marginal** distributions.

Assumption 4

- The uncertainty set $\mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_K$
- For any joint distribution F in \mathcal{F} , its marginal distributions F_1, \dots, F_K are **pairwise independent**.

$$\mathcal{F}_k = \{F_k \mid E_{F_k}[c^k] = \mu^k, \text{Cov}_{F_k}[c^k] = \Gamma^k\}, \quad k = 0, 1, \dots, K.$$

- We assume that $\mu^k \geq 0$, Γ^k is **positive definite matrix**, and $\sigma_{i,j}^k \geq 0$, $k = 0, 1, \dots, K$.

Uncertainty with known moments

Joint chance constraints case

Theorem 7

Given Assumption 4, (JRGP) is equivalent to

$$\begin{aligned} & \text{(JRGP}_1\text{)} \\ & \min_{t \in \mathbb{R}_{++}^M, y \in \mathbb{R}_{++}^K} \sum_{i=1}^{l_0} \mu_i^0 \prod_{j=1}^M t_j^{a_{ij}^0} \\ & \text{s. t.} \quad \sum_{i=1}^{l_k} \mu_i^k \prod_{j=1}^M t_j^{a_{ij}^k} + \sqrt{\frac{y_k}{1-y_k}} \sqrt{\sum_{i=1}^{l_k} \sum_{p=1}^{l_k} \sigma_{i,p}^k \prod_{j=1}^M t_j^{a_{ij}^k + a_{pj}^k}} \leq 1, \quad k = 1, \dots, K. \\ & \quad \prod_k y_k \geq 1 - \epsilon. \end{aligned}$$

Non convex due to the bi-linear term.

Uncertainty with known moments

Joint chance constraints case

- We transform $(JRGP_1)$ using $r_j = \log(t_j)$, $j = 1, \dots, M$ and $x_k = \log(y_k)$, $k = 1, \dots, K$:

$$\begin{aligned} (JRGP_{1s}) \quad & \min_{r \in \mathbb{R}^M, x \in \mathbb{R}^K} \sum_{i=1}^{l_0} \mu_i^0 \exp \left\{ \sum_{j=1}^M a_{ij}^0 r_j \right\} \\ \text{s.t.} \quad & \sqrt{\sum_{i=1}^{l_k} \sum_{p=1}^{l_k} \sigma_{i,p}^k \exp \left\{ \sum_{j=1}^M ((a_{ij}^k + a_{pj}^k) r_j) + \log \left(\frac{e^{x_k}}{1 - e^{x_k}} \right) \right\}} \\ & + \sum_{i=1}^{l_k} \mu_i^k \exp \left\{ \sum_{j=1}^M a_{ij}^k r_j \right\} \leq 1, \quad k = 1, \dots, K, \\ & \sum_{k=1}^K x_k \geq \log(1 - \epsilon), \quad x_k \leq 0, \quad k = 1, \dots, K. \end{aligned}$$

- $(JRGP_{1s})$ is a convex programming problem as $\log \left(\frac{e^{x_k}}{1 - e^{x_k}} \right)$ is convex. Can be rewritten as $x_k - \log(1 - e^{x_k})$ to meet the rules of disciplined convex programming of CVX.

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Uncertainty with unknown moments

Joint chance constraints case

Assumption 5

The uncertainty set $\mathcal{F} = \mathcal{F}_1 \times \dots \times \mathcal{F}_K$, and for any distribution F in \mathcal{F} , its marginal distributions F_1, \dots, F_K are pairwise independent.

Moreover,

$$\mathcal{F}_k = \left\{ F_k \mid \begin{array}{l} (\mathbb{E}_{F_k}[c^k] - \mu^k)^\top (\Gamma^k)^{-1} (\mathbb{E}_{F_k}[c^k] - \mu^k) \leq \pi_1^k, \\ \text{Cov}_{F_k}[c^k] \preceq_D \pi_2^k \Gamma^k. \end{array} \right\},$$

- We assume that $\mu^k \geq 0$, Γ^k is *positive definite matrix*, and $\sigma_{i,j}^k \geq 0$, $k = 0, 1, \dots, K$.

Uncertainty with unknown moments

Joint chance constraints case

Theorem 8

Given Assumption 5, (JRGP) is equivalent to

$$\begin{aligned} (JRGP_2) \quad & \min_{t \in \mathbb{R}_{++}^M, y \in \mathbb{R}_{++}^K} \sum_{i=1}^{l_0} \mu_i^0 \prod_{j=1}^M t_j^{a_{ij}^0} + \sqrt{\pi_1^0} \sqrt{\sum_{i=1}^{l_0} \sum_{p=1}^{l_0} \sigma_{i,p}^0 \prod_{j=1}^M t_j^{a_{ij}^0 + a_{pj}^0}} \\ \text{s. t.} \quad & \sum_{i=1}^{l_k} \mu_i^k \prod_{j=1}^M t_j^{a_{ij}^k} + \sqrt{\pi_1^k} \sqrt{\sum_{i=1}^{l_k} \sum_{p=1}^{l_k} \sigma_{i,p}^k \prod_{j=1}^M t_j^{a_{ij}^k + a_{pj}^k}} \\ & + \sqrt{\frac{y_k}{1-y_k}} \sqrt{\pi_2^k} \sqrt{\sum_{i=1}^{l_k} \sum_{p=1}^{l_k} \sigma_{i,p}^k \prod_{j=1}^M t_j^{a_{ij}^k + a_{pj}^k}} \leq 1, \quad k = 1, \dots, K. \\ & \prod_k y_k \geq 1 - \epsilon. \end{aligned}$$

(JRGP₂) can be reformulated as a convex programming problem by changing the decision variables.

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Uncertainty sets with density uncertainty

Joint chance constraints case

We consider uncertainty sets for (JRGP) with a reference distribution.

Assumption 6

The uncertainty sets are

$$\mathcal{F}_0 = \{F_0 \mid D_{DL}(F_0 \parallel F_0^0) \leq \kappa_0\} \text{ and } \mathcal{F} = \{F \mid D_{DL}(F \parallel F^0) \leq \kappa\},$$

where

- D_{DL} is defined in Assumption 3,
- F_0^0 is the reference distribution for c^0 ,
- F^0 is the reference joint distribution for c^1, c^2, \dots, c^K , such that $F^0 = F_1^0 \times \dots \times F_K^0$ and the marginal distributions F_1^0, \dots, F_K^0 are pairwise independent.

Uncertainty sets with density uncertainty

Joint chance constraints case

Theorem 9

Given Assumption 7, we assume that $F_0^0, F_1^0, \dots, F_K^0$ are normal distributions with mean vector $\mu^k = [\mu_1^k, \mu_2^k, \dots, \mu_{l_k}^k]^\top$ and covariance matrix $\Gamma^k = \{\sigma_{i,j}^k, i, j = 1, \dots, l_k\}$, $k = 0, 1, \dots, K$. Then (JRGP) is equivalent to (JRGP_{3N})

$$\begin{aligned} \min_{t \in \mathbb{R}_{++}^M, y \in \mathbb{R}_{++}^K} & \sum_{i=1}^{l_0} \mu_i^0 \prod_{j=1}^M t_j^{a_{ij}^0} + \sqrt{2\kappa_0 \sum_{i=1}^{l_0} \sum_{p=1}^{l_0} \sigma_{i,p}^0 \prod_{j=1}^M t_j^{a_{ij}^0 + a_{pj}^0}} \\ \text{s.t.} & \sum_{i=1}^{l_k} \mu_i^k \prod_{j=1}^M t_j^{a_{ij}^k} + \Phi^{-1}(y_k) \sqrt{\sum_{i=1}^{l_k} \sum_{p=1}^{l_k} \sigma_{i,p}^k \prod_{j=1}^M t_j^{a_{ij}^k + a_{pj}^k}} \leq 1, \quad k = 1, \dots, K, \\ & \prod_k y_k \geq 1 - \epsilon'. \end{aligned}$$

$$\text{where } \epsilon' = 1 - \inf_{x \in (0,1)} \left\{ \frac{e^{-\kappa} x^{1-\epsilon} - 1}{x-1} \right\}.$$

Convex reformulation can be obtained when $\epsilon' \leq 0.5$ and $\sigma_{i,p}^k \geq 0$.

Theorem 10

Given Assumption 7, we assume that F_k^0 is a discrete distribution with H possible values $\tilde{c}^k(h)$, $h = 1, \dots, H$, associated with their probabilities $\frac{1}{H}$, $k = 0, 1, \dots, K$. Then (JRGP) is equivalent to (JRGP_{3D})

$$\begin{aligned} \min_{r \in \mathbb{R}^M, \alpha \in (0, \infty), \varsigma} \quad & \alpha \log \left(\frac{1}{H} \sum_{h=1}^H \exp \left\{ \left(\sum_{i=1}^I \tilde{c}_i^0(h) \exp \left\{ \sum_{j=1}^M a_{ij}^0 r_j \right\} \right) / \alpha \right\} \right) + \alpha \kappa_0, \\ \text{s.t.} \quad & \frac{1}{H} \sum_{h=1}^H (1 - \varsigma_h) \geq 1 - \epsilon', \\ & \sum_{i=1}^I \tilde{c}_i^k(h) \exp \left\{ \sum_{j=1}^M a_{ij}^k r_j \right\} \leq M \varsigma_h + 1, \quad h = 1, \dots, H, \quad k = 1, 2, \dots, K, \\ & \varsigma_h \in \{0, 1\}, \quad h = 1, \dots, H. \end{aligned}$$

Outline of the talk

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Joint uncertainty set of a_{ij}^k and c_i^k

Individual chance constraints case

We consider uncertainty sets for (IRGP) with a reference distribution.

Assumption 7

The uncertainty sets are

$$\mathcal{G}_k = \left\{ G_k \left| \begin{array}{l} G_k(a^k = a^k(m)) = p_m^k, m = 1, \dots, M_k, \sum_{m=1}^{M_k} p_m^k = 1, \\ G_k(c|a^k = a^k(m)) = F_k^m(c), \forall c \in \text{supp}(F_k), F_k^m \in \mathcal{F}_k^m, m = 1, \dots, M_k. \end{array} \right. \right\},$$

$k = 0, 1, \dots, K.$

- where \mathcal{F}_k^m could be an uncertainty set defined in Assumptions 1, 2, or 3.

Joint uncertainty set of a_{ij}^k and c_i^k

Individual chance constraints case

Theorem 11

Given Assumption 7, (IRGP) is equivalent to the following geometric program:

$$(IRGP_a) \quad \min_{t \in \mathbb{R}_{++}^M, z} \sum_{m=1}^{M_0} p_m^0 \left(\sup_{F_0^m \in \mathcal{F}_0^m} \mathbb{E}_{F_0^m} \left[\sum_{m=1}^{I_0} c_m^0 \prod_{j=1}^M t_j^{a_{ij}^0(m)} \right] \right) \quad (1)$$

$$\text{s.t.} \quad \sum_{m=1}^{M_k} p_m^k z_m^k \geq 1 - \epsilon_k, \quad k = 1, \dots, K, \quad (2)$$

$$\inf_{F_k^m \in \mathcal{F}_k^m} \mathbb{P}_{F_k^m} \left(\sum_{m=1}^{I_k} c_m^k \prod_{j=1}^M t_j^{a_{ij}^k(m)} \leq 1 \right) \geq z_m^k, \quad (3)$$
$$m = 1, \dots, M_k, \quad k = 1, \dots, K,$$

$$z_m^k \in [0, 1], \quad m = 1, \dots, M_k, \quad k = 1, \dots, K. \quad (4)$$

Joint uncertainty set of a_{ij}^k and c_i^k

Individual chance constraints case

Choosing \mathcal{F}_k^m to be an uncertainty sets with known first two order moments, we can reformulate (IRGP_a) as

$$\min_{t \in \mathbb{R}_{++}^M, z} \sum_{i=1}^{l_0} \mu_i^0 \prod_{j=1}^M t_j^{a_{ij}^0(m)} \quad (5)$$

$$\text{s.t.} \quad \sum_{m=1}^{M_k} p_m^k z_m^k \geq 1 - \epsilon_k, \quad k = 1, \dots, K, \quad (6)$$

$$\sum_{i=1}^{l_k} \mu_i^k(m) \prod_{j=1}^M t_j^{a_{ij}^k(m)} + \sqrt{\frac{z_m^k}{1 - z_m^k}} \sqrt{\sum_{i=1}^{l_k} \sum_{p=1}^{l_k} \sigma_{i,p}^k(m) \prod_{j=1}^M t_j^{a_{ij}^k(m) + a_{pj}^k(m)}} \leq 1, \quad (7)$$
$$m = 1, \dots, M_k, k = 1, \dots, K,$$

$$z_m^k \in [0, 1], \quad m = 1, \dots, M_k, k = 1, \dots, K. \quad (8)$$

Proposition 12

If $\epsilon_k \leq \frac{1}{2} \min_m \{p_m^k\}$, and $\sigma_{i,p}^k \geq 0$, for any $i, p = 1, \dots, l_k$, $k = 1, \dots, K$, (IRGP_{a1r}) is a convex programming problem.

Joint uncertainty set of a_{ij}^k and c_i^k

Ambiguity of the distribution of a^k

Assumption 8

The uncertainty sets for the joint distribution of c^k and a^k are

$$\mathcal{G}_k = \left\{ G_k \mid \begin{array}{l} G_k(a^k = a^k(m)) = p_m^k, m = 1, \dots, M_k, p^k \in \mathcal{P}_k, \\ G_k(c | a^k = a^k(m)) = F_k^m(c), \forall c \in \text{supp}(F_k), F_k^m \in \mathcal{F}_k^m, m = 1, \dots, \end{array} \right.$$

here, \mathcal{P}_k is the uncertainty set of the distribution of a^k .

- box uncertainty,

$$\mathcal{P}_k = \{p^k \mid p^k = \tilde{p}^k + \eta_k, e^\top \eta_k = 0, \underline{\eta}_k \leq \eta_k \leq \bar{\eta}_k\},$$

- ellipsoidal uncertainty,

$$\mathcal{P}_k = \{p^k \mid p^k = \tilde{p}^k + A_k \eta_k, e^\top A_k \eta_k = 0, \tilde{p}^k + A_k \eta_k \geq 0, \|\eta_k\| \leq 1\}.$$

- Similar convex reformulations.

Joint uncertainty set of a_{ij}^k and c_i^k

Joint chance constraints case

Assumption 9

The uncertainty set $\mathcal{G} = \mathcal{G}_1 \times \cdots \times \mathcal{G}_K$, and for any distribution G in \mathcal{G} , its marginal distributions G_1, \dots, G_K are pairwise independent. Where, \mathcal{G}_k is a marginal uncertainty set defined in Assumption 7.

Theorem 13

Given Assumption 9, (JRGP) is equivalent to the following program:

$$\begin{aligned} (JRGP_a) \quad & \min_{t \in \mathbb{R}_{++}^M, y, z} \sum_{m=1}^{M_0} p_m^0 \left(\sup_{F_0^m \in \mathcal{F}_0^m} \mathbb{E}_{F_0^m} \left[\sum_{m=1}^{l_0} c_m^0 \prod_{j=1}^M t_j^{a_{ij}^0} \right] \right) \\ & \text{s.t.} \quad \prod_{k=1}^K y_k \geq (1 - \epsilon), \quad 0 \leq y_k \leq 1, \quad k = 1, \dots, K, \\ & \quad \sum_{m=1}^{M_k} p_m^k z_m^k \geq y_k, \quad k = 1, \dots, K, \\ & \quad (3) - (4). \end{aligned}$$

Joint uncertainty set of a_{ij}^k and c_i^k

Joint chance constraints case

Similarly, choosing unknown moments uncertainty set \mathcal{F}_k^m , We can reformulate ($JRGP_a$) as

$$\begin{aligned} (JRGP_{a1r}) \quad & \min_{r, x, z} \sum_{i=1}^{I_0} \mu_i^0 \exp \left\{ \sum_{j=1}^M a_{ij}^0 r_j \right\} \\ & \text{s.t.} \quad \sum_{k=1}^K x_k \geq \log(1 - \epsilon), \quad x_k \leq 0, \quad k = 1, \dots, K, \\ & \quad \sum_{m=1}^{M_k} p_m^k z_m^k \geq e^{x_k}, \quad k = 1, \dots, K, \\ & (7) - (8). \end{aligned}$$

Similarly to (12), a sufficient condition for the convexity of ($JRGP_{a1r}$) is $\epsilon \leq \frac{1}{2} \min_{k,m} \{p_m^k\}$ and $\sigma_{i,p}^k \geq 0$, for any $i, p = 1, \dots, I_k, k = 1, \dots, K$.

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Numerical results

Distributionally robust geometric optimization

- We consider a distributionally robust shape optimization problem with individual chance constraints

$$\begin{aligned} (RSOP_I) \quad & \min_{x_1, \dots, x_m} \prod_{i=1}^m x_i^{-1} \\ & \text{s.t.} \quad P \left[\sum_{j=1}^{m-1} \left(\frac{m-1}{A_j} x_1 \prod_{i=2, i \neq j}^m x_i \right) \leq \beta_{wall} \right] \geq 1 - \epsilon_{wall} \\ & \quad P \left[\frac{1}{A_{flr}} \prod_{j=2}^m x_j \leq \beta_{flr} \right] \geq 1 - \epsilon_{flr} \\ & \quad x_i x_j^{-1} \leq \gamma_{i,j}, \quad \forall i \neq j. \end{aligned}$$

Numerical results

Distributionally robust geometric optimization

- and a distributionally robust shape optimization problem with joint chance constraints

$$\begin{aligned} (RSOP_J) \quad & \min_{x_1, \dots, x_m} \prod_{i=1}^m x_i^{-1} \\ & \text{s.t.} \quad P \left[\sum_{j=1}^{m-1} \left(\frac{m-1}{A_j} x_1 \prod_{i=2, i \neq j}^m x_i \right) \leq \beta_{wall}, \frac{1}{A_{flr}} \prod_{j=2}^m x_j \leq \beta_{flr} \right] \geq 1 - \epsilon \\ & \quad x_i x_j^{-1} \leq \gamma_{i,j}, \quad \forall i \neq j, \end{aligned}$$

Numerical results

Distributionally robust geometric optimization

- $1/A_{flr}$ and $1/A_j$, $j = 1, \dots, m - 1$, are considered as random variables.
- We assume $1/A_{fl}$ to be independent to $1/A_j$, $j = 1, \dots, m - 1$.
- F_{wall} and F_j are the distributions of $1/A_{flr}$ and $1/A_j$, $j = 1, \dots, m - 1$.
- F is the joint distribution of $1/A_{wall}$ and $1/A_j$, $j = 1, \dots, m - 1$.
- Mean values and covariances in all uncertainty sets are set the same.

Numerical results

Distributionally robust geometric optimization

- MOSEK solver from CVX package with Matlab R2012b; PC with a 2.6 Ghz Intel Core i7-5600U CPU and 12.0 GB RAM.
- $m = 100$
- Mean value of $1/A_{flr}$ is 0.02; the variance of $1/A_{flr}$ is 0.02; the mean value of $1/A_j$ is 0.01, $j = 1, \dots, m - 1$; the variance of $1/A_j$ is 0.01, $j = 1, \dots, m - 1$; all the covariance between $1/A_{flr}$ and $1/A_j$, $j = 1, \dots, m - 1$, are zero.
- for $(IRGP_2)$ and $(JRGP_2)$, we set $\pi_1^k = 0.0001$, $\pi_2^k = 1.2$, $k = 1, 2$;
- for $(IRGP_{3N})$ and $(JRGP_{3N})$, we set $\kappa_0 = \kappa = \kappa_1 = \kappa_2 = 0.02$.

Numerical results

Distributionally robust geometric optimization

- Individual vs Joint geometric chance constraints.
- Test 9 groups of instances with ϵ_{wall} and ϵ_{flr} such that $(1 - \epsilon_{wall})(1 - \epsilon_{flr}) = 1 - \epsilon$.

Table: Optimal values of (IRGP) and (JRGP)

ϵ	ϵ_{wall}	ϵ_{flr}	(IRGP ₁)	(JRGP ₁)	(IRGP ₂)	(JRGP ₂)	(ISGP)	(JSGP)	(IRGP _{3N})	(JRGP _{3N})
0.05	0.045	0.0052	289.98	277.05	313.50	299.35	138.24	135.61	162.96	159.35
0.05	0.040	0.0104	305.27	277.05	330.27	299.35	141.10	135.61	167.03	159.35
0.05	0.035	0.0155	323.66	277.05	350.44	299.35	144.29	135.61	171.66	159.35
0.05	0.030	0.0206	346.47	277.05	375.44	299.35	147.87	135.61	177.08	159.35
0.05	0.025	0.0256	375.74	277.05	407.52	299.35	151.97	135.61	183.61	159.35
0.05	0.020	0.0306	415.34	277.05	450.97	299.35	156.85	135.61	191.87	159.35
0.05	0.015	0.0355	473.29	277.05	514.50	299.35	162.89	135.61	203.06	159.35
0.05	0.010	0.0404	570.38	277.05	620.92	299.35	171.02	135.61	220.46	159.35
0.05	0.005	0.0452	789.58	277.05	861.26	299.35	184.01	135.61	257.39	159.35

Numerical results

Distributionally robust geometric optimization

- Comparisons between DRO and SP for geometric optimization
- We generate 50 groups of normal distributions with different mean values and variances $1/A_{flr}$ and $1/A_j, j = 1, \dots, m - 1$.
- Compute the satisfaction probabilities by optimal solutions of $(ISGP)$, $(IRGP_1)$, $(IRGP_2)$ and $(IRGP_{3N})$

Table 3: Values of $\mathbb{P}_{F_{wall}}$ and $\mathbb{P}_{F_{flr}}$ with $\epsilon_{wall} = 0.02$ and $\epsilon_{flr} = 0.0306$

Real distribution			$(ISGP)$		$(IRGP_1)$		$(IRGP_2)$		$(IRGP_{3N})$	
$1/A_{wall}$	$1/A_1$	\dots	$\mathbb{P}_{F_{wall}}$	$\mathbb{P}_{F_{flr}}$	$\mathbb{P}_{F_{wall}}$	$\mathbb{P}_{F_{flr}}$	$\mathbb{P}_{F_{wall}}$	$\mathbb{P}_{F_{flr}}$	$\mathbb{P}_{F_{wall}}$	$\mathbb{P}_{F_{flr}}$
N(0.0297 , 0.0337)	N(0.0356 , 0.0232)	\dots	0.9971	1.0000	0.9991	1.0000	0.5176	1.0000	0.6601	1.0000
N(0.0171 , 0.0212)	N(0.0208 , 0.0261)	\dots	0.9976	1.0000	0.9993	1.0000	0.5486	1.0000	0.6879	1.0000
N(0.0228 , 0.0355)	N(0.0264 , 0.0105)	\dots	0.9981	1.0000	0.9995	1.0000	0.5801	1.0000	0.7151	1.0000
N(0.0417 , 0.0117)	N(0.0329 , 0.0259)	\dots	0.9985	1.0000	0.9996	1.0000	0.6103	1.0000	0.7406	1.0000
N(0.0362 , 0.0235)	N(0.0355 , 0.0267)	\dots	0.9987	1.0000	0.9997	1.0000	0.5886	1.0000	0.7268	1.0000
N(0.0307 , 0.0357)	N(0.0155 , 0.0177)	\dots	0.9988	1.0000	0.9997	1.0000	0.6015	1.0000	0.7372	1.0000
N(0.0144 , 0.0157)	N(0.0357 , 0.0214)	\dots	0.9990	1.0000	0.9997	1.0000	0.6199	1.0000	0.7530	1.0000
N(0.0106 , 0.0371)	N(0.0124 , 0.0319)	\dots	0.9992	1.0000	0.9998	1.0000	0.6294	1.0000	0.7629	1.0000

Numerical results

Distributionally robust geometric optimization

Individual DRG Chance Constraints

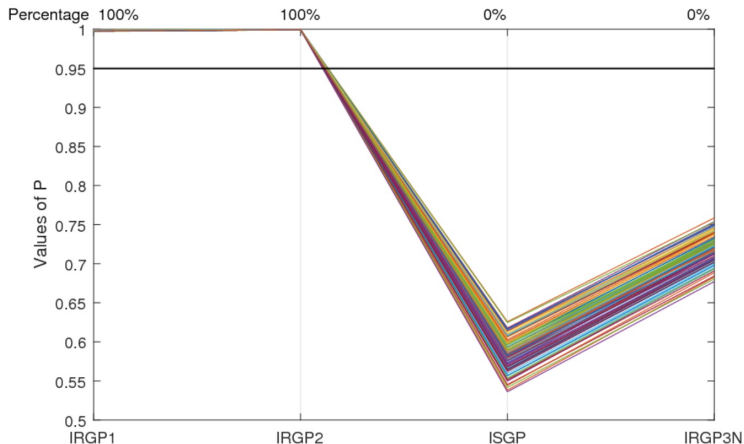


Figure 1 shows the values of $\mathbb{P} = \mathbb{P}_{F_{wall}} \times \mathbb{P}_{F_{fir}}$, the product of the satisfaction probabilities of individual chance constraints.

Joint DRG Chance Constraints

Table: Values of joint satisfaction probabilities \mathbb{P} with $\epsilon = 0.05$

Real distribution			Values of \mathbb{P}			
$1/A_{wall}$	$1/A_1$...	(JSGP)	(JRGP ₁)	(JRGP ₂)	(JRGP _{3N})
N(0.0199 , 0.0385)	N(0.0317 , 0.0338)	...	0.9047	0.9383	0.4332	0.5321
N(0.0220 , 0.0457)	N(0.0182 , 0.0353)	...	0.9177	0.9475	0.4670	0.5655
N(0.0279 , 0.0521)	N(0.0157 , 0.0203)	...	0.9208	0.9499	0.4688	0.5684
N(0.0112 , 0.0502)	N(0.0404 , 0.0229)	...	0.9281	0.9549	0.4911	0.5900
N(0.0279 , 0.0344)	N(0.0413 , 0.0441)	...	0.9318	0.9579	0.4914	0.5920
N(0.0396 , 0.0173)	N(0.0295 , 0.0528)	...	0.9373	0.9622	0.4910	0.5945
N(0.0158 , 0.0335)	N(0.0202 , 0.0478)	...	0.9483	0.9691	0.5432	0.6422
N(0.0162 , 0.0229)	N(0.0318 , 0.0136)	...	0.9508	0.9709	0.5469	0.6466

Numerical results

Distributionally robust geometric optimization

Joint DRG Chance Constraints

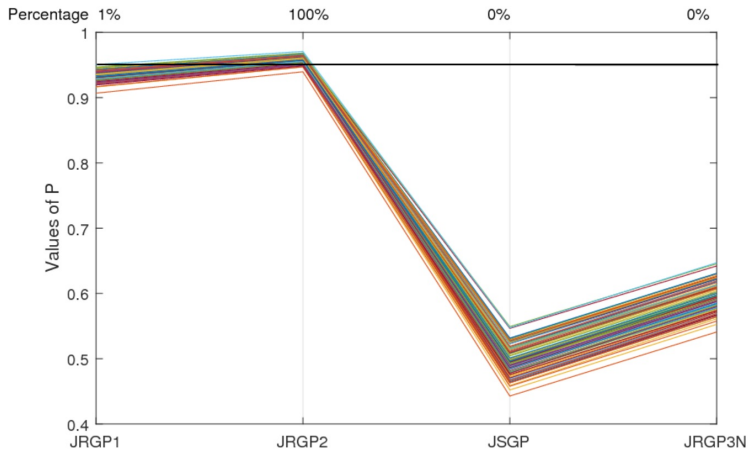


Figure 2 shows the values of salification probability of the joint constraint

\mathbb{P}

- Propose tractable reformulations for distributionally robust chance constrained geometric optimization problems with 3 different ambiguity sets.
- Show numerical feasibility on a stochastic optimization shape problem.
- Our results might be more conservative generally speaking due to some strong assumptions.
- Further research should be the extension of our results to more general (standard) geometric optimization under uncertainty.

Thank you!