Distributionally robust chance constrained geometric optimization

Liu Jia
Xi'an Jiaotong University

Joint work with Abdel Lisser[a] and Zhiping Chen[b]
[a] LRI, Université Paris Sud, France
[b] Xi'an Jiaotong University

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Outline of the talk

1. **Introduction**
2. Individual robust geometric chance constraints
3. Joint robust geometric chance constraints
4. Joint uncertainty of a_{ij}^k and c_i^k
5. Numerical results

Distributionally robust geometric optimization

A geometric program can be formulated as:

$$\min_t g_0(t) \text{ s.t. } g_k(t) \leq 1,\ k = 1, \ldots, K,\ t \in \mathbb{R}^M_{++}$$

with

$$g_k(t) = \sum_{i=1}^{I_k} c_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k},\ k = 0, \ldots, K.$$

- $c_i^k \prod_{j=1}^M t_j^{a_{ij}^k}$ is called a monomial, and $g_k(t)$ is called posynomial.
- The posynomials might have different parameters $c_i^k \geq 0$ and a_{ij}^k.
- Geometric programs are not convex with respect to t
- They are convex with respect to $\{r : r_j = \log\ t_j,\ j = 1, \ldots, M\}$.

Applications (cf. S. Boyd, 2007, R. Wiebking, 1977, M. Luptáčik, 1981, S. Kim et al., 2007)

- Wireless communications
- Semiconductor device engineering
- Floor planning
- Digital circuit gate sizing
- Economic and managerial problems
- Wire sizing
- ...

Individual chance constraints are studied in S. Rao (1996) and J. Dupačová (2009):

$$
\begin{aligned}
(ISGP) \quad & \min_{t \in \mathbb{R}^M_{++}} \quad \mathbb{E}_{F_0}\left[\sum_{i=1}^{I_0} c_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0}\right] \\
& \text{s.t.} \quad \mathbb{P}_{F_k}\Big(\sum_{i=1}^{I_k} c_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} \leq 1\Big) \geq 1 - \epsilon_k, \ \ k = 1, \ldots K
\end{aligned}
$$

- c_i^k is random (normally distributed), a_{ij}^k is constant.
- Reformulated as geometric programs.

Dupacova J (2009) Stochastic geometric programming: approaches and applications. In Brozova V,Kvasnicka R, eds. Proceedings of MME09, 63-66.

Distributionally robust geometric optimization

Recently, L. and A. Lisser (2016) studied stochastic geometric problems with joint chance constraints; i.e.;

$$
\begin{aligned}
(JSGP) \quad & \min_{t \in \mathbb{R}^M_{++}} \quad \mathbb{E}_{F_0}\left[\sum_{i=1}^{I_0} c_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0}\right] \\
& \text{s.t.} \quad \mathbb{P}_F\left(\sum_{i=1}^{I_k} c_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} \leq 1,\ k = 1, ...K\right) \geq 1 - \epsilon,
\end{aligned}
$$

- c_i^k is random (normally distributed), a_{ij}^k is constant.
- Reformulated as convex programming problems when $\epsilon \leq 0.5$.

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   - Uncertainty with unknown moments
   - Uncertainty with density uncertainty (Continuous/Data-driven cases)
3. Joint robust geometric chance constraints
   - Uncertainty with known moments (Outer/inner approximations)
   - Uncertainty with unknown moments
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4. Joint uncertainty of a_{ij}^k and c_i^k
5. Numerical results

We consider the distributionally robust geometric programs with individual chance constraints and the ambiguity of F_0, F_k or F:

$$
\begin{aligned}
(IRGP) \quad & \min_{t\in\mathbb{R}^M_{++}} \quad \sup_{F\in\mathscr{F}_0} \mathbb{E}_{F_0}\left[\sum_{i=1}^{I_0} c_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0}\right] \\
& \text{s.t.} \quad \inf_{F_k\in\mathscr{F}_k} \mathbb{P}_{F_k}\left(\sum_{i=1}^{I_k} c_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} \leq 1\right) \geq 1-\epsilon_k, \ \ k=1,...K.
\end{aligned}
$$

where $\mathscr{F}_0$, $\mathscr{F}_k$, $k=1,\ldots,K$ are the uncertainty sets, which contain all the distributions of F_0, F_k, $k=1,\ldots,K$.

- c_i^k is random, a_{ij}^k is constant.

Uncertainty sets with known moments

- We consider the uncertainty sets $\mathscr{F}_k$, $k = 0, ..., K$, with known two first order moments information (L. El Ghaoui et al, 2003, L. Chen et al, 2011).

Assumption 1

The uncertainty sets are

$$\mathscr{F}_k = \{F_k \mid \mathbb{E}_{F_k}[c^k] = \mu^k, Cov_{F_k}[c^k] = \Gamma^k\}, \; k = 0, ..., K.$$

where $Cov_F[c^k] = \mathbb{E}_{F_k}\left[\left(c^k - \mathbb{E}_{F_k}[c^k]\right)\left(c^k - \mathbb{E}_{F_k}[c^k]\right)^\top\right]$,

- $\mu^k = [\mu_1^k, \mu_2^k, \ldots, \mu_{I_k}^k]^\top$ *and* $\Gamma^k = \{\sigma_{i,j}^k\}$.
- μ_i^k *is the reference value of the expected value of* c_i^k.
- $\sigma_{i,j}^k$ *is the reference value of the covariance between* c_i^k *and* c_j^k,
- *We assume that* $\mu^k \geq 0$, Γ^k *is positive definite matrix, and* $\sigma_{i,j}^k \geq 0$.

Distributionally robust geometric optimization

Theorem 1

Suppose that Assumption 1 holds. Then (IRGP) is equivalent to

$$(IRGP_1) \quad \min_{t \in \mathbb{R}^M_{++}} \quad \sum_{i=1}^{I_0} \mu_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0}$$

$$s.t. \quad \sum_{i=1}^{I_k} \mu_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} + \sqrt{\frac{1-\epsilon_k}{\epsilon_k}} \sqrt{\sum_{i=1}^{I_k} \sum_{p=1}^{I_k} \sigma_{i,p}^k \prod_{j=1}^{M} t_j^{a_{ij}^k + a_{pj}^k}} \leq 1,$$

$$k = 1, \ldots K.$$

Sketch of the proof:

- Lemma 2.1 (L. Chen et al., 2011)
- Chebytchev inequality (N. Rujeerapaiboon et al., 2016)

Distributionally robust geometric optimization

Problem $(IRGP_1)$ is not convex w.r.t t, it can be transformed into a convex problem using $r_j = \log(t_j),\ j = 1, \ldots, M$

$$
(IRGP_{1s}) \quad \min_{r \in \mathbb{R}^M} \quad \sum_{i=1}^{I_0} \mu_i^0 \exp\left\{\sum_{j=1}^{M} a_{ij}^0 r_j\right\}
$$

$$
s.t. \quad \sqrt{\frac{1-\epsilon_k}{\epsilon_k}} \sqrt{\sum_{i=1}^{I_k}\sum_{p=1}^{I_k} \sigma_{i,p}^k \exp\left\{\sum_{j=1}^{M}(a_{ij}^k + a_{pj}^k) r_j\right\}} + \sum_{i=1}^{I_k} \mu_i^k \exp\left\{\sum_{j=1}^{M} a_{ij}^k r_j\right\} \leq 1,\ k = 1, \ldots, K,
$$

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5. Numerical results

We consider the uncertainty sets $\mathscr{F}_k$, $k = 0, ..., K$, for $(IRGP)$. (E. Delage and Y. Ye, 2010; J. Cheng et al. 2003; N. Rujeerapaiboon et al., 2015).

Assumption 2

- *The uncertainty sets are*

$$\mathscr{F}_k = \left\{ F_k \left| \begin{array}{l} \left(\mathbb{E}_{F_k}[c^k] - \mu^k\right)^\top (\Gamma^k)^{-1} \left(\mathbb{E}_{F_k}[c^k] - \mu^k\right) \leq \pi_1^k, \\ Cov_{F_k}[c^k] \preceq_D \pi_2^k \Gamma^k. \end{array} \right. \right\},$$

- *We assume that $\mu^k \geq 0$, Γ^k is positive definite matrix, and $\sigma_{i,j}^k \geq 0$, $k = 0, 1, ..., K$.*
- *$\pi_1^k, \pi_2^k \in \mathbb{R}$ are two scale parameters controlling the size of the uncertainty sets.*
- *$A \preceq_D B$ means that for any $x \in \mathbb{R}^n$, we have $x^\top A x \leq x^\top B x$.*

Theorem 2

Given Assumption 2, (IRGP) is equivalent to

$$(IRGP_2) \quad \min_{t \in \mathbb{R}^M_{++}} \quad \sum_{i=1}^{I_0} \mu_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0} + \sqrt{\pi_1^0} \sqrt{\sum_{i=1}^{I_0} \sum_{p=1}^{I_0} \sigma_{i,p}^0 \prod_{j=1}^{M} t_j^{a_{ij}^0 + a_{pj}^0}}$$

$$s.t. \quad \sum_{i=1}^{I_k} \mu_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} +$$

$$\left(\sqrt{\frac{1-\epsilon_k}{\epsilon_k}} \sqrt{\pi_2^k} + \sqrt{\pi_1^k} \right) \sqrt{\sum_{i=1}^{I_k} \sum_{p=1}^{I_k} \sigma_{i,p}^k \prod_{j=1}^{M} t_j^{a_{ij}^k + a_{pj}^k}} \leq 1, \; k = 1, \ldots K.$$

Sketch of the proof:

- Same ingredients as the previous Theorem extended to this case.

With the standard variable transformation $r_j = \log(t_j),\ j = 1, \ldots, M$, we can transform $(IRGP_2)$ into

$$\begin{aligned} \min_{r\in\mathbb{R}^M} \quad & \sum_{i=1}^{I_0} \mu_i^0 \exp\left\{\sum_{j=1}^{M} a_{ij}^0 r_j\right\} + \sqrt{\pi_1^0}\sqrt{\sum_{i=1}^{I_0}\sum_{p=1}^{I_0} \sigma_{i,p}^0 \exp\left\{\sum_{j=1}^{M}(a_{ij}^0 + a_{pj}^0) r_j\right\}} \\ s.t. \quad & \left(\sqrt{\frac{1-\epsilon_k}{\epsilon_k}}\sqrt{\pi_2^k} + \sqrt{\pi_1^k}\right)\sqrt{\sum_{i=1}^{I_k}\sum_{p=1}^{I_k} \sigma_{i,p}^k \exp\left\{\sum_{j=1}^{M}(a_{ij}^k + a_{pj}^k) r_j\right\}} \\ & + \sum_{i=1}^{I_k} \mu_i^k \exp\left\{\sum_{j=1}^{M} a_{ij}^k r_j\right\} \leq 1,\ k = 1, \ldots, K. \end{aligned}$$

which is a convex optimization problem.

- Avoid SDP reformulation (Delage and Ye 2010; Cheng, Delage and Lisser 2014)

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Uncertainty sets controlled by the distance between the true distribution and the reference distribution of c^k (A. Ben-Tal, 2013; Z. Hu and J. Hong, 2013; R. Jiang and Y. Guan, 2016).

Assumption 3

The uncertainty sets are

$$\mathscr{F}_k = \{F_k \mid D_{DL}(F_k||F_k^0) \leq \kappa_k\}, \; k = 0, ..., K.$$

where D_{DL} is the Kullback-Leibler divergence distance

$$D_{DL}(F_k||F_k^0) = \int_\Omega \phi\left(\frac{f_{F_k}(c^k)}{f_{F_k^0}(c^k)}\right) f_{F_k^0}(c^k)dc^k,$$

- *F_k^0 is the reference distribution of c^k, $f_{F_k}(c^k)$ and $f_{F_k^0}(c^k)$ are the density functions of the true distribution and the reference distribution of c^k on Ω,*
- *κ_k is a parameter controlling the size of the uncertainty set, $k = 0, ..., K$. $\phi(t) = t \log t - t + 1$, for $t \geq 0$, and $\phi(t) = \infty$, ow.*

We use Theorem 1 in Z. Hu and J. Hong (2013) for the following proposition:

Proposition 3

Given Assumption 3, the objective function is equivalent to

$$\inf_{\alpha\in(0,\infty)} \alpha \log \mathbb{E}_{F_0^0}\left[\exp\left\{\left(\sum_{i=1}^{I_0} c_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0}\right)/\alpha\right\}\right] + \alpha\kappa_0.$$

and Theorem 1 and Proposition 4 in R. Jiang and Y. Guan (2016),

Proposition 4

Given Assumption 3, the constraint is equivalent to

$$\mathbb{P}_{F_k^0}(\sum_{i=1}^{I_k} c_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} \leq 1) \geq 1-\epsilon_k', \; k=1,\ldots K,$$

where $\epsilon_k' = 1 - \inf_{x\in(0,1)}\left\{\frac{e^{-\kappa_k}x^{1-\epsilon_k}-1}{x-1}\right\}$, $k=1,\ldots K$.

Assume that the reference distribution F_k^0 follows a normal distribution:

- mean vector $\mu^k = [\mu_1^k, \mu_2^k, \ldots, \mu_{I_k}^k]^\top \geq 0$
- positive definite covariance matrix $\Gamma^k = \{\sigma_{i,j}^k,\ i,j = 1, \ldots, I_k\}, \forall k.$

Theorem 5

Given Assumption 3 and normal distribution assumption for F_k^0, $k = 0, 1, \ldots, K$, (IRGP) is equivalent to

$$(IRGP_{3N}) \quad \min_{t \in \mathbb{R}_{++}^M} \sum_{i=1}^{I_0} \mu_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0} + \sqrt{2\kappa_0 \sum_{i=1}^{I_0} \sum_{p=1}^{I_0} \sigma_{i,p}^0 \prod_{j=1}^{M} t_j^{a_{ij}^0 + a_{pj}^0}}$$

$$s.t. \quad \sum_{i=1}^{I_k} \mu_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} + \Phi^{-1}(1 - \epsilon_k') \sqrt{\sum_{i=1}^{I_k} \sum_{p=1}^{I_k} \sigma_{i,p}^k \prod_{j=1}^{M} t_j^{a_{ij}^k + a_{pj}^k}} \leq 1, \ k = 1, \ldots K.$$

Here, $\Phi^{-1}(\cdot)$ is the quantile of the standard normal distribution $N(0, 1)$.

Proof: Based on previous theorems

Historical data based discrete reference distribution F_k^0.

Theorem 6

Given Assumption 3, we further assume F_k^0 follows a discrete distribution with H possible scenarios $\tilde{c}^k(1)$, $\tilde{c}^k(2)$, $\tilde{c}^k(H)$, associated with their probabilities $\frac{1}{H}$, $k = 0, 1, \ldots, K$. Then, problem (IRGP) is equivalent to ($IRGP_{3D}$)

$$
\begin{aligned}
\min_{r\in\mathbb{R}^M,\alpha\in(0,\infty),\varsigma} \quad & \alpha \log\left(\frac{1}{H}\sum_{h=1}^{H} \exp\left\{\left(\sum_{i=1}^{I_0} \tilde{c}_i^0(h)\exp\left\{\sum_{j=1}^{M} a_{ij}^0 r_j\right\}\right)/\alpha\right\}\right) + \alpha\kappa_0, \\
\text{s.t.} \quad & \frac{1}{H}\sum_{h=1}^{H}(1-\varsigma_h^k) \geq 1-\epsilon'_k,\ k = 1, 2, ..., K, \\
& \sum_{i=1}^{I_k} \tilde{c}_i^k(h)\exp\left\{\sum_{j=1}^{M} a_{ij}^k r_j\right\} \leq M\varsigma_h^k + 1,\ h = 1, \ldots, H,\ k = 1, 2, ..., K, \\
& \varsigma_h^k \in \{0, 1\},\ h = 1, \ldots, H,\ k = 1, 2, ..., K.
\end{aligned}
$$

Proof: Propositions 4 and 5 and previous theorems.

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We consider the distributionally robust geometric programs with joint chance constraints and the ambiguity of F_0 or F:

$$
\begin{aligned}
(JRGP) \quad & \min_{t\in\mathbb{R}^M_{++}} \quad \sup_{F_0\in\mathscr{F}_0} \mathbb{E}_{F_0}\left[\sum_{i=1}^{I_0} c_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0}\right] \\
& \text{s.t.} \quad \inf_{F\in\mathscr{F}} \mathbb{P}_F\left(\sum_{i=1}^{I_k} c_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} \leq 1,\ k=1,...K\right) \geq 1-\epsilon,
\end{aligned}
$$

where $\mathscr{F}_0$ and $\mathscr{F}$ are the uncertainty sets, which contain all the distributions of F_0 and F.

- c_i^k is random, a_{ij}^k is constant.

- We consider an uncertainty sets $\mathscr{F}$ for $(JRGP)$, with known two first order moments, and with pairwise independent marginal distributions.

Assumption 4

- *The uncertainty set* $\mathscr{F} = \mathscr{F}_1 \times \cdots \times \mathscr{F}_K$
- *For any joint distribution* F *in* $\mathscr{F}$, *its marginal distributions* $F_1, \ldots, F_K$ *are pairwise independent.*

$$\mathscr{F}_k = \{F_k \mid E_{F_k}[c^k] = \mu^k, Cov_{F_k}[c^k] = \Gamma^k\},\ \ k = 0, 1, \ldots, K.$$

- *We assume that* $\mu^k \geq 0$, Γ^k *is positive definite matrix, and* $\sigma^k_{i,j} \geq 0$, $k = 0, 1, ..., K$.

Theorem 7

Given Assumption 4, (JRGP) is equivalent to

$$(JRGP_1)$$

$$\min_{t\in\mathbb{R}^M_{++},y\in\mathbb{R}^K_{++}} \quad \sum_{i=1}^{I_0}\mu_i^0\prod_{j=1}^{M}t_j^{a_{ij}^0}$$

$$s.t. \quad \sum_{i=1}^{I_k}\mu_i^k\prod_{j=1}^{M}t_j^{a_{ij}^k}+\sqrt{\frac{y_k}{1-y_k}}\sqrt{\sum_{i=1}^{I_k}\sum_{p=1}^{I_k}\sigma_{i,p}^k\prod_{j=1}^{M}t_j^{a_{ij}^k+a_{pj}^k}}\le 1,\ \ k=1,\ldots K.$$

$$\prod_k y_k \ge 1-\epsilon.$$

Non convex due to the bi-linear term.

- We transform $(JRGP_1)$ using $r_j = \log(t_j),\ j = 1, \ldots, M$ and $x_k = \log(y_k),\ k = 1, \ldots, K$:

$$(JRGP_{1s}) \quad \min_{r \in \mathbb{R}^M, x \in \mathbb{R}^K} \sum_{i=1}^{I_0} \mu_i^0 \exp\left\{ \sum_{j=1}^{M} a_{ij}^0 r_j \right\}$$

$$s.t. \quad \sqrt{\sum_{i=1}^{I_k} \sum_{p=1}^{I_k} \sigma_{i,p}^k \exp\left\{ \sum_{j=1}^{M} \left((a_{ij}^k + a_{pj}^k) r_j \right) + \log\left(\frac{e^{x_k}}{1 - e^{x_k}} \right) \right\}}$$

$$+ \sum_{i=1}^{I_k} \mu_i^k \exp\left\{ \sum_{j=1}^{M} a_{ij}^k r_j \right\} \leq 1,\ k = 1, \ldots, K,$$

$$\sum_{k=1}^{K} x_k \geq \log(1 - \epsilon),\ x_k \leq 0,\ k = 1, \ldots, K.$$

- $(JRGP_{1s})$ is a convex programming problem as $\log\left(\frac{e^{x_k}}{1-e^{x_k}}\right)$ is convex. Can be rewritten as $x_k - \log\left(1 - e^{x_k}\right)$ to meet the rules of disciplined convex programming of CVX.

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Assumption 5

The uncertainty set $\mathscr{F} = \mathscr{F}_1 \times \cdots \times \mathscr{F}_K$, *and for any distribution* F *in* $\mathscr{F}$, *its marginal distributions* $F_1, \ldots, F_K$ *are pairwise independent. Moreover,*

$$\mathscr{F}_k = \left\{ F_k \;\middle|\; \begin{array}{l} \left(\mathbb{E}_{F_k}[c^k] - \mu^k\right)^\top (\Gamma^k)^{-1} \left(\mathbb{E}_{F_k}[c^k] - \mu^k\right) \leq \pi_1^k, \\ Cov_{F_k}[c^k] \preceq_D \pi_2^k \Gamma^k. \end{array} \right\},$$

- *We assume that* $\mu^k \geq 0$, Γ^k *is positive definite matrix, and* $\sigma_{i,j}^k \geq 0$, $k = 0, 1, ..., K$.

Theorem 8

Given Assumption 5, (JRGP) is equivalent to

$$
\begin{aligned}
(JRGP_2) \quad \min_{t\in\mathbb{R}^M_{++}, y\in\mathbb{R}^K_{++}} \quad & \sum_{i=1}^{I_0} \mu_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0} + \sqrt{\pi_1^0}\sqrt{\sum_{i=1}^{I_0}\sum_{p=1}^{I_0} \sigma_{i,p}^0 \prod_{j=1}^{M} t_j^{a_{ij}^0+a_{pj}^0}} \\
s.t. \quad & \sum_{i=1}^{I_k} \mu_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} + \sqrt{\pi_1^k}\sqrt{\sum_{i=1}^{I_k}\sum_{p=1}^{I_k} \sigma_{i,p}^k \prod_{j=1}^{M} t_j^{a_{ij}^k+a_{pj}^k}} \\
& + \sqrt{\frac{y_k}{1-y_k}}\sqrt{\pi_2^k}\sqrt{\sum_{i=1}^{I_k}\sum_{p=1}^{I_k} \sigma_{i,p}^k \prod_{j=1}^{M} t_j^{a_{ij}^k+a_{pj}^k}} \le 1, \ k=1,\dots K. \\
& \prod_{k} y_k \ge 1-\epsilon.
\end{aligned}
$$

$(JRGP_2)$ can be reformulated as a convex programming problem by changing the decision variables.

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We consider uncertainty sets for $(JRGP)$ with a reference distribution.

Assumption 6

The uncertainty sets are

$$\mathscr{F}_0 = \{F_0 \mid D_{DL}(F_0 || F_0^0) \le \kappa_0\} \text{ and } \mathscr{F} = \{F \mid D_{DL}(F || F^0) \le \kappa\},$$

where

- D_{DL} *is defined in Assumption 3,*
- F_0^0 *is the reference distribution for* c^0,
- F^0 *is the reference joint distribution for* c^1, $c^2, \ldots, c^K$, *such that* $F^0 = F_1^0 \times \cdots \times F_K^0$ *and the marginal distributions* $F_1^0, \ldots, F_K^0$ *are pairwise independent.*

Theorem 9

Given Assumption 7, we assume that $F_0^0, F_1^0, \ldots, F_K^0$ *are normal distributions with mean vector* $\mu^k = [\mu_1^k, \mu_2^k, \ldots, \mu_{I_k}^k]^\top$ *and covariance matrix* $\Gamma^k = \{\sigma_{i,j}^k,\ i,j = 1, \ldots, I_k\}$, $k = 0, 1, \ldots, K$. *Then* $(JRGP)$ *is equivalent to* $(JRGP_{3N})$

$$\begin{aligned}
\min_{t\in\mathbb{R}^M_{++},\, y\in\mathbb{R}^K_{++}} \quad & \sum_{i=1}^{I_0} \mu_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0} + \sqrt{2\kappa_0 \sum_{i=1}^{I_0} \sum_{p=1}^{I_0} \sigma_{i,p}^0 \prod_{j=1}^{M} t_j^{a_{ij}^0 + a_{pj}^0}} \\
\text{s.t.} \quad & \sum_{i=1}^{I_k} \mu_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} + \Phi^{-1}(y_k) \sqrt{\sum_{i=1}^{I_k} \sum_{p=1}^{I_k} \sigma_{i,p}^k \prod_{j=1}^{M} t_j^{a_{ij}^k + a_{pj}^k}} \leq 1,\ k = 1, \ldots K, \\
& \prod_k y_k \geq 1 - \epsilon'.
\end{aligned}$$

where $\epsilon' = 1 - \inf_{x\in(0,1)} \left\{ \frac{e^{-\kappa} x^{1-\epsilon} - 1}{x - 1} \right\}$.

Convex reformulation can be obtained when $\epsilon' \leq 0.5$ and $\sigma_{i,p}^k \geq 0$.

Theorem 10

Given Assumption 7, we assume that F_k^0 is a discrete distribution with H possible values $\tilde{c}^k(h)$, $h = 1, \ldots, H$, associated with their probabilities $\frac{1}{H}$, $k = 0, 1, \ldots, K$. Then $(JRGP)$ is equivalent to $(JRGP_{3D})$

$$
\begin{aligned}
\min_{r\in\mathbb{R}^M,\alpha\in(0,\infty),\varsigma} \quad & \alpha \log\left(\frac{1}{H}\sum_{h=1}^{H}\exp\left\{\left(\sum_{i=1}^{I_0}\tilde{c}_i^0(h)\exp\left\{\sum_{j=1}^{M}a_{ij}^0 r_j\right\}\right)/\alpha\right\}\right) + \alpha\kappa_0, \\
\text{s.t.} \quad & \frac{1}{H}\sum_{h=1}^{H}(1-\varsigma_h) \geq 1-\epsilon', \\
& \sum_{i=1}^{I_k}\tilde{c}_i^k(h)\exp\left\{\sum_{j=1}^{M}a_{ij}^k r_j\right\} \leq M\varsigma_h + 1,\ h = 1, \ldots, H,\ k = 1, 2, \ldots, K, \\
& \varsigma_h \in \{0, 1\},\ h = 1, \ldots, H.
\end{aligned}
$$

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   - Uncertainty with known moments (Outer/inner approximations)
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4. Joint uncertainty of a_{ij}^k and c_i^k
5. Numerical results

We consider uncertainty sets for $(IRGP)$ with a reference distribution.

Assumption 7

The uncertainty sets are

$$\mathscr{G}_k = \left\{ G_k \left| \begin{array}{l} G_k(a^k = a^k(m)) = p_m^k,\ m = 1, \ldots, M_k,\ \sum_{m=1}^{M_k} p_m^k = 1, \\ G_k(c|a^k = a^k(m)) = F_k^m(c), \forall c \in supp(F_k),\ F_k^m \in \mathscr{F}_k^m,\ m = 1, \ldots, M_k. \end{array} \right. \right\},$$

$$k = 0, 1, \ldots, K.$$

- *where $\mathscr{F}_k^m$ could be an uncertainty set defined in Assumptions 1, 2, or 3.*

Joint uncertainty set of a_{ij}^k and c_i^k

Individual chance constraints case

Theorem 11

Given Assumption 7, (IRGP) is equivalent to the following geometric program:

$$
\begin{aligned}
(IRGP_a) \quad \min_{t\in\mathbb{R}^M_{++}, z} \quad & \sum_{m=1}^{M_0} p_m^0 \left(\sup_{F_0^m \in \mathscr{F}_0^m} \mathbb{E}_{F_0^m} \left[\sum_{m=1}^{I_0} c_m^0 \prod_{j=1}^{M} t_j^{a_{ij}^0(m)} \right] \right) && (1)\\
\text{s.t.} \quad & \sum_{m=1}^{M_k} p_m^k z_m^k \geq 1 - \epsilon_k, \ k = 1, \ldots, K, && (2)\\
& \inf_{F_k^m \in \mathscr{F}_k^m} \mathbb{P}_{F_k^m} \left(\sum_{m=1}^{I_k} c_m^k \prod_{j=1}^{M} t_j^{a_{ij}^k(m)} \leq 1 \right) \geq z_m^k,\\
& \qquad m = 1, \ldots, M_k, k = 1, \ldots, K, && (3)\\
& z_m^k \in [0,1], \ m = 1, \ldots, M_k, k = 1, \ldots K. && (4)
\end{aligned}
$$

Choosing $\mathscr{F}_k^m$ to be an uncertainty sets with known first two order moments, we can reformulate $(IRGP_a)$ as

$$\min_{t\in\mathbb{R}^M_{++},z} \quad \sum_{i=1}^{I_0} \mu_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0(m)} \tag{5}$$

$$\text{s.t.} \quad \sum_{m=1}^{M_k} p_m^k z_m^k \geq 1-\epsilon_k, \ \ k=1,\dots,K, \tag{6}$$

$$\sum_{i=1}^{I_k} \mu_i^k(m) \prod_{j=1}^{M} t_j^{a_{ij}^k(m)} + \sqrt{\frac{z_m^k}{1-z_m^k}} \sqrt{\sum_{i=1}^{I_k}\sum_{p=1}^{I_k} \sigma_{i,p}^k(m) \prod_{j=1}^{M} t_j^{a_{ij}^k(m)+a_{pj}^k(m)}} \leq 1,$$

$$m=1,\dots,M_k, k=1,\dots,K, \tag{7}$$

$$z_m^k \in [0,1], \ \ m=1,\dots,M_k, k=1,\dots K. \tag{8}$$

Proposition 12

If $\epsilon_k \leq \frac{1}{2}\min_m\{p_m^k\}$, and $\sigma_{i,p}^k \geq 0$, for any $i,p=1,\dots,I_k$, $k=1,\dots,K$, $(IRGP_{a1r})$ is a convex programming problem.

Assumption 8

The uncertainty sets for the joint distribution of c^k and a^k are

$$\mathscr{G}_k = \left\{ G_k \left| \begin{array}{l} G_k(a^k = a^k(m)) = p_m^k,\ m = 1, \ldots, M_k,\ p^k \in \mathscr{P}_k, \\ G_k(c|a^k = a^k(m)) = F_k^m(c), \forall c \in supp(F_k),\ F_k^m \in \mathscr{F}_k^m,\ m = 1, \ldots, \end{array} \right. \right.$$

here, $\mathscr{P}_k$ is the uncertainty set of the distribution of a^k.

- box uncertainty,

$$\mathscr{P}_k = \{p^k | p^k = \tilde{p}^k + \eta_k,\ e^\top \eta_k = 0,\ \underline{\eta}_k \leq \eta_k \leq \bar{\eta}_k\},$$

- ellipsoidal uncertainty,

$$\mathscr{P}_k = \{p^k | p^k = \tilde{p}^k + A_k \eta_k,\ e^\top A_k \eta_k = 0,\ \tilde{p}^k + A_k \eta_k \geq 0,\ \|\eta_k\| \leq 1\}.$$

- Similar convex reformulations.

Joint uncertainty set of a_{ij}^k and c_i^k

Joint chance constraints case

Assumption 9

The uncertainty set $\mathscr{G} = \mathscr{G}_1 \times \cdots \times \mathscr{G}_K$, and for any distribution G in $\mathscr{G}$, its marginal distributions $G_1, \ldots, G_K$ are pairwise independent. Where, $\mathscr{G}_k$ is a marginal uncertainty set defined in Assumption 7.

Theorem 13

Given Assumption 9, $(JRGP)$ is equivalent to the following program:

$$
\begin{aligned}
(JRGP_a) \quad & \min_{t\in\mathbb{R}^M_{++}, y, z} \quad \sum_{m=1}^{M_0} p_m^0 \left(\sup_{F_0^m \in \mathscr{F}_0^m} \mathbb{E}_{F_0^m} \left[\sum_{m=1}^{I_0} c_m^0 \prod_{j=1}^{M} t_j^{a_{ij}^0} \right] \right) \\
& \text{s.t.} \quad \prod_{k=1}^{K} y_k \geq (1-\epsilon),\ 0 \leq y_k \leq 1,\ k = 1, \ldots, K, \\
& \qquad \sum_{m=1}^{M_k} p_m^k z_m^k \geq y_k,\ k = 1, \ldots, K, \\
& \qquad (3) - (4).
\end{aligned}
$$

Joint uncertainty set of a_{ij}^k and c_i^k

Joint chance constraints case

Similarly, choosing unknown moments uncertainty set $\mathscr{F}_k^m$, We can reformulate $(JRGP_a)$ as

$$
\begin{aligned}
(JRGP_{a1r}) \quad & \min_{r,x,z} \quad \sum_{i=1}^{I_0} \mu_i^0 \exp\left\{\sum_{j=1}^{M} a_{ij}^0 r_j\right\} \\
& \text{s.t.} \quad \sum_{k=1}^{K} x_k \geq \log(1-\epsilon),\ x_k \leq 0,\ k=1,\ldots,K, \\
& \qquad \sum_{m=1}^{M_k} p_m^k z_m^k \geq e^{x_k},\ k=1,\ldots,K, \\
& \qquad (7)-(8).
\end{aligned}
$$

Similarly to (12), a sufficient condition for the convexity of $(JRGP_{a1r})$ is $\epsilon \leq \frac{1}{2}\min_{k,m}\{p_m^k\}$ and $\sigma_{i,p}^k \geq 0$, for any $i,p=1,\ldots,I_k,\ k=1,\ldots,K$.

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5. **Numerical** results

- We consider a distributionally robust shape optimization problem with individual chance constraints

$$
\begin{aligned}
(RSOP_I) \quad & \min_{x_1,\ldots,x_m} \quad \prod_{i=1}^{m} x_i^{-1} \\
& s.t. \quad P\left[\sum_{j=1}^{m-1}\left(\frac{m-1}{A_j} x_1 \prod_{i=2, i\neq j}^{m} x_i\right) \leq \beta_{wall}\right] \geq 1-\epsilon_{wall} \\
& \qquad P\left[\frac{1}{A_{flr}} \prod_{j=2}^{m} x_j \leq \beta_{flr}\right] \geq 1-\epsilon_{flr} \\
& \qquad x_i x_j^{-1} \leq \gamma_{i,j}, \ \forall i \neq j.
\end{aligned}
$$

- and a distributionally robust shape optimization problem with joint chance constraints

$$
\begin{aligned}
(RSOP_J) \quad & \min_{x_1,\ldots,x_m} \quad \prod_{i=1}^{m} x_i^{-1} \\
& s.t. \quad P\left[\sum_{j=1}^{m-1} \left(\frac{m-1}{A_j} x_1 \prod_{i=2, i\neq j}^{m} x_i\right) \le \beta_{wall},\ \frac{1}{A_{flr}} \prod_{j=2}^{m} x_j \le \beta_{flr}\right] \ge 1-\epsilon \\
& \qquad x_i x_j^{-1} \le \gamma_{i,j},\ \forall i \neq j,
\end{aligned}
$$

- $1/A_{flr}$ and $1/A_j$, $j = 1, \ldots, m-1$, are considered as random variables.
- We assume $1/A_{fl}$ to be independent to $1/A_j$, $j = 1, \ldots, m-1$.
- F_{wall} and F_j are the distributions of $1/A_{flr}$ and $1/A_j$, $j = 1, \ldots, m-1$.
- F is the joint distribution of $1/A_{wall}$ and $1/A_j$, $j = 1, \ldots, m-1$.
- Mean values and covariances in all uncertainty sets are set the same.

- MOSEK solver from CVX package with Matlab R2012b; PC with a 2.6 Ghz Intel Core i7-5600U CPU and 12.0 GB RAM.
- $m = 100$
- Mean value of $1/A_{flr}$ is 0.02; the variance of $1/A_{flr}$ is 0.02; the mean value of $1/A_j$ is 0.01, $j = 1, \ldots, m-1$; the variance of $1/A_j$ is 0.01, $j = 1, \ldots, m-1$; all the covariance between $1/A_{flr}$ and $1/A_j$, $j = 1, \ldots, m-1$, are zero.
- for $(IRGP_2)$ and $(JRGP_2)$, we set $\pi_1^k = 0.0001$, $\pi_2^k = 1.2$, $k = 1, 2$;
- for $(IRGP_{3N})$ and $(JRGP_{3N})$, we set $\kappa_0 = \kappa = \kappa_1 = \kappa_2 = 0.02$.

- Individual vs Joint geometric chance constraints.
- Test 9 groups of instances with ϵ_{wall} and ϵ_{flr} such that $(1-\epsilon_{wall})(1-\epsilon_{flr}) = 1-\epsilon$.

Table: Optimal values of ($IRGP$) and ($JRGP$)

ϵ	ϵ_{wall}	ϵ_{flr}	($IRGP_1$)	($JRGP_1$)	($IRGP_2$)	($JRGP_2$)	($ISGP$)	($JSGP$)	($IRGP_{3N}$)	($JRGP_{3N}$)
0.05	0.045	0.0052	289.98	277.05	313.50	299.35	138.24	135.61	162.96	159.35
0.05	0.040	0.0104	305.27	277.05	330.27	299.35	141.10	135.61	167.03	159.35
0.05	0.035	0.0155	323.66	277.05	350.44	299.35	144.29	135.61	171.66	159.35
0.05	0.030	0.0206	346.47	277.05	375.44	299.35	147.87	135.61	177.08	159.35
0.05	0.025	0.0256	375.74	277.05	407.52	299.35	151.97	135.61	183.61	159.35
0.05	0.020	0.0306	415.34	277.05	450.97	299.35	156.85	135.61	191.87	159.35
0.05	0.015	0.0355	473.29	277.05	514.50	299.35	162.89	135.61	203.06	159.35
0.05	0.010	0.0404	570.38	277.05	620.92	299.35	171.02	135.61	220.46	159.35
0.05	0.005	0.0452	789.58	277.05	861.26	299.35	184.01	135.61	257.39	159.35

- Comparisons between DRO and SP for geometric optimization
- We generate 50 groups of normal distributions with different mean values and variances $1/A_{flr}$ and $1/A_j$, $j = 1, \ldots, m-1$.
- Compute the satisfaction probabilities by optimal solutions of $(ISGP)$, $(IRGP_1)$, $(IRGP_2)$ and $(IRGP_{3N})$

Table 3: Values of $\mathbb{P}_{F_{wall}}$ and $\mathbb{P}_{F_{flr}}$ with $\epsilon_{wall} = 0.02$ and $\epsilon_{flr} = 0.0306$

Real distribution			$(ISGP)$		$(IRGP_1)$		$(IRGP_2)$		$(IRGP_{3N})$	
$1/A_{wall}$	$1/A_1$	$\cdots$	$\mathbb{P}_{F_{wall}}$	$\mathbb{P}_{F_{flr}}$	$\mathbb{P}_{F_{wall}}$	$\mathbb{P}_{F_{flr}}$	$\mathbb{P}_{F_{wall}}$	$\mathbb{P}_{F_{flr}}$	$\mathbb{P}_{F_{wall}}$	$\mathbb{P}_{F_{flr}}$
N(0.0297 , 0.0337)	N(0.0356 , 0.0232)	$\cdots$	0.9971	1.0000	0.9991	1.0000	0.5176	1.0000	0.6601	1.0000
N(0.0171 , 0.0212)	N(0.0208 , 0.0261)	$\cdots$	0.9976	1.0000	0.9993	1.0000	0.5486	1.0000	0.6879	1.0000
N(0.0228 , 0.0355)	N(0.0264 , 0.0105)	$\cdots$	0.9981	1.0000	0.9995	1.0000	0.5801	1.0000	0.7151	1.0000
N(0.0417 , 0.0117)	N(0.0329 , 0.0259)	$\cdots$	0.9985	1.0000	0.9996	1.0000	0.6103	1.0000	0.7406	1.0000
N(0.0362 , 0.0235)	N(0.0355 , 0.0267)	$\cdots$	0.9987	1.0000	0.9997	1.0000	0.5886	1.0000	0.7268	1.0000
N(0.0307 , 0.0357)	N(0.0155 , 0.0177)	$\cdots$	0.9988	1.0000	0.9997	1.0000	0.6015	1.0000	0.7372	1.0000
N(0.0144 , 0.0157)	N(0.0357 , 0.0214)	$\cdots$	0.9990	1.0000	0.9997	1.0000	0.6199	1.0000	0.7530	1.0000
N(0.0106 , 0.0371)	N(0.0124 , 0.0319)	$\cdots$	0.9992	1.0000	0.9998	1.0000	0.6294	1.0000	0.7629	1.0000

Individual DRG Chance Constraints

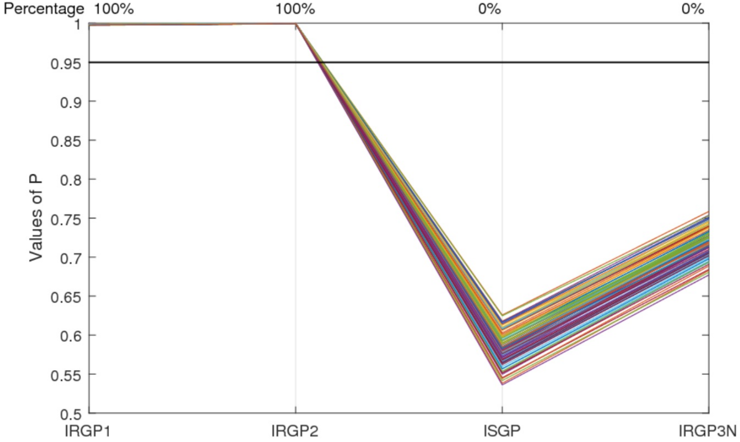

Figure 1 shows the values of $\mathbb{P} = \mathbb{P}_{F_{wall}} \times \mathbb{P}_{F_{flr}}$, the product of the satisfaction probabilities of individual chance constraints.

Joint DRG Chance Constraints

Table: Values of joint satisfaction probabilities $\mathbb{P}$ with $\epsilon = 0.05$

Real distribution			Values of $\mathbb{P}$			
$1/A_{wall}$	$1/A_1$	$\cdots$	$(JSGP)$	$(JRGP_1)$	$(JRGP_2)$	$(JRGP_{3N})$
N(0.0199 , 0.0385)	N(0.0317 , 0.0338)	$\cdots$	0.9047	0.9383	0.4332	0.5321
N(0.0220 , 0.0457)	N(0.0182 , 0.0353)	$\cdots$	0.9177	0.9475	0.4670	0.5655
N(0.0279 , 0.0521)	N(0.0157 , 0.0203)	$\cdots$	0.9208	0.9499	0.4688	0.5684
N(0.0112 , 0.0502)	N(0.0404 , 0.0229)	$\cdots$	0.9281	0.9549	0.4911	0.5900
N(0.0279 , 0.0344)	N(0.0413 , 0.0441)	$\cdots$	0.9318	0.9579	0.4914	0.5920
N(0.0396 , 0.0173)	N(0.0295 , 0.0528)	$\cdots$	0.9373	0.9622	0.4910	0.5945
N(0.0158 , 0.0335)	N(0.0202 , 0.0478)	$\cdots$	0.9483	0.9691	0.5432	0.6422
N(0.0162 , 0.0229)	N(0.0318 , 0.0136)	$\cdots$	0.9508	0.9709	0.5469	0.6466

Joint DRG Chance Constraints

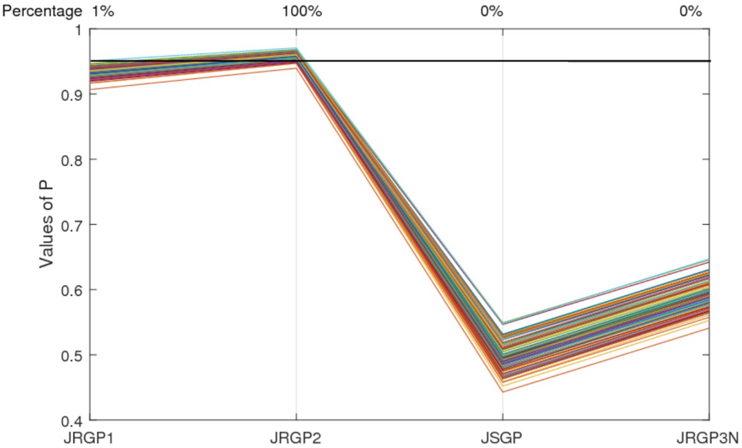

Figure 2 shows the values of salification probability of the joint constraint $\mathbb{P}$

We consider variations of the distributionally robust individual/joint chance constrained shape optimization problem,

$$(RSOP_I^a) \quad \min_{x_1,\dots,x_m} \quad \prod_{i=1}^{m} x_i^{-1}$$

$$s.t. \quad \mathbb{P}\left[\sum_{j=1}^{m-1}\left(\frac{m-1}{A_j} x_1^{a_{wall}} \prod_{i=2, i\neq j}^{m} x_i^{a_{wall}}\right) \leq \beta_{wall}\right] \geq 1-\epsilon_{wall}$$

$$\mathbb{P}\left[\frac{1}{A_{flr}} \prod_{j=2}^{m} x_j^{a_{flr}} \leq \beta_{flr}\right] \geq 1-\epsilon_{flr}$$

$$x_i x_j^{-1} \leq \gamma_{i,j},\ \forall i \neq j,$$

$$(RSOP_J^a) \quad \min_{x_1,\dots,x_m} \quad \prod_{i=1}^{m} x_i^{-1}$$

$$s.t. \quad \mathbb{P}\left[\sum_{j=1}^{m-1}\left(\frac{m-1}{A_j} x_1^{a_{wall}} \prod_{i=2, i\neq j}^{m} x_i^{a_{wall}}\right) \leq \beta_{wall},\ \frac{1}{A_{flr}} \prod_{j=2}^{m} x_j^{a_{flr}} \leq \beta_{flr}\right] \geq 1-\epsilon$$

$$x_i x_j^{-1} \leq \gamma_{i,j},\ \forall i \neq j.$$

Table: Optimal values of $(IRGP_{a1})$ and $(JRGP_{a1})$

ϵ	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
ϵ_{wall}	0.045	0.04	0.035	0.03	0.025	0.02	0.015	0.01
ϵ_{flr}	0.0052	0.0104	0.0155	0.0206	0.0256	0.0306	0.0355	0.0404
$(IRGP_{a1})$	20513.7	25315.0	31586.8	39822.6	50692.7	65106.5	84670.4	120878.0
$(JRGP_{a1})$	16822.6	16822.6	16822.6	16822.6	16822.6	16822.6	16822.6	16822.6

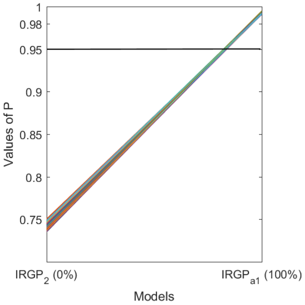

Figure: Values of $\mathbb{P}_{F_{wall}} \times \mathbb{P}_{F_{flr}}$ for $IRGP_2$ and $IRGP_{a1}$

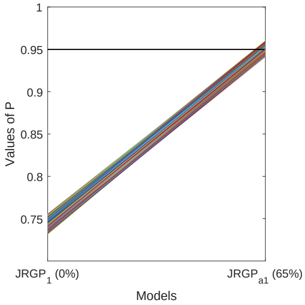

Figure: Values of $\mathbb{P}_{joint}$ for $JRGP_2$ and $JRGP_{a1}$

Conclusions

- Propose tractable reformulations for distributionally robust chance constrained geometric optimization problems with 3 different ambiguity sets.
- Show numerical feasibility on a stochastic optimization shape problem.
- Our results might be more conservative generally speaking due to some strong assumptions.
- Further research should be the extension of our results to more general (standard) geometric optimization under uncertainty.

Thank you!