Distributionally robust chance constrained geometric optimization

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Outline of the talk

1. **Introduction**
2. Individual robust geometric chance constraints
3. Joint robust geometric chance constraints
4. Joint uncertainty of a_{ij}^k and c_i^k
5. Numerical results

Distributionally robust geometric optimization

A geometric program can be formulated as:

$$\min_t g_0(t) \text{ s.t. } g_k(t) \leq 1,\ k = 1, \ldots, K,\ t \in \mathbb{R}^M_{++}$$

with

$$g_k(t) = \sum_{i=1}^{I_k} c_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k},\ k = 0, \ldots, K.$$

- $c_i^k \prod_{j=1}^M t_j^{a_{ij}^k}$ is called a monomial, and $g_k(t)$ is called posynomial.
- The posynomials might have different parameters $c_i^k \geq 0$ and a_{ij}^k.
- Geometric programs are not convex with respect to t
- They are convex with respect to $\{r : r_j = \log\ t_j,\ j = 1, \ldots, M\}$.

Distributionally robust geometric optimization

Applications (cf. S. Boyd, 2007, R. Wiebking, 1977, M. Luptáčik, 1981, S. Kim et al., 2007)

- Wireless communications
- Semiconductor device engineering
- Floor planning
- Digital circuit gate sizing
- Economic and managerial problems
- Wire sizing
- ...

Distributionally robust geometric optimization

Individual chance constraints are studied in S. Rao (1996) and J. Dupačová (2009):

$$
\begin{aligned}
(ISGP) \quad & \min_{t\in\mathbb{R}^M_{++}} \quad \mathbb{E}_{F_0}\left[\sum_{i=1}^{I_0} c_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0}\right] \\
& \text{s.t.} \quad \mathbb{P}_{F_k}\Big(\sum_{i=1}^{I_k} c_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} \leq 1\Big) \geq 1-\epsilon_k, \;\; k = 1, \ldots K
\end{aligned}
$$

- c_i^k is random (normally distributed), a_{ij}^k is constant.
- Reformulated as geometric programs.

Dupacova J (2009) Stochastic geometric programming: approaches and applications. In Brozova V,Kvasnicka R, eds. Proceedings of MME09, 63-66.

Distributionally robust geometric optimization

Recently, L. and A. Lisser (2016) studied stochastic geometric problems with joint chance constraints; i.e.;

$$(JSGP) \quad \min_{t \in \mathbb{R}^M_{++}} \quad \mathbb{E}_{F_0}\left[\sum_{i=1}^{I_0} c_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0}\right]$$

$$\text{s.t.} \quad \mathbb{P}_F\left(\sum_{i=1}^{I_k} c_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} \leq 1,\ k = 1, \ldots K\right) \geq 1 - \epsilon,$$

- c_i^k is random (normally distributed), a_{ij}^k is constant.
- Reformulated as convex programming problems when $\epsilon \leq 1 - \Phi(1)$.

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4. Joint uncertainty of a_{ij}^k and c_i^k
5. Numerical results

We consider the distributionally robust geometric programs with individual chance constraints and the ambiguity of F_0, F_k or F:

$$(IRGP) \quad \min_{t\in\mathbb{R}^M_{++}} \quad \sup_{F\in\mathscr{F}_0} \mathbb{E}_{F_0}\left[\sum_{i=1}^{I_0} c_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0}\right]$$

$$\text{s.t.} \quad \inf_{F_k\in\mathscr{F}_k} \mathbb{P}_{F_k}\left(\sum_{i=1}^{I_k} c_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} \le 1\right) \ge 1-\epsilon_k,\ \ k=1,...K.$$

where $\mathscr{F}_0$, $\mathscr{F}_k$, $k=1,\ldots,K$ are the uncertainty sets, which contain all the distributions of F_0, F_k, $k=1,\ldots,K$.

- c_i^k is random, a_{ij}^k is constant.

Uncertainty sets with known moments

- We consider the uncertainty sets $\mathscr{F}_k$, $k = 0, ..., K$, with known two first order moments information (L. El Ghaoui et al, 2003, L. Chen et al, 2011).

Assumption 1

The uncertainty sets are

$$\mathscr{F}_k = \{F_k \mid \mathbb{E}_{F_k}[c^k] = \mu^k, Cov_{F_k}[c^k] = \Gamma^k\}, \ k = 0, ..., K.$$

where $Cov_F[c^k] = \mathbb{E}_{F_k}\left[\left(c^k - \mathbb{E}_{F_k}[c^k]\right)\left(c^k - \mathbb{E}_{F_k}[c^k]\right)^\top\right]$,

- $\mu^k = [\mu_1^k, \mu_2^k, \ldots, \mu_{I_k}^k]^\top$ *and* $\Gamma^k = \{\sigma_{i,j}^k\}$.
- μ_i^k *is the reference value of the expected value of* c_i^k.
- $\sigma_{i,j}^k$ *is the reference value of the covariance between* c_i^k *and* c_j^k,
- *We assume that* $\mu^k \geq 0$, Γ^k *is positive definite matrix, and* $\sigma_{i,j}^k \geq 0$.

Distributionally robust geometric optimization

Theorem 1

Suppose that Assumption 1 holds. Then (IRGP) is equivalent to

$$(IRGP_1) \quad \min_{t \in \mathbb{R}^M_{++}} \quad \sum_{i=1}^{I_0} \mu_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0}$$

$$s.t. \quad \sum_{i=1}^{I_k} \mu_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} + \sqrt{\frac{1-\epsilon_k}{\epsilon_k}} \sqrt{\sum_{i=1}^{I_k} \sum_{p=1}^{I_k} \sigma_{i,p}^k \prod_{j=1}^{M} t_j^{a_{ij}^k + a_{pj}^k}} \leq 1,$$

$$k = 1, \ldots K.$$

Sketch of the proof:

- Lemma 2.1 (L. Chen et al., 2011)
- Chebytchev inequality (N. Rujeerapaiboon et al., 2016)

Distributionally robust geometric optimization

Problem $(IRGP_1)$ is not convex w.r.t t, it can be transformed into a convex problem using $r_j = \log(t_j),\ j = 1, \ldots, M$

$$(IRGP_{1s}) \quad \min_{r \in \mathbb{R}^M} \quad \sum_{i=1}^{I_0} \mu_i^0 \exp\left\{\sum_{j=1}^{M} a_{ij}^0 r_j\right\}$$

$$s.t. \quad \sqrt{\frac{1-\epsilon_k}{\epsilon_k}} \sqrt{\sum_{i=1}^{I_k} \sum_{p=1}^{I_k} \sigma_{i,p}^k \exp\left\{\sum_{j=1}^{M} (a_{ij}^k + a_{pj}^k) r_j\right\}}$$

$$+ \sum_{i=1}^{I_k} \mu_i^k \exp\left\{\sum_{j=1}^{M} a_{ij}^k r_j\right\} \le 1,\ k = 1, \ldots, K,$$

We consider the uncertainty sets $\mathscr{F}_k$, $k = 0, ..., K$, for $(IRGP)$. (E. Delage and Y. Ye, 2010; J. Cheng et al. 2003; N. Rujeerapaiboon et al., 2015).

Assumption 2

- *The uncertainty sets are*

$$\mathscr{F}_k = \left\{ F_k \middle| \begin{array}{l} \left(\mathbb{E}_{F_k}[c^k] - \mu^k\right)^\top (\Gamma^k)^{-1} \left(\mathbb{E}_{F_k}[c^k] - \mu^k\right) \le \pi_1^k, \\ Cov_{F_k}[c^k] \preceq_D \pi_2^k \Gamma^k. \end{array} \right\},$$

- *We assume that $\mu^k \ge 0$, Γ^k is positive definite matrix, and $\sigma_{i,j}^k \ge 0$, $k = 0, 1, ..., K$.*
- $\pi_1^k, \pi_2^k \in \mathbb{R}$ *are two scale parameters controlling the size of the uncertainty sets.*
- *$A \preceq_D B$ means that for any $x \in \mathbb{R}^n$, we have $x^\top Ax \le x^\top Bx$.*

Theorem 2

Given Assumption 2, (IRGP) is equivalent to

$$(IRGP_2) \quad \min_{t\in\mathbb{R}^M_{++}} \quad \sum_{i=1}^{I_0} \mu_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0} + \sqrt{\pi_1^0}\sqrt{\sum_{i=1}^{I_0}\sum_{p=1}^{I_0} \sigma_{i,p}^0 \prod_{j=1}^{M} t_j^{a_{ij}^0+a_{pj}^0}}$$

$$s.t. \quad \sum_{i=1}^{I_k} \mu_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} + \left(\sqrt{\frac{1-\epsilon_k}{\epsilon_k}}\sqrt{\pi_2^k} + \sqrt{\pi_1^k}\right)\sqrt{\sum_{i=1}^{I_k}\sum_{p=1}^{I_k} \sigma_{i,p}^k \prod_{j=1}^{M} t_j^{a_{ij}^k+a_{pj}^k}} \le 1, \; k = 1, \dots K.$$

Sketch of the proof:

- Same ingredients as the previous Theorem extended to this case.

With the standard variable transformation $r_j = \log(t_j),\ j = 1, \ldots, M$, we can transform $(IRGP_2)$ into

$$\begin{aligned}\min_{r\in\mathbb{R}^M} \quad & \sum_{i=1}^{I_0} \mu_i^0 \exp\left\{\sum_{j=1}^{M} a_{ij}^0 r_j\right\} + \sqrt{\pi_1^0}\sqrt{\sum_{i=1}^{I_0}\sum_{p=1}^{I_0} \sigma_{i,p}^0 \exp\left\{\sum_{j=1}^{M}(a_{ij}^0 + a_{pj}^0) r_j\right\}} \\ s.t. \quad & \left(\sqrt{\frac{1-\epsilon_k}{\epsilon_k}}\sqrt{\pi_2^k} + \sqrt{\pi_1^k}\right)\sqrt{\sum_{i=1}^{I_k}\sum_{p=1}^{I_k} \sigma_{i,p}^k \exp\left\{\sum_{j=1}^{M}(a_{ij}^k + a_{pj}^k) r_j\right\}} \\ & + \sum_{i=1}^{I_k} \mu_i^k \exp\left\{\sum_{j=1}^{M} a_{ij}^k r_j\right\} \le 1,\ k = 1, \ldots, K.\end{aligned}$$

which is a convex optimization problem.

- Avoid SDP reformulation (Delage and Ye 2010; Cheng, Delage and Lisser 2014)

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Uncertainty sets controlled by the distance between the true distribution and the reference distribution of c^k (A. Ben-Tal, 2013; Z. Hu and J. Hong, 2013; R. Jiang and Y. Guan, 2016).

Assumption 3

The uncertainty sets are

$$\mathscr{F}_k = \{F_k \mid D_{DL}(F_k||F_k^0) \leq \kappa_k\},\ \ k = 0, ..., K.$$

where D_{DL} is the Kullback-Leibler divergence distance

$$D_{DL}(F_k||F_k^0) = \int_\Omega \phi\left(\frac{f_{F_k}(c^k)}{f_{F_k^0}(c^k)}\right) f_{F_k^0}(c^k)dc^k,$$

- *F_k^0 is the reference distribution of c^k, $f_{F_k}(c^k)$ and $f_{F_k^0}(c^k)$ are the density functions of the true distribution and the reference distribution of c^k on Ω,*
- *κ_k is a parameter controlling the size of the uncertainty set, $k = 0, ..., K$. $\phi(t) = t\log t - t + 1$, for $t \geq 0$, and $\phi(t) = \infty$, ow.*

We use Theorem 1 in Z. Hu and J. Hong (2013) for the following proposition:

Proposition 3

Given Assumption 3, the objective function is equivalent to

$$\inf_{\alpha \in (0,\infty)} \alpha \log \mathbb{E}_{F_0^0} \left[\exp \left\{ \left(\sum_{i=1}^{I_0} c_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0} \right) / \alpha \right\} \right] + \alpha \kappa_0.$$

and Theorem 1 and Proposition 4 in R. Jiang and Y. Guan (2016),

Proposition 4

Given Assumption 3, the constraint is equivalent to

$$\mathbb{P}_{F_k^0}\left(\sum_{i=1}^{I_k} c_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} \leq 1\right) \geq 1 - \epsilon_k', \; k = 1, \ldots K,$$

where $\epsilon_k' = 1 - \inf_{x \in (0,1)} \left\{ \frac{e^{-\kappa_k} x^{1-\epsilon_k} - 1}{x - 1} \right\}$, $k = 1, \ldots K.$

Assume that the reference distribution F_k^0 follows a normal distribution:

- mean vector $\mu^k = [\mu_1^k, \mu_2^k, \ldots, \mu_{I_k}^k]^\top \geq 0$
- positive definite covariance matrix $\Gamma^k = \{\sigma_{i,j}^k,\ i,j = 1, \ldots, I_k\}, \forall k.$

Theorem 5

Given Assumption 3 and normal distribution assumption for F_k^0, $k = 0, 1, \ldots, K$, (IRGP) is equivalent to

$$(IRGP_{3N}) \quad \min_{t \in \mathbb{R}_{++}^M} \sum_{i=1}^{I_0} \mu_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0} + \sqrt{2\kappa_0 \sum_{i=1}^{I_0} \sum_{p=1}^{I_0} \sigma_{i,p}^0 \prod_{j=1}^{M} t_j^{a_{ij}^0 + a_{pj}^0}}$$

$$s.t. \quad \sum_{i=1}^{I_k} \mu_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} + \Phi^{-1}(1 - \epsilon_k') \sqrt{\sum_{i=1}^{I_k} \sum_{p=1}^{I_k} \sigma_{i,p}^k \prod_{j=1}^{M} t_j^{a_{ij}^k + a_{pj}^k}} \leq 1, \ \ k = 1, \ldots K.$$

Here, $\Phi^{-1}(\cdot)$ is the quantile of the standard normal distribution $N(0,1)$.

Proof: Based on previous theorems

Historical data based discrete reference distribution F_k^0.

Theorem 6

Given Assumption 3, we further assume F_k^0 follows a discrete distribution with H possible scenarios $\tilde{c}^k(1)$, $\tilde{c}^k(2)$, $\tilde{c}^k(H)$, associated with their probabilities $\frac{1}{H}$, $k = 0, 1, \ldots, K$. Then, problem (IRGP) is equivalent to ($IRGP_{3D}$)

$$\min_{r\in\mathbb{R}^M,\alpha\in(0,\infty),\varsigma} \quad \alpha \log\left(\frac{1}{H}\sum_{h=1}^{H}\exp\left\{\left(\sum_{i=1}^{I_0}\tilde{c}_i^0(h)\exp\left\{\sum_{j=1}^{M}a_{ij}^0 r_j\right\}\right)/\alpha\right\}\right)+\alpha\kappa_0,$$

$$\text{s.t.} \quad \frac{1}{H}\sum_{h=1}^{H}(1-\varsigma_h^k)\geq 1-\epsilon_k',\ k=1,2,...,K,$$

$$\sum_{i=1}^{I_k}\tilde{c}_i^k(h)\exp\left\{\sum_{j=1}^{M}a_{ij}^k r_j\right\}\leq M\varsigma_h^k+1,\ h=1,\ldots,H,\ k=1,2,...,K,$$

$$\varsigma_h^k\in\{0,1\},\ h=1,\ldots,H,\ k=1,2,...,K.$$

Proof: Propositions 4 and 5 and previous theorems.

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- We consider the uncertainty sets $\mathscr{F}_k$, $k = 0, ..., K$, with known first order moments information on the nonnegative support.

Assumption 4

The uncertainty sets are

$$\mathscr{F}_k = \{F_k \mid \mathbb{E}_{F_k}[c^k] = \mu^k, \mathbb{P}[c^k \geq 0] = 1\},\ k = 0, 1, ..., K,$$

- $\mu^k = [\mu_1^k, \mu_2^k, \ldots, \mu_{I_k}^k]^\top$ *and* $\Gamma^k = \{\sigma_{i,j}^k\}$.
- μ_i^k *is the reference value of the expected value of* c_i^k
- *Support information: nonnegative cone! K. Natarajan, M. Sim (2008,2010)*
- *W. Wiesemann, D. Kuhn, M. Sim (2014); G. Hanasusanto, V. Roitch, D. Kuhn, W. Wiesemann (2017), support on convex, closed, and solid cone*

Distributionally robust geometric optimization

Theorem 7

Suppose that Assumption 4 holds. Then (IRGP) is equivalent to

$$
\begin{aligned}
(IRGP_4) \quad \min_{t\in\mathbb{R}^M_{++},\lambda,\alpha,\beta} \quad & \sum_{i=1}^{I_0} \mu_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0} \\
s.t. \quad & (1-\epsilon_k)\lambda_k^{-1} - \lambda_k^{-1}\beta_k^\top \mu^k \le 1,\ k=1,...,K, \\
& \beta_k < 0, 0 < \lambda_k \le 1,\ k=1,...,K, \\
& \lambda_k^{-1}\alpha_k \ge 1,\ k=1,...,K, \\
& (-\beta_i^k)^{-1}\alpha_k \prod_{j=1}^{M} t_j^{a_{ij}^k} \le 1,\ i=1,\ldots,I_k,\ k=1,...,K.
\end{aligned}
$$

Sketch of the proof:

- Duality; Classification;

Distributionally robust geometric optimization

Problem $(IRGP_4)$ is not convex w.r.t t, it can be transformed into a convex problem using new variables

$$
\begin{aligned}
(IRGP_{4r}) \quad & \min_{r\in\mathbb{R}^M,\tilde{\lambda},\tilde{\alpha},\tilde{\beta}} \quad \sum_{i=1}^{I_0} \mu_i^0 \exp\left\{\sum_{j=1}^{M} a_{ij}^0 r_j\right\} \\
& s.t. \quad (1-\epsilon_k)e^{-\tilde{\lambda_k}} + \sum_{i=1}^{I_k} \exp\left\{-\tilde{\lambda_k} + \tilde{\beta_i^k} + \log(\mu_i^k)\right\} \leq 1,\ k=1,...,K, \\
& \qquad \tilde{\lambda_k} \leq 0,\ k=1,...,K, \\
& \qquad \tilde{\lambda_k} \leq \tilde{\alpha_k},\ k=1,...,K, \\
& \qquad \tilde{\alpha_k} + \sum_{j=1}^{M} a_{ij}^k r_j - \tilde{\beta_i^k} \leq 0,\ i=1,\dots,I_k,\ k=1,...,K.
\end{aligned}
$$

- convex programming form

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We consider the uncertainty sets $\mathscr{F}_k$, $k = 0, ..., K$, with known first two order moments information and nonnegative support.

Assumption 5

The uncertainty sets are

$$\mathscr{F}_k = \{F_k \mid \mathbb{E}_{F_k}[c^k] = \mu^k,\ Cov_{F_k}[c^k] \preceq_S \Gamma^k,\ \mathbb{P}[c^k \geq 0] = 1\},$$
$$k = 0, 1, ..., K.$$

- $\mu^k > 0$ and Γ^k is a positive definite matrix,
- $A \succeq_S B$ means $A - B \in \mathbb{S}^n_+$, where $\mathbb{S}^n_+$ is the n-dimensional positive semi-definite cone.
- Checking for the existence of multivariate distributions with nonnegative support satisfying a given mean and second-moment matrix is, however, a difficult problem (Bertsimas and Popescu 2005, Kemperman and Skibinsky 1993, Murty and Kabadi 1987).
- Always lead to NP-hard. Research on distributionally robust mixed 0-1 linear programs: K. Natarajan, C. Teo, Z. Zheng (2011)

Given Assumption 5, ($IRGP$) is equivalent to the following optimization problem ($IRGP_5$):

$$\min_{t,\tilde{Y},\tilde{\beta},\tilde{\varepsilon},\tilde{\lambda},w} \quad \sum_{i=1}^{I_0} \mu_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0} \tag{1}$$

$$s.t. \quad \prod_{j=1}^{M} t_j^{a_{ij}^k} \le w_k^i,\ i = 1, \dots, I_k,\ k = 1, ..., K, \tag{2}$$

$$\begin{pmatrix} -\tilde{Y}_k & -\frac{1}{2}\tilde{\beta}_k \\ -\frac{1}{2}\tilde{\beta}_k^\top & \tilde{\varepsilon}_k - \tilde{\lambda}_k \end{pmatrix} \in \mathbb{C}_n,\ k = 1, ..., K, \tag{3}$$

$$\begin{pmatrix} -\tilde{Y}_k & -\frac{1}{2}(\tilde{\beta}_k + w^k),\ k = 1, ..., K, \\ -\frac{1}{2}(\tilde{\beta}_k + w^k)^\top & 1 - \tilde{\lambda}_k \end{pmatrix} \in \mathbb{C}_n,\ k = 1, ..., K, \tag{4}$$

$$\tilde{\beta}_k^\top \mu^k + \tilde{\lambda}_k + < \tilde{Y}_k, \Gamma^k + \mu^k(\mu^k)^\top > \ge (1-\epsilon_k)\tilde{\varepsilon}_k,\ k = 1, ..., K, \tag{5}$$

$$-\tilde{Y}_k \succeq_S 0,\ k = 1, ..., K. \tag{6}$$

$\mathbb{C}_n$ is the co-positive cone, say $\mathbb{C}_n = \{A \in \mathbb{S}_n | x^\top A x \ge 0,\ \forall x \ge 0\}$.

We can further reformulate $(IRGP_{5r})$ in a convex programming form, by bringing new auxiliary variables, $r_j = \log(t_j), j = 1, \ldots, M$.

$$
\begin{aligned}
(IRGP_{5r}) \quad & \min_{r,\tilde{Y},\tilde{\beta},\tilde{\varepsilon},\tilde{\lambda},w} \quad \sum_{i=1}^{I_0} \mu_i^0 \exp\left\{ \sum_{j=1}^{M} a_{ij}^0 r_j \right\} \\
& s.t. \quad \exp\left\{ \sum_{j=1}^{M} a_{ij}^k r_j \right\} \le w_k^i, \; i = 1, \ldots, I_k, \; k = 1, ..., K, \\
& \qquad (3)\text{-}(6).
\end{aligned}
$$

- Although $(IRGP_{5r})$ is a convex optimization problem,
- $(IRGP_{5r})$ is a NP-hard problem due to the co-positive cone (Hiriart-Urruty and Seeger, 2010).
- $\mathbb{C}_n$ approximated by doubly nonnegative cone $\mathbb{S}_+^n \bigcap \mathbb{N}_+$.

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Distributionally robust geometric optimization

We consider the distributionally robust geometric programs with joint chance constraints and the ambiguity of F_0 or F:

$$
\begin{aligned}
(JRGP) \quad & \min_{t \in \mathbb{R}^M_{++}} \quad \sup_{F_0 \in \mathscr{F}_0} \mathbb{E}_{F_0} \left[\sum_{i=1}^{I_0} c_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0} \right] \\
& \text{s.t.} \quad \inf_{F \in \mathscr{F}} \mathbb{P}_F \left(\sum_{i=1}^{I_k} c_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} \leq 1, \ k = 1, \ldots K \right) \geq 1 - \epsilon,
\end{aligned}
$$

where $\mathscr{F}_0$ and $\mathscr{F}$ are the uncertainty sets, which contain all the distributions of F_0 and F.

- c_i^k is random, a_{ij}^k is constant.

- We consider an uncertainty sets $\mathscr{F}$ for $(JRGP)$, with known two first order moments, and with pairwise independent marginal distributions.

Assumption 6

- *The uncertainty set* $\mathscr{F} = \mathscr{F}_1 \times \cdots \times \mathscr{F}_K$
- *For any joint distribution F in $\mathscr{F}$, its marginal distributions $F_1, \ldots, F_K$ are pairwise independent.*

$$\mathscr{F}_k = \{F_k \mid E_{F_k}[c^k] = \mu^k, Cov_{F_k}[c^k] = \Gamma^k\},\ \ k = 0, 1, \ldots, K.$$

- *We assume that $\mu^k \geq 0$, Γ^k is positive definite matrix, and $\sigma^k_{i,j} \geq 0$, $k = 0, 1, ..., K$.*

Theorem 8

Given Assumption 6, (JRGP) is equivalent to

$$(JRGP_1)$$

$$\min_{t\in\mathbb{R}^M_{++},\, y\in\mathbb{R}^K_{++}} \quad \sum_{i=1}^{I_0} \mu_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0}$$

$$s.t. \quad \sum_{i=1}^{I_k} \mu_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} + \sqrt{\frac{y_k}{1-y_k}} \sqrt{\sum_{i=1}^{I_k}\sum_{p=1}^{I_k} \sigma_{i,p}^k \prod_{j=1}^{M} t_j^{a_{ij}^k + a_{pj}^k}} \le 1,\ \ k = 1, \ldots K.$$

$$\prod_k y_k \ge 1 - \epsilon.$$

Non convex due to the bi-linear term.

- We transform $(JRGP_1)$ using $r_j = \log(t_j),\ j = 1, \ldots, M$ and $x_k = \log(y_k),\ k = 1, \ldots, K$:

$$
(JRGP_{1s}) \quad \min_{r \in \mathbb{R}^M, x \in \mathbb{R}^K} \sum_{i=1}^{I_0} \mu_i^0 \exp\left\{ \sum_{j=1}^{M} a_{ij}^0 r_j \right\}
$$

$$
s.t. \quad \sqrt{\sum_{i=1}^{I_k} \sum_{p=1}^{I_k} \sigma_{i,p}^k \exp\left\{ \sum_{j=1}^{M} \left((a_{ij}^k + a_{pj}^k) r_j \right) + \log\left(\frac{e^{x_k}}{1 - e^{x_k}} \right) \right\}}
$$

$$
+ \sum_{i=1}^{I_k} \mu_i^k \exp\left\{ \sum_{j=1}^{M} a_{ij}^k r_j \right\} \leq 1,\ k = 1, \ldots, K,
$$

$$
\sum_{k=1}^{K} x_k \geq \log(1 - \epsilon),\ x_k \leq 0,\ k = 1, \ldots, K.
$$

- $(JRGP_{1s})$ is a convex programming problem as $\log\left(\frac{e^{x_k}}{1-e^{x_k}}\right)$ is convex. Can be rewritten as $x_k - \log\left(1 - e^{x_k}\right)$ to meet the rules of disciplined convex programming of CVX.

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Assumption 7

The uncertainty set $\mathscr{F} = \mathscr{F}_1 \times \cdots \times \mathscr{F}_K$, and for any distribution F in $\mathscr{F}$, its marginal distributions $F_1, \ldots, F_K$ are pairwise independent. Moreover,

$$\mathscr{F}_k = \left\{ F_k \left| \begin{array}{l} \left(\mathbb{E}_{F_k}[c^k] - \mu^k\right)^\top (\Gamma^k)^{-1} \left(\mathbb{E}_{F_k}[c^k] - \mu^k\right) \le \pi_1^k, \\ Cov_{F_k}[c^k] \preceq_D \pi_2^k \Gamma^k. \end{array} \right. \right\},$$

- *We assume that $\mu^k \geq 0$, Γ^k is positive definite matrix, and $\sigma_{i,j}^k \geq 0$, $k = 0, 1, ..., K$.*

Theorem 9

Given Assumption 7, $(JRGP)$ is equivalent to

$$
\begin{aligned}
(JRGP_2)\quad & \min_{t\in\mathbb{R}^M_{++},\, y\in\mathbb{R}^K_{++}} \sum_{i=1}^{I_0}\mu_i^0\prod_{j=1}^M t_j^{a_{ij}^0} + \sqrt{\pi_1^0}\sqrt{\sum_{i=1}^{I_0}\sum_{p=1}^{I_0}\sigma_{i,p}^0\prod_{j=1}^M t_j^{a_{ij}^0+a_{pj}^0}} \\
& s.t. \quad \sum_{i=1}^{I_k}\mu_i^k\prod_{j=1}^M t_j^{a_{ij}^k} + \sqrt{\pi_1^k}\sqrt{\sum_{i=1}^{I_k}\sum_{p=1}^{I_k}\sigma_{i,p}^k\prod_{j=1}^M t_j^{a_{ij}^k+a_{pj}^k}} \\
& \qquad + \sqrt{\frac{y_k}{1-y_k}}\sqrt{\pi_2^k}\sqrt{\sum_{i=1}^{I_k}\sum_{p=1}^{I_k}\sigma_{i,p}^k\prod_{j=1}^M t_j^{a_{ij}^k+a_{pj}^k}} \le 1,\ \ k=1,\dots K. \\
& \qquad \prod_k y_k \ge 1-\epsilon.
\end{aligned}
$$

$(JRGP_2)$ can be reformulated as a convex programming problem by changing the decision variables.

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We consider uncertainty sets for $(JRGP)$ with a reference distribution.

Assumption 8

The uncertainty sets are

$$\mathscr{F}_0 = \{F_0 \mid D_{DL}(F_0 || F_0^0) \leq \kappa_0\} \text{ and } \mathscr{F} = \{F \mid D_{DL}(F || F^0) \leq \kappa\},$$

where

- D_{DL} *is defined in Assumption 3,*
- F_0^0 *is the reference distribution for* c^0*,*
- F^0 *is the reference joint distribution for* c^1*,* $c^2, \ldots, c^K$*, such that* $F^0 = F_1^0 \times \cdots \times F_K^0$ *and the marginal distributions* $F_1^0, \ldots, F_K^0$ *are pairwise independent.*

Theorem 10

Given Assumption 11, we assume that $F_0^0, F_1^0, \ldots, F_K^0$ *are normal distributions with mean vector* $\mu^k = [\mu_1^k, \mu_2^k, \ldots, \mu_{I_k}^k]^\top$ *and covariance matrix* $\Gamma^k = \{\sigma_{i,j}^k,\ i,j = 1, \ldots, I_k\}$, $k = 0, 1, ..., K$. *Then* $(JRGP)$ *is equivalent to* $(JRGP_{3N})$

$$\begin{aligned}
\min_{t\in\mathbb{R}^M_{++}, y\in\mathbb{R}^K_{++}} \quad & \sum_{i=1}^{I_0} \mu_i^0 \prod_{j=1}^M t_j^{a_{ij}^0} + \sqrt{2\kappa_0 \sum_{i=1}^{I_0} \sum_{p=1}^{I_0} \sigma_{i,p}^0 \prod_{j=1}^M t_j^{a_{ij}^0 + a_{pj}^0}} \\
s.t. \quad & \sum_{i=1}^{I_k} \mu_i^k \prod_{j=1}^M t_j^{a_{ij}^k} + \Phi^{-1}(y_k) \sqrt{\sum_{i=1}^{I_k} \sum_{p=1}^{I_k} \sigma_{i,p}^k \prod_{j=1}^M t_j^{a_{ij}^k + a_{pj}^k}} \leq 1,\ k = 1, \ldots K, \\
& \prod_k y_k \geq 1 - \epsilon'.
\end{aligned}$$

where $\epsilon' = 1 - \inf_{x\in(0,1)} \left\{ \frac{e^{-\kappa} x^{1-\epsilon} - 1}{x-1} \right\}$.

Convex reformulation can be obtained when $\epsilon' \leq 1 - \Phi(1)$ and $\sigma_{i,p}^k \geq 0$.

Theorem 11

Given Assumption 11, we assume that F_k^0 is a discrete distribution with H possible values $\tilde{c}^k(h)$, $h = 1, \ldots, H$, associated with their probabilities $\frac{1}{H}$, $k = 0, 1, \ldots, K$. Then $(JRGP)$ is equivalent to $(JRGP_{3D})$

$$
\begin{aligned}
\min_{r\in\mathbb{R}^M,\alpha\in(0,\infty),\varsigma} \quad & \alpha \log\left(\frac{1}{H}\sum_{h=1}^{H}\exp\left\{\left(\sum_{i=1}^{I_0}\tilde{c}_i^0(h)\exp\left\{\sum_{j=1}^{M}a_{ij}^0 r_j\right\}\right)/\alpha\right\}\right) + \alpha\kappa_0, \\
\text{s.t.} \quad & \frac{1}{H}\sum_{h=1}^{H}(1-\varsigma_h) \geq 1-\epsilon', \\
& \sum_{i=1}^{I_k}\tilde{c}_i^k(h)\exp\left\{\sum_{j=1}^{M}a_{ij}^k r_j\right\} \leq M\varsigma_h + 1,\ h = 1,\ldots,H,\ k = 1,2,\ldots,K, \\
& \varsigma_h \in \{0,1\},\ h = 1,\ldots,H.
\end{aligned}
$$

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5. Numerical results

- We consider an uncertainty sets $\mathscr{F}$ for $(JRGP)$, with known first order moments and nonnegative support, and with pairwise independent marginal distributions.

Assumption 9

- *The uncertainty set* $\mathscr{F} = \mathscr{F}_1 \times \cdots \times \mathscr{F}_K$
- *For any joint distribution* F *in* $\mathscr{F}$, *its marginal distributions* $F_1, \ldots, F_K$ *are pairwise independent.*

$$\mathscr{F}_k = \{F_k \mid \mathbb{E}_{F_k}[c^k] = \mu^k, \mathbb{P}[c^k \geq 0] = 1\}, \; k = 0, 1, ..., K,$$

- *We assume that* $\mu^k \geq 0$, Γ^k *is positive definite matrix, and* $\sigma^k_{i,j} \geq 0$, $k = 0, 1, ..., K$.

Theorem 12

Given Assumption 9, (JRGP) is equivalent to the following geometric program ($JRGP_4$)*:*

$$
\begin{aligned}
\min_{t\in\mathbb{R}^M_{++}, y\in\mathbb{R}^K_{+}, \alpha,\beta,\lambda} \quad & \sum_{i=1}^{I_0} \mu_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0} \\
s.t. \quad & \prod_{k=1}^{K} y_k \geq 1-\epsilon,\ 0 \leq y_k \leq 1,\ k=1,\dots,K. \\
& y_k \lambda_k^{-1} - \lambda_k^{-1} \beta_k^\top \mu^k \leq 1,\ k=1,...,K, \\
& \beta_k < 0, 0 < \lambda_k \leq 1,\ k=1,...,K, \\
& \lambda_k^{-1} \alpha_k \geq 1,\ k=1,...,K, \\
& (-\beta_i^k)^{-1} \alpha_k \prod_{j=1}^{M} t_j^{a_{ij}^k} \leq 1,\ i=1,\dots,I_k,\ k=1,...,K.
\end{aligned}
$$

By bringing some auxiliary variables, we can transform the $(JRGP_4)$ into the following convex programming problem.

$$(JRGP_{4r}) \quad \min_{r,x,\tilde{\lambda},\tilde{\alpha},\tilde{\beta}} \sum_{i=1}^{I_0} \mu_i^0 \exp\left\{\sum_{j=1}^{M} a_{ij}^0 r_j\right\}$$

$$s.t. \quad \sum_{k=1}^{K} x_k \geq \log(1-\epsilon),\ x_k \leq 0,\ k=1,\ldots,K,$$

$$\exp\left\{x_k - \tilde{\lambda_k}\right\} + \sum_{i=1}^{I_k} \exp\left\{-\tilde{\lambda_k} + \tilde{\beta_i^k} + \log(\mu_i^k)\right\} \leq 1,\ k=1,...,K,$$

$$\tilde{\lambda_k} \leq 0,\ k=1,...,K,$$

$$\tilde{\lambda_k} \leq \tilde{\alpha_k},\ k=1,...,K,$$

$$\tilde{\alpha_k} + \sum_{j=1}^{M} a_{ij}^k r_j - \tilde{\beta_i^k} \leq 0,\ i=1,\ldots,I_k,\ k=1,...,K.$$

Constraints with exponential terms.

Assumption 10

The uncertainty set $\mathscr{F} = \mathscr{F}_1 \times \cdots \times \mathscr{F}_K$, *and for any distribution* F *in* $\mathscr{F}$, *its marginal distributions* $F_1, \ldots, F_K$ *are pairwise independent. Moreover,*

$$\mathscr{F}_k = \{F_k \mid \mathbb{E}_{F_k}[c^k] = \mu^k,\ Cov_{F_k}[c^k] \preceq_S \Gamma^k,\ \mathbb{P}[c^k \geq 0] = 1\},\ k = 0, 1, \ldots,$$

where $\mu^k > 0$, $k = 0, 1, \ldots, K$.

Theorem 13

Given Assumption 10, (JRGP) is equivalent to the following program ($JRGP_4$):

$$\min_{t,y,\tilde{Y},\tilde{\beta},\tilde{\varepsilon},\tilde{\lambda},w} \quad \sum_{i=1}^{I_0} \mu_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0}$$

$$\text{s.t.} \quad \prod_{j=1}^{M} t_j^{a_{ij}^k} \leq w_k^i, \; i = 1, \ldots, I_k, \; k = 1, ..., K,$$

$$\begin{pmatrix} -\tilde{Y}_k & -\frac{1}{2}\tilde{\beta}_k \\ -\frac{1}{2}\tilde{\beta}_k^\top & \tilde{\varepsilon}_k - \tilde{\lambda}_k \end{pmatrix} \in \mathbb{C}_n, \; k = 1, ..., K,$$

$$\begin{pmatrix} -\tilde{Y}_k & -\frac{1}{2}(\tilde{\beta}_k + w^k), \; k = 1, ..., K, \\ -\frac{1}{2}(\tilde{\beta}_k + w^k)^\top & 1 - \tilde{\lambda}_k \end{pmatrix} \in \mathbb{C}_n, \; k = 1, ..., K,$$

$$-\tilde{Y}_k \succeq_S 0, \; k = 1, ..., K,$$

$$\tilde{\beta}^\top \mu^k + \tilde{\lambda} + < \tilde{Y}, \Gamma^k + \mu^k (\mu^k)^\top > \geq y_k \tilde{\varepsilon}, \; k = 1, \ldots, K,$$

$$\prod_k y_k \geq 1 - \epsilon, \; 0 \leq y_k \leq 1, \; k = 1, \ldots, K.$$

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Joint uncertainty set of a_{ij}^k and c_i^k

Individual chance constraints case

We consider uncertainty sets for $(IRGP)$ with a reference distribution.

Assumption 11

The uncertainty sets are

$$\mathscr{G}_k = \left\{ G_k \middle| \begin{array}{l} G_k(a^k = a^k(m)) = p_m^k,\ m = 1, \ldots, M_k,\ \sum_{m=1}^{M_k} p_m^k = 1, \\ G_k(c|a^k = a^k(m)) = F_k^m(c), \forall c \in supp(F_k),\ F_k^m \in \mathscr{F}_k^m,\ m = 1, \ldots, M_k. \end{array} \right\},$$

$$k = 0, 1, ..., K.$$

- *where $\mathscr{F}_k^m$ could be an uncertainty set defined in Assumptions 1, 2, or 3.*

Theorem 14

Given Assumption 7, (IRGP) is equivalent to the following geometric program:

$$(IRGP_a) \quad \min_{t\in\mathbb{R}^M_{++},z} \quad \sum_{m=1}^{M_0} p_m^0 \left(\sup_{F_0^m\in\mathscr{F}_0^m} \mathbb{E}_{F_0^m} \left[\sum_{m=1}^{I_0} c_m^0 \prod_{j=1}^{M} t_j^{a_{ij}^0(m)} \right] \right) \tag{7}$$

$$\text{s.t.} \quad \sum_{m=1}^{M_k} p_m^k z_m^k \geq 1 - \epsilon_k, \;\; k = 1, \ldots, K, \tag{8}$$

$$\inf_{F_k^m\in\mathscr{F}_k^m} \mathbb{P}_{F_k^m} \left(\sum_{m=1}^{I_k} c_m^k \prod_{j=1}^{M} t_j^{a_{ij}^k(m)} \leq 1 \right) \geq z_m^k,$$

$$m = 1, \ldots, M_k, k = 1, \ldots, K, \tag{9}$$

$$z_m^k \in [0, 1], \;\; m = 1, \ldots, M_k, k = 1, \ldots K. \tag{10}$$

Choosing $\mathscr{F}_k^m$ to be an uncertainty sets with known first two order moments, we can reformulate ($IRGP_a$) as

$$\min_{t\in\mathbb{R}^M_{++},z} \quad \sum_{i=1}^{I_0} \mu_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0(m)} \tag{11}$$

$$\text{s.t.} \quad \sum_{m=1}^{M_k} p_m^k z_m^k \geq 1-\epsilon_k,\ \ k=1,\dots,K, \tag{12}$$

$$\sum_{i=1}^{I_k} \mu_i^k(m) \prod_{j=1}^{M} t_j^{a_{ij}^k(m)} + \sqrt{\frac{z_m^k}{1-z_m^k}} \sqrt{\sum_{i=1}^{I_k}\sum_{p=1}^{I_k} \sigma_{i,p}^k(m) \prod_{j=1}^{M} t_j^{a_{ij}^k(m)+a_{pj}^k(m)}} \leq 1,$$

$$m=1,\dots,M_k, k=1,\dots,K, \tag{13}$$

$$z_m^k \in [0,1],\ \ m=1,\dots,M_k, k=1,\dots K. \tag{14}$$

Proposition 15

If $\epsilon_k \leq \frac{1}{2}\min_m\{p_m^k\}$, and $\sigma_{i,p}^k \geq 0$, for any $i,p=1,\dots,I_k$, $k=1,\dots,K$, ($IRGP_{a1r}$) is a convex programming problem.

Assumption 12

The uncertainty sets for the joint distribution of c^k and a^k are

$$\mathscr{G}_k = \left\{ G_k \middle| \begin{array}{l} G_k(a^k = a^k(m)) = p_m^k,\ m = 1, \ldots, M_k,\ p^k \in \mathscr{P}_k, \\ G_k(c|a^k = a^k(m)) = F_k^m(c), \forall c \in supp(F_k),\ F_k^m \in \mathscr{F}_k^m,\ m = 1, \ldots, \end{array} \right.$$

here, $\mathscr{P}_k$ is the uncertainty set of the distribution of a^k.

- box uncertainty,

$$\mathscr{P}_k = \{p^k | p^k = \tilde{p}^k + \eta_k,\ e^\top \eta_k = 0,\ \underline{\eta}_k \leq \eta_k \leq \bar{\eta}_k\},$$

- ellipsoidal uncertainty,

$$\mathscr{P}_k = \{p^k | p^k = \tilde{p}^k + A_k \eta_k,\ e^\top A_k \eta_k = 0,\ \tilde{p}^k + A_k \eta_k \geq 0,\ \|\eta_k\| \leq 1\}.$$

- Similar convex reformulations.

Joint uncertainty set of a_{ij}^k and c_i^k

Joint chance constraints case

Assumption 13

The uncertainty set $\mathscr{G} = \mathscr{G}_1 \times \cdots \times \mathscr{G}_K$, and for any distribution G in $\mathscr{G}$, its marginal distributions $G_1, \ldots, G_K$ are pairwise independent. Where, $\mathscr{G}_k$ is a marginal uncertainty set defined in Assumption 7.

Theorem 16

Given Assumption 13, $(JRGP)$ is equivalent to the following program:

$$
\begin{aligned}
(JRGP_a) \quad & \min_{t \in \mathbb{R}^M_{++}, y, z} \quad \sum_{m=1}^{M_0} p_m^0 \left(\sup_{F_0^m \in \mathscr{F}_0^m} \mathbb{E}_{F_0^m} \left[\sum_{m=1}^{I_0} c_m^0 \prod_{j=1}^{M} t_j^{a_{ij}^0} \right] \right) \\
& \text{s.t.} \quad \prod_{k=1}^{K} y_k \geq (1-\epsilon), \; 0 \leq y_k \leq 1, \; k = 1, \ldots, K, \\
& \qquad \sum_{m=1}^{M_k} p_m^k z_m^k \geq y_k, \; k = 1, \ldots, K, \\
& \qquad (9) - (10).
\end{aligned}
$$

Similarly, choosing unknown moments uncertainty set $\mathscr{F}_k^m$, We can reformulate $(JRGP_a)$ as

$$
\begin{aligned}
(JRGP_{a1r}) \quad \min_{r,x,z} \quad & \sum_{i=1}^{I_0} \mu_i^0 \exp\left\{ \sum_{j=1}^{M} a_{ij}^0 r_j \right\} \\
\text{s.t.} \quad & \sum_{k=1}^{K} x_k \geq \log(1-\epsilon),\ x_k \leq 0,\ k = 1, \ldots, K, \\
& \sum_{m=1}^{M_k} p_m^k z_m^k \geq e^{x_k},\ k = 1, \ldots, K, \\
& (13) - (14).
\end{aligned}
$$

Similarly to (15), a sufficient condition for the convexity of $(JRGP_{a1r})$ is $\epsilon \leq \frac{1}{2} \min_{k,m}\{p_m^k\}$ and $\sigma_{i,p}^k \geq 0$, for any $i, p = 1, \ldots, I_k$, $k = 1, \ldots, K$.

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5. **Numerical** results

- We consider a distributionally robust shape optimization problem with individual chance constraints

$$
\begin{aligned}
(RSOP_I) \quad & \min_{x_1,\ldots,x_m} \quad \prod_{i=1}^{m} x_i^{-1} \\
& s.t. \quad P\left[\sum_{j=1}^{m-1}\left(\frac{m-1}{A_j} x_1 \prod_{i=2, i\neq j}^{m} x_i\right) \leq \beta_{wall}\right] \geq 1-\epsilon_{wall} \\
& \qquad\quad P\left[\frac{1}{A_{flr}} \prod_{j=2}^{m} x_j \leq \beta_{flr}\right] \geq 1-\epsilon_{flr} \\
& \qquad\quad x_i x_j^{-1} \leq \gamma_{i,j}, \ \forall i \neq j.
\end{aligned}
$$

- and a distributionally robust shape optimization problem with joint chance constraints

$$
\begin{aligned}
(RSOP_J) \quad & \min_{x_1,\ldots,x_m} \quad \prod_{i=1}^{m} x_i^{-1} \\
& s.t. \quad P\left[\sum_{j=1}^{m-1} \left(\frac{m-1}{A_j} x_1 \prod_{i=2, i\neq j}^{m} x_i \right) \leq \beta_{wall}, \ \frac{1}{A_{flr}} \prod_{j=2}^{m} x_j \leq \beta_{flr} \right] \geq 1-\epsilon \\
& \qquad x_i x_j^{-1} \leq \gamma_{i,j}, \ \forall i \neq j,
\end{aligned}
$$

- $1/A_{flr}$ and $1/A_j$, $j = 1, \ldots, m-1$, are considered as random variables.
- We assume $1/A_{fl}$ to be independent to $1/A_j$, $j = 1, \ldots, m-1$.
- F_{wall} and F_j are the distributions of $1/A_{flr}$ and $1/A_j$, $j = 1, \ldots, m-1$.
- F is the joint distribution of $1/A_{wall}$ and $1/A_j$, $j = 1, \ldots, m-1$.
- Mean values and covariances in all uncertainty sets are set the same.

- MOSEK solver from CVX package with Matlab R2012b; PC with a 2.6 Ghz Intel Core i7-5600U CPU and 12.0 GB RAM.
- $m = 100$
- Mean value of $1/A_{flr}$ is 0.02; the variance of $1/A_{flr}$ is 0.02; the mean value of $1/A_j$ is 0.01, $j = 1, \dots, m-1$; the variance of $1/A_j$ is 0.01, $j = 1, \dots, m-1$; all the covariance between $1/A_{flr}$ and $1/A_j$, $j = 1, \dots, m-1$, are zero.
- for $(IRGP_2)$ and $(JRGP_2)$, we set $\pi_1^k = 0.0001$, $\pi_2^k = 1.2$, $k = 1, 2$;
- for $(IRGP_{3N})$ and $(JRGP_{3N})$, we set $\kappa_0 = \kappa = \kappa_1 = \kappa_2 = 0.02$.

- Individual vs Joint geometric chance constraints.
- Test 9 groups of instances with ϵ_{wall} and ϵ_{flr} such that $(1-\epsilon_{wall})(1-\epsilon_{flr}) = 1-\epsilon$.

Table: Optimal values of $(IRGP)$ and $(JRGP)$

ϵ	ϵ_{wall}	ϵ_{flr}	$(IRGP_1)$	$(JRGP_1)$	$(IRGP_2)$	$(JRGP_2)$	$(ISGP)$	$(JSGP)$	$(IRGP_{3N})$	$(JRGP_{3N})$
0.05	0.045	0.0052	289.98	277.05	313.50	299.35	138.24	135.61	162.96	159.35
0.05	0.040	0.0104	305.27	277.05	330.27	299.35	141.10	135.61	167.03	159.35
0.05	0.035	0.0155	323.66	277.05	350.44	299.35	144.29	135.61	171.66	159.35
0.05	0.030	0.0206	346.47	277.05	375.44	299.35	147.87	135.61	177.08	159.35
0.05	0.025	0.0256	375.74	277.05	407.52	299.35	151.97	135.61	183.61	159.35
0.05	0.020	0.0306	415.34	277.05	450.97	299.35	156.85	135.61	191.87	159.35
0.05	0.015	0.0355	473.29	277.05	514.50	299.35	162.89	135.61	203.06	159.35
0.05	0.010	0.0404	570.38	277.05	620.92	299.35	171.02	135.61	220.46	159.35
0.05	0.005	0.0452	789.58	277.05	861.26	299.35	184.01	135.61	257.39	159.35

- Comparisons between DRO and SP for geometric optimization
- We generate 50 groups of normal distributions with different mean values and variances $1/A_{flr}$ and $1/A_j$, $j = 1, \ldots, m-1$.
- Compute the satisfaction probabilities by optimal solutions of $(ISGP)$, $(IRGP_1)$, $(IRGP_2)$ and $(IRGP_{3N})$

Table 3: Values of $\mathbb{P}_{F_{wall}}$ and $\mathbb{P}_{F_{flr}}$ with $\epsilon_{wall} = 0.02$ and $\epsilon_{flr} = 0.0306$

Real distribution			$(ISGP)$		$(IRGP_1)$		$(IRGP_2)$		$(IRGP_{3N})$	
$1/A_{wall}$	$1/A_1$	$\cdots$	$\mathbb{P}_{F_{wall}}$	$\mathbb{P}_{F_{flr}}$	$\mathbb{P}_{F_{wall}}$	$\mathbb{P}_{F_{flr}}$	$\mathbb{P}_{F_{wall}}$	$\mathbb{P}_{F_{flr}}$	$\mathbb{P}_{F_{wall}}$	$\mathbb{P}_{F_{flr}}$
N(0.0297 , 0.0337)	N(0.0356 , 0.0232)	$\cdots$	0.9971	1.0000	0.9991	1.0000	0.5176	1.0000	0.6601	1.0000
N(0.0171 , 0.0212)	N(0.0208 , 0.0261)	$\cdots$	0.9976	1.0000	0.9993	1.0000	0.5486	1.0000	0.6879	1.0000
N(0.0228 , 0.0355)	N(0.0264 , 0.0105)	$\cdots$	0.9981	1.0000	0.9995	1.0000	0.5801	1.0000	0.7151	1.0000
N(0.0417 , 0.0117)	N(0.0329 , 0.0259)	$\cdots$	0.9985	1.0000	0.9996	1.0000	0.6103	1.0000	0.7406	1.0000
N(0.0362 , 0.0235)	N(0.0355 , 0.0267)	$\cdots$	0.9987	1.0000	0.9997	1.0000	0.5886	1.0000	0.7268	1.0000
N(0.0307 , 0.0357)	N(0.0155 , 0.0177)	$\cdots$	0.9988	1.0000	0.9997	1.0000	0.6015	1.0000	0.7372	1.0000
N(0.0144 , 0.0157)	N(0.0357 , 0.0214)	$\cdots$	0.9990	1.0000	0.9997	1.0000	0.6199	1.0000	0.7530	1.0000
N(0.0106 , 0.0371)	N(0.0124 , 0.0319)	$\cdots$	0.9992	1.0000	0.9998	1.0000	0.6294	1.0000	0.7629	1.0000

Individual DRG Chance Constraints

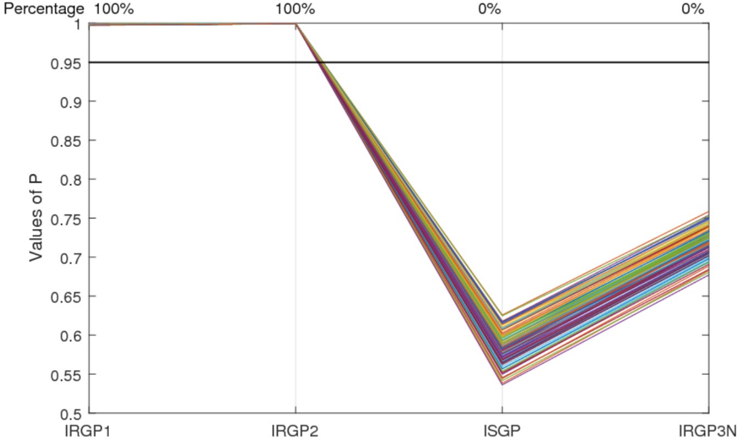

Figure 1 shows the values of $\mathbb{P} = \mathbb{P}_{F_{wall}} \times \mathbb{P}_{F_{flr}}$, the product of the satisfaction probabilities of individual chance constraints.

Joint DRG Chance Constraints

Table: Values of joint satisfaction probabilities $\mathbb{P}$ with $\epsilon = 0.05$

Real distribution			Values of $\mathbb{P}$			
$1/A_{wall}$	$1/A_1$	$\cdots$	$(JSGP)$	$(JRGP_1)$	$(JRGP_2)$	$(JRGP_{3N})$
N(0.0199 , 0.0385)	N(0.0317 , 0.0338)	$\cdots$	0.9047	0.9383	0.4332	0.5321
N(0.0220 , 0.0457)	N(0.0182 , 0.0353)	$\cdots$	0.9177	0.9475	0.4670	0.5655
N(0.0279 , 0.0521)	N(0.0157 , 0.0203)	$\cdots$	0.9208	0.9499	0.4688	0.5684
N(0.0112 , 0.0502)	N(0.0404 , 0.0229)	$\cdots$	0.9281	0.9549	0.4911	0.5900
N(0.0279 , 0.0344)	N(0.0413 , 0.0441)	$\cdots$	0.9318	0.9579	0.4914	0.5920
N(0.0396 , 0.0173)	N(0.0295 , 0.0528)	$\cdots$	0.9373	0.9622	0.4910	0.5945
N(0.0158 , 0.0335)	N(0.0202 , 0.0478)	$\cdots$	0.9483	0.9691	0.5432	0.6422
N(0.0162 , 0.0229)	N(0.0318 , 0.0136)	$\cdots$	0.9508	0.9709	0.5469	0.6466

Joint DRG Chance Constraints

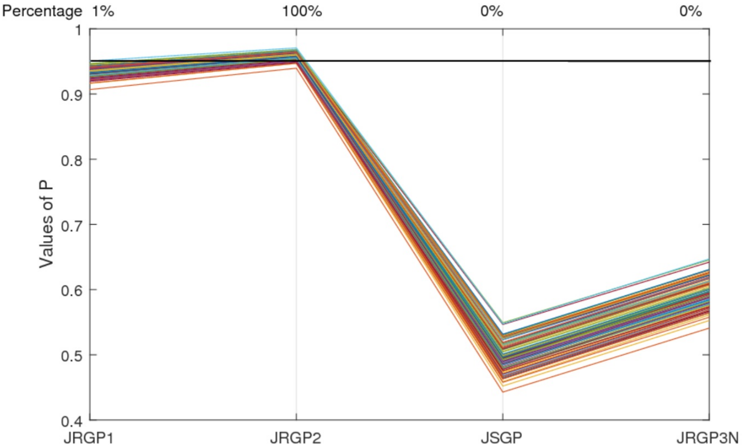

Figure 2 shows the values of salification probability of the joint constraint $\mathbb{P}$

Table: Optimal values of ($IRGP_4$) and ($JRGP_4$) with or without non-negative support constraints

ϵ		0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
ϵ_{wall}		0.045	0.04	0.035	0.03	0.025	0.02	0.015	0.01	0.005
ϵ_{flr}		0.0052	0.0104	0.0155	0.0206	0.0256	0.0306	0.0355	0.0404	0.0452
With NS	($IRGP_4$)	2337.4	2632.7	3012.9	3520.5	4232.4	5302.4	7090.5	10679.4	21508.8
	($JRGP_4$)	2101.8	2101.8	2101.8	2101.8	2101.8	2101.8	2101.8	2101.8	2101.8
Without NS	individual*					+Inf				
	joint*					+Inf				

*: with only first order moment constraint

We consider variations of the distributionally robust individual/joint chance constrained shape optimization problem,

$$
\begin{aligned}
(RSOP_I^a) \quad & \min_{x_1,\ldots,x_m} \prod_{i=1}^m x_i^{-1} \\
& s.t. \quad \mathbb{P}\left[\sum_{j=1}^{m-1}\left(\frac{m-1}{A_j} x_1^{a_{wall}} \prod_{i=2, i\neq j}^m x_i^{a_{wall}}\right) \leq \beta_{wall}\right] \geq 1-\epsilon_{wall} \\
& \qquad \mathbb{P}\left[\frac{1}{A_{flr}} \prod_{j=2}^m x_j^{a_{flr}} \leq \beta_{flr}\right] \geq 1-\epsilon_{flr} \\
& \qquad x_i x_j^{-1} \leq \gamma_{i,j}, \ \forall i \neq j,
\end{aligned}
$$

$$
\begin{aligned}
(RSOP_J^a) \quad & \min_{x_1,\ldots,x_m} \prod_{i=1}^m x_i^{-1} \\
& s.t. \quad \mathbb{P}\left[\sum_{j=1}^{m-1}\left(\frac{m-1}{A_j} x_1^{a_{wall}} \prod_{i=2, i\neq j}^m x_i^{a_{wall}}\right) \leq \beta_{wall}, \ \frac{1}{A_{flr}} \prod_{j=2}^m x_j^{a_{flr}} \leq \beta_{flr}\right] \geq 1-\epsilon \\
& \qquad x_i x_j^{-1} \leq \gamma_{i,j}, \ \forall i \neq j.
\end{aligned}
$$

Table: Optimal values of $(IRGP_{a1})$ and $(JRGP_{a1})$

ϵ	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
ϵ_{wall}	0.045	0.04	0.035	0.03	0.025	0.02	0.015	0.01
ϵ_{flr}	0.0052	0.0104	0.0155	0.0206	0.0256	0.0306	0.0355	0.0404
$(IRGP_{a1})$	20513.7	25315.0	31586.8	39822.6	50692.7	65106.5	84670.4	120878.0
$(JRGP_{a1})$	16822.6	16822.6	16822.6	16822.6	16822.6	16822.6	16822.6	16822.6

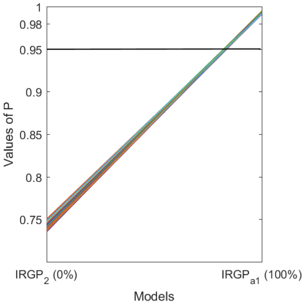

Figure: Values of $\mathbb{P}_{F_{wall}} \times \mathbb{P}_{F_{flr}}$ for $IRGP_2$ and $IRGP_{a1}$

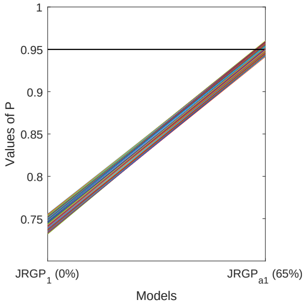

Figure: Values of $\mathbb{P}_{joint}$ for $JRGP_2$ and $JRGP_{a1}$

Conclusions

- Propose tractable reformulations for distributionally robust chance constrained geometric optimization problems with 3 different ambiguity sets.
- Show numerical feasibility on a stochastic optimization shape problem.
- Our results might be more conservative generally speaking due to some strong assumptions.
- Further research should be the extension of our results to more general (standard) geometric optimization under uncertainty.

Thank you!