New formulations and relaxations for mixed-binary quadratic optimization

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joint work with

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Mixed-binary QPs, key condition

Consider (for indefinite Q and $\{\mathsf{c}, \mathsf{a}_i, \mathsf{b}\} \subset \mathbb{R}^n$)

$$\left.\begin{array}{lll} z^* = \min & \mathsf{x}^\top \mathsf{Q}\mathsf{x} + 2\mathsf{c}^\top \mathsf{x} & \\ \text{s.t.} & \mathsf{a}_i^\top \mathsf{x} = b_i & \text{for } i \in [1:m] \\ & \mathsf{x} \in \mathbb{R}^n_+ & \\ & x_j \in \{0,1\} & \text{for } j \in B\,, \end{array}\right\} (P)$$

B ... binary variables

$[1:n] \setminus B$ ... continuous variables

$\mathsf{A} = [\mathsf{a}_1^\top, \ldots, \mathsf{a}_m^\top]^\top$ $m \times n$ constraint matrix.

Polyhedron $Z = \left\{\mathsf{x} \in \mathbb{R}^n_+ : \mathsf{A}\mathsf{x} = \mathsf{b}\right\}$ contains feasible set.

Assume Burer's *key condition:* Z gives linear relaxation,

$$\mathsf{x} \in Z \quad \Rightarrow \quad x_j \in [0,1] \text{ for all } j \in B\,. \qquad (K)$$

Burer's reformulation: enter copositivity

Linearizing by squaring equalities like RLT: put $X_{ij} = x_i x_j$.

If (K) holds, then (P) is equivalent to the *copositive problem*

$$\left.\begin{array}{lll} \min & \langle \mathsf{Q}, \mathsf{X}\rangle + 2\mathsf{c}^\top \mathsf{x} & \\ \text{s.t.} & \mathsf{a}_i^\top \mathsf{x} = b_i & \text{for } i \in [1:m] \\ & \langle \mathsf{a}_i \mathsf{a}_i^\top, \mathsf{X}\rangle = b_i^2 & \text{for } i \in [1:m] \\ & X_{jj} = x_j & \text{for } j \in B \\ & \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{C}P^{n+1}, & \end{array}\right\} (B)$$

where $\langle \mathsf{Q}, \mathsf{X}\rangle = \mathrm{trace}(\mathsf{QX})$ and

$$\mathcal{C}P^n = \mathrm{conv}\left\{\mathsf{x}\mathsf{x}^\top : \mathsf{x} \in \mathbb{R}^n_+\right\}$$

is the *cp cone*. Its dual is the *copositive cone*

$$[\mathcal{C}P^n]^* = \left\{\mathsf{C} = \mathsf{C}^\top\ n \times n\text{-matrix} : \mathsf{x}^\top \mathsf{C}\mathsf{x} \geq 0 \text{ if } \mathsf{x} \in \mathbb{R}^n_+\right\}.$$

Counting variables/constraints

Original formulation: $2m + |B|$ constraints,

symmetric matrix variable of order $n + 1$ (one entry fixed).

Aggregation [Arima/Kim/Kojima '14]:

$$\left.\begin{array}{ll} \min & \langle \mathsf{Q}, \mathsf{X} \rangle + 2\mathsf{c}^\top \mathsf{x} \\ \text{s.t.} & \sum_{i=1}^{m} \left(\mathsf{a}_i^\top \mathsf{X} \mathsf{a}_i - 2b_i\, \mathsf{a}_i^\top \mathsf{x} + b_i^2 \right) = 0\,, \\ & \sum_{j \in B} (X_{jj} - x_j) = 0\,, \\ & \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{CP}^{n+1}\,, \end{array}\right\} (A)$$

gives equivalent reformulation with only two constraints.

Doubly NonNegative relaxation

In both formulations, cone constraint can be relaxed

– *DNN-relaxation*:

$$\text{replace} \quad \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{C}P^{n+1} \quad \text{with} \quad \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{S}_+^{n+1} \cap \mathcal{N}^{n+1}\,.$$

Motivation: any cp matrix is psd and has no negative entries.

$$(B) \to (DB), \quad (A) \to (DA)$$

Leads to SDP with $\binom{n}{2}$ additional nonnegativity constraints, for both formulations: (DB) and (DA).

Exploiting linear constraints

Idea: use equality constraints to reduce order of matrix variable:

choose $\mathsf{x}_0 \in Z$ (only need $\mathsf{A}\mathsf{x}_0 = \mathsf{b}$) and linearly independent

$$\left\{\mathsf{a}_{m+1}, \ldots, \mathsf{a}_n\right\} \subset \{\mathsf{a}_1, \ldots, \mathsf{a}_m\}^{\perp}$$

and form

$$\mathsf{B} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \mathsf{x}_0 & \mathsf{a}_{m+1} & \mathsf{a}_{m+2} & \cdots & \mathsf{a}_n \end{bmatrix}.$$

Have

$$\mathsf{A}\mathsf{z} = \zeta \mathsf{b} \quad \Leftrightarrow \quad \begin{bmatrix} \zeta \\ \mathsf{z} \end{bmatrix} = \mathsf{B}\mathsf{y} \quad \text{for some } \mathsf{y} \in \mathbb{R}^{n+1-m}.$$

Reduced equivalent formulation

[Burer]: (DB) is equivalent to

$$\left.\begin{array}{ll} \min & \langle \mathsf{Q}, \mathsf{X} \rangle + 2\mathsf{c}^\top \mathsf{x} \\ \text{s.t.} & \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} = \mathsf{B}\mathsf{Y}\mathsf{B}^\top, \\ & X_{jj} = x_j, \text{ for } j \in B, \\ & \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{N}^{n+1}, \mathsf{Y} \in \mathcal{S}_+^{n+1-m}. \end{array}\right\} (DR)$$

... has smaller psd Y but additional equalities.

Same holds for original reformulation (R) of (B) where

$\begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{CP}^{n+1}$ replaces weaker $\begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{N}^{n+1}, \mathsf{Y} \in \mathcal{S}_+^{n+1-m}$.

Motivation

We shall prove the equivalence of problems (B), (A) and (R) and similar results for the relaxations, from which, we aim at finding new reformulations.

- Reformulations of linear constraints

- Reformulations of binary constraints

From the combinations of the reformulations of different constraints, we find new reformulations of (B).

Linear constraints: aggregation and facial reduction

We shall consider the following four linear subspaces in $\mathcal{S}^{n+1}$:

$$\mathcal{L}_1 = \left\{ \begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{S}^{n+1} : \exists \mathsf{Y} \in \mathcal{S}^{n+1-m} \text{ s.t. } \mathsf{B}\mathsf{Y}\mathsf{B}^\top = \begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \right\},$$

$$\mathcal{L}_2 = \left\{ \begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{S}^{n+1} : \begin{array}{ll} \mathsf{a}_i^\top \mathsf{x} = b_i x_0 & \text{for } i \in [1:m], \\ \mathsf{a}_i^\top \mathsf{X} = b_i \mathsf{x}^\top & \text{for } i \in [1:m] \end{array} \right\},$$

$$\mathcal{L}_3 = \left\{ \begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{S}^{n+1} : \begin{array}{ll} \mathsf{a}_i^\top \mathsf{x} = b_i x_0 & \text{for } i \in [1:m], \\ \mathsf{a}_i^\top \mathsf{X} \mathsf{a}_i = b_i^2 x_0 & \text{for } i \in [1:m] \end{array} \right\},$$

$$\mathcal{L}_4 = \left\{ \begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{S}^{n+1} : \textstyle\sum_{i=1}^m \left(\mathsf{a}_i^\top \mathsf{X} \mathsf{a}_i - 2 b_i \, \mathsf{a}_i^\top \mathsf{x} + b_i^2 x_0 \right) = 0 \right\}.$$

Equivalence when intersecting with psd cone

Notice that $\mathcal{L}_1 = \mathsf{B}\mathcal{S}^{n+1-m}\mathsf{B}^\top$.

We will now show that in fact when intersecting with the positive semidefinite cone these four cones coincide.

Theorem 1. *We have*

(a) $\mathcal{L}_1 = \mathcal{L}_2 \subseteq \mathcal{L}_3 \subseteq \mathcal{L}_4\,,$

(b) $\mathcal{L}_1 \cap \mathcal{S}_+^{n+1} = \mathcal{L}_2 \cap \mathcal{S}_+^{n+1} = \mathcal{L}_3 \cap \mathcal{S}_+^{n+1} = \mathcal{L}_4 \cap \mathcal{S}_+^{n+1}$

(c) $\mathcal{L}_1 \cap \mathcal{S}_+^{n+1} = \mathsf{B}\mathcal{S}_+^{n+1-m}\mathsf{B}^\top\,.$

Binary constraints: aggregation

We further look at what can be done with the constraints related to the binary constraints. We consider the following cones:

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{S}^{n+1} : X_{jj} = x_j \quad \text{for } j \in B \right\},$$

$$\mathcal{B}_2 = \left\{ \begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{S}^{n+1} : \textstyle\sum_{j \in B} (X_{jj} - x_j) = 0 \right\}.$$

Equivalence when intersecting with psd cone and linear constrained set

We shall show the following results:

Lemma 2. *Let* $i\in[1:4]$ *and* $\begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{L}_i \cap \mathcal{S}_+^{n+1} \cap \mathcal{N}^{n+1}$. *Then* $X_{jk} \leq x_k$ *for all* $k\in[1:n]$, $j \in B$.

Theorem 3. *For all* $i,j\in[1:4]$ *we have*

$$\mathcal{B}_1 \cap \mathcal{L}_i \cap \mathcal{CP}^{n+1} = \mathcal{B}_2 \cap \mathcal{L}_j \cap \mathcal{CP}^{n+1},$$
$$\mathcal{B}_1 \cap \mathcal{L}_i \cap \mathcal{S}_+^{n+1} \cap \mathcal{N}^{n+1} = \mathcal{B}_2 \cap \mathcal{L}_j \cap \mathcal{S}_+^{n+1} \cap \mathcal{N}^{n+1}.$$

Reformulations of (B)

From the equivalences between $\mathcal{L}_i,\ i,j\in[1:4]$, and $\mathcal{B}_k,\ k=1,2$, we have

$$(B) \Leftrightarrow \min\left\{\langle \mathsf{Q},\mathsf{X}\rangle + 2\mathsf{c}^\top\mathsf{x} : \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X}\end{pmatrix} \in \mathcal{B}_1 \cap \mathcal{L}_3 \cap \mathcal{CP}^{n+1}\right\}$$

$$(A) \Leftrightarrow \min\left\{\langle \mathsf{Q},\mathsf{X}\rangle + 2\mathsf{c}^\top\mathsf{x} : \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X}\end{pmatrix} \in \mathcal{B}_2 \cap \mathcal{L}_4 \cap \mathcal{CP}^{n+1}\right\}$$

$$(R) \Leftrightarrow \min\left\{\langle \mathsf{Q},\mathsf{X}\rangle + 2\mathsf{c}^\top\mathsf{x} : \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X}\end{pmatrix} \in \mathcal{B}_1 \cap \mathcal{L}_1 \cap \mathcal{CP}^{n+1}\right\}$$

$$(B) \Leftrightarrow (A) \Leftrightarrow (R)$$

Reformulations of (DB)

$$(DB) \Leftrightarrow \min \left\{ \langle \mathsf{Q}, \mathsf{X} \rangle + 2\mathsf{c}^\top \mathsf{x} : \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{B}_1 \cap \mathcal{L}_3 \cap \mathcal{S}_+^{n+1} \cap \mathcal{N}^{n+1} \right\}$$

means no merging

$$(DA) \Leftrightarrow \min \left\{ \langle \mathsf{Q}, \mathsf{X} \rangle + 2\mathsf{c}^\top \mathsf{x} : \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{B}_2 \cap \mathcal{L}_4 \cap \mathcal{S}_+^{n+1} \cap \mathcal{N}^{n+1} \right\}$$

means merging both linear and binary constraints

$$(DR) \Leftrightarrow \min \left\{ \langle \mathsf{Q}, \mathsf{X} \rangle + 2\mathsf{c}^\top \mathsf{x} : \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{B}_1 \cap (\mathsf{B}\mathcal{S}_+^{n+1-m}\mathsf{B}^\top) \cap \mathcal{N}^{n+1} \right\}$$

means reduced psd without merging binary constraints

$$(DB) \Leftrightarrow (DA) \Leftrightarrow (DR)$$

Merging only linear constraints

Merging only linear constraints leans a new reformulations of (DB): $\min\{\langle \mathsf{Q}, \mathsf{X}\rangle + 2\mathsf{c}^\top \mathsf{x} : \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{B}_1 \cap \mathcal{L}_4 \cap \mathcal{S}_+^{n+1} \cap \mathcal{N}^{n+1}\}$, i.e.

$$\begin{array}{ll} \min & \langle \mathsf{Q}, \mathsf{X}\rangle + 2\mathsf{c}^\top \mathsf{x} \\ \text{s.t.} & \sum_{i=1}^m \left(\mathsf{a}_i^\top \mathsf{X} \mathsf{a}_i - 2b_i\, \mathsf{a}_i^\top \mathsf{x} + b_i^2\right) = 0 \\ & X_{jj} = x_j \quad \text{for } j \in B, \\ & \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{N}^{n+1} \cap \mathcal{S}_+^{n+1}. \end{array} \qquad \text{(DML)}$$

Meanwhile, it is a reformulation of (B) when

$$\mathcal{C}P^{n+1} \text{ replaces } \mathcal{N}^{n+1} \cap \mathcal{S}_+^{n+1}.$$

Merging only binary constraints

Merging only binary constraints also leans a new reformulations of (DB): $\min\{\langle \mathsf{Q}, \mathsf{X}\rangle + 2\mathsf{c}^\top \mathsf{x} : \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{B}_2 \cap \mathcal{L}_3 \cap \mathcal{S}_+^{n+1} \cap \mathcal{N}^{n+1}\}$,

$$
\begin{array}{lll}
\min & \langle \mathsf{Q}, \mathsf{X}\rangle + 2\mathsf{c}^\top \mathsf{x} & \\
\text{s.t.} & \mathsf{a}_i^\top \mathsf{x} = b_i & \text{for } i \in [1:m] \\
& \langle \mathsf{a}_i \mathsf{a}_i^\top, \mathsf{X}\rangle = b_i^2 & \text{for } i \in [1:m] \\
& \sum_{j\in B}(X_{jj} - x_j) = 0 & \\
& \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{S}_+^{n+1} \cap \mathcal{N}^{n+1}. &
\end{array}
\qquad \text{(DMB)}
$$

Meanwhile, it is a reformulation of (B) when

$$\mathcal{C}P^{n+1} \text{ replaces } \mathcal{N}^{n+1} \cap \mathcal{S}_+^{n+1}.$$

Merging strategy for reduced problem

Merging binary constraints of (R), we have a reformulation of (R): $\min \left\{ \langle \mathsf{Q}, \mathsf{X} \rangle + 2\mathsf{c}^\top \mathsf{x} : \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{B}_2 \cap \mathcal{L}_1 \cap \mathcal{C}P^{n+1} \right\}$

$$\begin{aligned} \min \quad & \langle \mathsf{Q}, \mathsf{X} \rangle + 2\mathsf{c}^\top \mathsf{x} \\ \text{s.t.} \quad & \textstyle\sum_{j \in B} (X_{jj} - x_j) = 0 \\ & \mathsf{BYB}^\top = \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \\ & \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{CP}^{n+1}, \ \mathsf{Y} \in \mathcal{S}^{n+1-m}. \end{aligned} \qquad \text{(MR)}$$

Reformulation of (DR)

Replacing $\begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{CP}^{n+1}$, $\mathsf{Y} \in \mathcal{S}^{n+1-m}$ in (DR) by $\begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{N}^{n+1}$, $\mathsf{Y} \in \mathcal{S}_+^{n+1-m}$ gives a reformulation of (DR).

$$\min \left\{ \langle \mathsf{Q}, \mathsf{X} \rangle + 2\mathsf{c}^\top \mathsf{x} : \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{B}_2 \cap (\mathsf{B}\mathcal{S}_+^{n+1-m}\mathsf{B}^\top) \cap \mathcal{N}^{n+1} \right\} \ (DMR)$$

$$(DB) \Leftrightarrow (DA) \Leftrightarrow (DR)$$

$$\Leftrightarrow (DML) \Leftrightarrow (DMB) \Leftrightarrow (DMR)$$

An example

Consider a multidimensional quadratic knapsack problems

$$\begin{array}{ll} \max & \mathsf{x}^\top \mathsf{Q}\mathsf{x} \\ \text{s.t.} & \tilde{\mathsf{a}}_i^\top \mathsf{x} \le b_i \quad \text{for } i \in [1:m] \\ & \mathsf{x} \in \{0,1\}^n. \end{array} \tag{1}$$

Adding slack variables to remove the inequality constraints

$$\begin{array}{ll} \max & \mathsf{x}^\top \mathsf{Q}\mathsf{x} \\ \text{s.t.} & \tilde{\mathsf{a}}_i^\top \mathsf{x} + v_i = b_i \quad \text{for } i \in [1:m] \\ & \mathsf{x} \in \{0,1\}^n \\ & \mathsf{v} \in \mathbb{R}_+^m. \end{array} \tag{2}$$

Add m slack variables!

Reformulation to satisfy condition (K)

Adding slack variables to guarantee the variables are restricted in $[0,1]$.

$$\begin{array}{ll} \max & \mathsf{x}^\top \mathsf{Q}\mathsf{x} \\ \text{s. t.} & \tilde{\mathsf{a}}_i^\top \mathsf{x} + v_i = b_i \quad \text{for } i \in [1:m] \\ & x_j + z_j = 1 \quad \text{for } j \in [1:n] \\ & \mathsf{x} \in \{0,1\}^n \\ & \mathsf{v} \in \mathbb{R}_+^m,\ \mathsf{z} \in \mathbb{R}_+^n,\ \mathsf{x} \in \mathbb{R}_+^n. \end{array} \tag{3}$$

Numerical results

A MQKP with 2 knapsacks and 5 goods.

Statistics of solution status

	(DB)	(DML)	(DMB)	(DA)	(DR)	(DMR)
SDPT3	25.0888	26.5766*	25.0888	28.744*	25.0888	25.0888
SeDuMi	25.0888	26.5767*	25.0888	28.744*	25.0888	25.0888
Mosek	25.0888	NaN**	25.0888	NaN**	25.0888	25.0888
Sdpnal+	NaN**	NaN**	NaN**	NaN**	25.0888	25.0888

*: Inaccurate solved, the algorithm converge only to a near optimal or dual feasible solution.

**: Failed, the sdp problem is unsolvable by solvers

When the problem is precisely solved, all sdp relaxation problems have the same optimal value.

Statistics of solution status

We choose 4 different sizes and run 100 groups of sdp problems with different parameters under the same size.

		(DB)	(DML)	(DMB)	(DA)	(DR)	(DMR)
6*10	'Solved'	12	0	10	0	100	100
	'Inacc./Solved'	88	100	90	100	0	0
	'Failed'	0	0	0	0	0	0
10*10	'Solved'	0	0	0	0	99	98
	'Inacc./Solved'	100	100	98	100	1	2
	'Failed'	0	0	2	0	0	0
15*10	'Solved'	0	0	0	0	100	100
	'Inacc./Solved'	6	100	13	100	0	0
	'Failed'	94	0	87	0	0	0
30*5	'Solved'	0	0	0	0	41	99
	'Inacc./Solved'	85	100	90	100	59	1
	'Failed'	15	0	10	0	0	0

Average solution times

		(DB)	(DML)	(DMB)	(DA)	(DR)	(DMR)
	'Solved'	1.01	-	1.06	-	0.28	0.28
6*10	'Inacc./Solved'	1.07	1.20	1.09	1.16	-	-
	'Failed'	-	-	-	-	-	-
	'Solved'	-	-	-	-	0.40	0.40
10*10	'Inacc./Solved'	2.51	3.01	2.62	3.04	0.45	0.43
	'Failed'	-	-	1.08	-	-	-
	'Solved'	-	-	-	-	0.73	0.73
15*10	'Inacc./Solved'	4.84	9.48	6.64	9.27	-	-
	'Failed'	2.59	-	2.65	-	-	-
	'Solved'	-	-	-	-	5.72	5.07
30*5	'Inacc./Solved'	217.35	380.23	234.98	324.25	5.75	5.31
	'Failed'	65.21	-	71.12	-	-	-

Results are derived with Sedumi.

Conclusions

- The sdp problems have no feasible points in the interior of the semi-definite cone. Still hard to solve! further improvement!

- Most of the reduced sdp problems can be solve precisely, while the non-reduced sdp problems can not.

- The CPU time of reduced sdp problems are much less.

- Compared the reduced sdp problem with no merging, the reduced sdp problem with merging is more tractable.

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Thank you!