Multistage Utility Preference Robust Optimization

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Outline

- Preference robust optimization
- Multistage preference robust optimization
- Rectangularity and time consistency
- Scenario tree approach
- Numerical results
- Conclusion

Introduction

Decision-making problem (e.g. Portfolio selection)

- utility preference $\max_x \mathbb{E}[u(r^\top x)]$
- risk preference $\max_x \mathbb{E}(r^\top x) - \lambda\rho(r^\top x)$
- probabilistic preference $\max_x \mathbb{E}[r^\top x] \quad \text{s.t.} \mathbb{P}(r^\top x \geq y) \geq 1 - \epsilon$

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Decision-making under preference ambiguity.

- Stochastic dominance (with a benchmark)
  - Dominance test (Levy, Post, Kuosmanen)
  - Optimization (Dentcheva, Ruszczyński, Luedtke, Schultz)
- Preference robust optimization
  - utility, risk measure
  - pairwise, moments, nominal,
  - Armbruster B, Delage E, Li JY, Xu HF, Guo SY, Wang W, Homen-de-Mello T, Hu J, Haskell W
- Preference learning

Introduction

Consider the following one-stage expected utility maximization problem

$$\max_{x \in X} \mathbb{E}_{\mathbb{P}}[u(h(x, \xi(\omega)))],$$

where

- $u : \mathbb{R} \to \mathbb{R}$ is an increasing real-valued utility function,
- h is a continuous function over $\mathbb{R}^n \times \mathbb{R}^k$,
- $x \in X$ is a decision vector, X is a compact and convex subset of $\mathbb{R}^n$
- $\xi : \Omega \to \mathbb{R}^n$ is a vector of random variables defined over probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a bounded support in $\mathbb{R}^n$

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One-stage utility preference robust optimization (PRO) problem

$$\max_{x\in X} \inf_{u\in\mathcal{U}} \mathbb{E}_{\mathbb{P}}[u(h(x,\xi(\omega)))],$$

$\mathcal{U}$: ambiguity set of the utility

Introduction

Construction of the ambiguity set

- \- Parametric utility functions, i.e., S-shaped utility

$$\mathcal{U} = \left\{ u \mid u(t) = \begin{cases} t^\alpha & \text{if } t \geq \theta \\ -\lambda(-t)^\beta & \text{otherwise}, \end{cases} \quad [\alpha, \beta, \theta] \in \mathcal{C} \subseteq \mathbb{R}^3 \right\}$$

- \- Mixture of some utility functions, i.e.,

$$\mathcal{U} = \{u \mid u(t) = \alpha u_1(t) + (1-\alpha)u_2(t), \text{ for } \alpha \in (0,1)\}$$

- \- Kantorovich ball

$$\mathcal{U} = \mathbb{B}\left(\tilde{u}(t), r\right) := \{u \in \mathscr{U} \mid \mathsf{dl}_{\mathscr{G}}(u, \tilde{u}) \leq r\}$$

- \- Pairwise comparisons

$$\mathcal{U} = \{u \mid \mathbb{E}[u(X_i)] \geq \mathbb{E}[u(Y_i)], \ i = 1, \dots, K\}$$

multistage expected utility maximization problem

We consider multistage expected utility maximization problem

$$\max_{x_1\in\mathscr{X}_1} \mathbb{E}\left[u_1(h_1(x_1,\xi_1)) + \max_{x_2\in\mathscr{X}_2(x_1,\xi_1)} \mathbb{E}_{|\mathcal{F}_1}\left[u_2(h_2(x_2,\xi_2),\xi_1) + \cdots + \max_{x_T\in\mathscr{X}_T(x_{[T-1]},\xi_{[T-1]})} \mathbb{E}_{|\mathcal{F}_{T-1}}\left[u_T(h_T(x_T,\xi_T),\xi_{[T-1]})\right]\right]\right] \quad (1)$$

where

- $h_t : \mathbb{R}^{n_t} \times \mathbb{R}^{d_t} \to \mathbb{R}$ is a continuous reward function at stage t,
- $u_t : \mathbb{R} \times \mathbb{R}^{\sum_{i=1}^{t-1} d_i} \to \mathbb{R}$ is the utility function characterizing the DM's utility value of the reward at stage t,
  - e.g. habit-formation period utility (C_t: $\mathcal{F}_t$-measurable consumption)

    $$u_t(C_t) = \frac{1}{(1-\sigma)}\left(\frac{C_t}{Z_t^\gamma}\right)^{(1-\sigma)}$$
  - habit-formation reference consumption level

    $$Z_t = \rho Z_{t-1} + (1-\rho)C_{t-1}$$

Reformulation

We may reformulate the multistage expected utility maximization problem (1) as

$$\begin{array}{ll} \max\limits_{\boldsymbol{x}_{[1,T]}} & \mathbb{E}\left[u_1(h_1(x_1,\xi_1)) + u_2(h_2(\boldsymbol{x}_2(\xi_1),\xi_2),\xi_1) + \cdots + u_T(h_T(\boldsymbol{x}_T(\xi_{[T-1]}),\xi_T),\xi_{[T-1]})\right] \\ \text{s.t.} & x_1 \in \mathscr{X}_1, \boldsymbol{x}_t(\xi_{[t-1]}) \in \mathscr{X}_t(\boldsymbol{x}_{[t-1]}(\xi_{[t-2]}),\xi_{[t-1]}), \text{ for } t = 2,\ldots,T, \end{array} \quad (2)$$

- the expectation is taken w.r.t. the distribution of $\xi_{[T]}$
- we write $\boldsymbol{x}_{[1,T]}$ (or $\boldsymbol{x}_{[T]}$ when the decision process starts from the initial stage) for a sequence of decisions $(x_1, \boldsymbol{x}_2(\cdot)\ldots, \boldsymbol{x}_T(\cdot))$, which is also known as an implementable policy
- denote $\boldsymbol{x}_{[t-1]}(\xi_{[t-2]}) := (x_1, \boldsymbol{x}_2(\xi_1), \ldots, \boldsymbol{x}_{t-1}(\xi_{[t-2]}))$ the $\xi_{[t-2]}$ scenario dependent historical decision process up to stage $t-1$.

State-independent model

A simplified version of (1) is that the utility functions at each stage are state independent, that is,

$$\max_{x_1 \in \mathscr{X}_1} \mathbb{E}\left[u_1(h_1(x_1,\xi_1)) + \max_{x_2 \in \mathscr{X}_2(x_1,\xi_1)} \mathbb{E}_{|\mathcal{F}_1}\left[u_2(h_2(x_2,\xi_2)) + \cdots + \max_{x_T \in \mathscr{X}_T(x_{[T-1]},\xi_{[T-1]})} \mathbb{E}_{|\mathcal{F}_{T-1}}\left[u_T(h_T(x_T,\xi_T)) \right] \right] \right]. \quad (3)$$

Likewise, we can reformulate (3) as

$$\begin{array}{cl} \max\limits_{x_{[1,T]}} & \mathbb{E}\left[u_1(h_1(x_1,\xi_1)) + u_2(h_2(\boldsymbol{x}_2(\xi_1),\xi_2)) + \cdots + u_T(h_T(\boldsymbol{x}_T(\xi_{[T-1]}),\xi_T)) \right] \\ \text{s.t.} & x_1 \in \mathscr{X}_1, \boldsymbol{x}_t(\xi_{[t-1]}) \in \mathscr{X}_t\left(\boldsymbol{x}_{[t-1]}(\xi_{[t-2]}),\xi_{[t-1]}\right),\ t = 2,\ldots,T. \end{array} \quad (4)$$

Ambiguity set of utility functions

The DM's preference can be represented by von Neumann-Morgenstern expected utility theory and is consistent at each state.

Definition 1 (Ambiguity set of utility functions)

Let $\mathbb{U}$ be the set of all continuous, bounded and monotonically increasing functions in $\mathcal{L}^p(\mathbb{R})$ and $\mathcal{U}_t$ be a $\mathcal{F}_{t-1}$-measurable set-valued mapping. For given $\xi_{[t-1]}$, $\mathcal{U}_t(\xi_{[t-1]})$ is a subset of $\mathbb{U}$, $t = 1, \cdots, T$. Define the ambiguity set

$$\mathcal{U} := \{\vec{\mathfrak{u}} \mid \vec{\mathfrak{u}} = [\mathfrak{u}_1, \mathfrak{u}_2, \ldots, \mathfrak{u}_T]^\top,\ \mathfrak{u}_t(\cdot, \xi_{[t-1]}) \in \mathcal{U}_t(\xi_{[t-1]}), \text{ for any } \xi_{[t-1]},\ t = 1, \ldots, T\}, \tag{5}$$

where $\mathfrak{u}_1(\cdot, \xi_{[0]}) = u_1(\cdot)$ is a real-valued function in the deterministic ambiguity set U_1. We say that the sequence of utility functions $\{\mathfrak{u}_t(\cdot, \xi_{[t]})\}$ is state independent if $\mathcal{U}_t$ is $\mathcal{F}_0$-measurable, i.e., a deterministic set, for $t = 1, \ldots, T$. In this case, we write $u_t(\cdot)$ for $\mathfrak{u}_t(\cdot, \xi_{[t-1]})$ and $\mathcal{U}_t$ for $\mathcal{U}_t(\xi_{[t-1]})$.

Multistage preference robust optimization model

Multistage preference robust optimization model with state-dependent utility (MS-PRO-SD)

$$\begin{aligned}\max_{\boldsymbol{x}_{[1,T]}} \quad & \inf_{\vec{\mathfrak{u}}\in\mathcal{U}} \mathbb{E}[u_1(h_1(x_1,\xi_1)) + \mathfrak{u}_2(h_2(\boldsymbol{x}_2(\xi_1),\xi_2),\xi_1) + \cdots \\ & + \mathfrak{u}_T(h_T(x_T(\xi_{[T-1]}),\xi_T),\xi_{[T-1]})] \\ \text{s.t.} \quad & x_1 \in \mathscr{X}_1, \boldsymbol{x}_t(\xi_{[t-1]}) \in \mathscr{X}_t(\boldsymbol{x}_{[t-1]}(\xi_{[t-2]}),\xi_{[t-1]}), t = 2,..,T \end{aligned} \tag{6}$$

Multistage preference robust optimization model with state-independent utility (MS-PRO-SID)

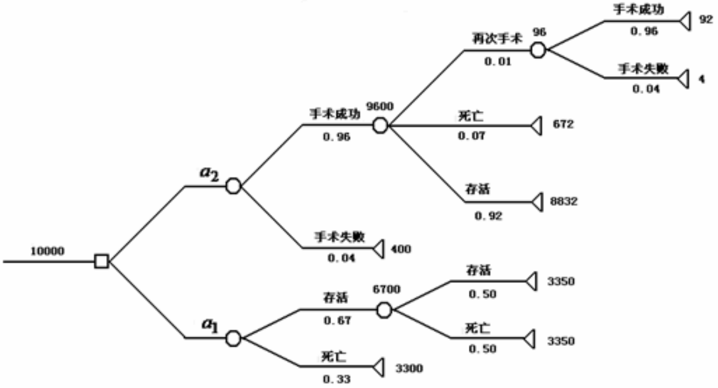
$$\begin{aligned}\max_{\boldsymbol{x}_{[1,T]}} \quad & \inf_{\vec{u}\in U} \mathbb{E}[u_1(h_1(x_1,\xi_1)) + u_2(h_2(\boldsymbol{x}_2(\xi_1),\xi_2)) + \cdots \\ & + u_T(h_T(x_T(\xi_{[T-1]}),\xi_T))] \\ \text{s.t.} \quad & x_1 \in \mathscr{X}_1, \boldsymbol{x}_t(\xi_{[t-1]}) \in \mathscr{X}_t(\boldsymbol{x}_{[t-1]}(\xi_{[t-2]}),\xi_{[t-1]}), \ t = 2,\ldots,T, \end{aligned}$$

where $\vec{u} = [u_1, u_2, \ldots, u_T]^\top$ and $U \subset \mathcal{L}^p(\mathbb{R}) \times \cdots \times \mathcal{L}^p(\mathbb{R})$ is an ambiguity set of the vectors of utility functions in product form.

Time consistency

Definition 2 (Time consistency of dynamic policy)

A multistage PRO model is said to be *time consistent* if any optimal policy for the multistage PRO model over the entire time horizon also satisfies the local optimality conditions of the sub-PRO model from period t to period T, for any given historical $\xi_{[t]}$, for all $t = 2, \ldots, T-1$.



Rectangularity

Definition 3 (Rectangularity of the ambiguity set)

Let $\mathcal{U}$ be a nonempty set of utility sequences $\vec{\mathfrak{u}}$, $\mathcal{U}$ is said to be *rectangular* if

$$\begin{aligned}&\inf_{\vec{\mathfrak{u}}\in\mathcal{U}} \mathbb{E}\left[\mathfrak{u}_1(Z_1)+\mathfrak{u}_2(Z_2,\xi_1)+\cdots+\mathfrak{u}_T(Z_T,\xi_{[T-1]})\right]\\ =\ &\inf_{u_1\in\mathcal{U}_1}\mathbb{E}\left[u_1(Z_1)+\inf_{u_2\in\mathcal{U}_2(\xi_{[1]})}\mathbb{E}_{|\mathcal{F}_1}\left[u_2(Z_2)+\cdots+\inf_{u_T\in\mathcal{U}_T(\xi_{[T-1]})}\mathbb{E}_{|\mathcal{F}_{T-1}}\left[u_T(Z_T)\right]\right]\right]\end{aligned} \tag{7}$$

holds for any $\{x_t\}$, $\{\xi_t\}$ and $\{Z_t := h_t(x_t(\xi_{[t-1]}),\xi_t)\}$, where

$$\begin{aligned}\mathcal{U}_t\left(\xi_{[t-1]}\right) &:= \mathcal{U}_t\left(\vec{\mathfrak{u}}_{[1,t-1]}(\cdot,\xi_{[t-1]}),\xi_{[t-1]}\right)\\ &= \left\{u_t\in\mathcal{L}_p(\mathbb{R})\,\middle|\, \begin{array}{l}\exists\vec{u}_{[t+1,T]}\in\mathcal{L}_p(\mathbb{R})\times\cdots\times\mathcal{L}_p(\mathbb{R})\\ \text{such that }\left[\vec{\mathfrak{u}}_{[1,t-1]}(\cdot,\xi_{[t-1]});\ u_t;\ \vec{u}_{[t+1,T]}\right]^\top\in\mathcal{U}\end{array}\right\},\\ &\qquad \forall\xi_{[t-1]}\in\mathcal{L}_p(\Omega,\mathcal{F}_{t-1},\mathbb{P};\mathbb{R}^{d_1}\times\ldots\mathbb{R}^{d_t-1}).\end{aligned} \tag{8}$$

Is the ambiguity set defined in Definition 1 rectangular?

Inter-exchangeability property

We say a random function $f : \mathbb{Z} \times \Omega \to \mathbb{R}$ is a *Carathédory function* if $\omega \to f(z,\omega)$ is $\mathcal{F}$-measurable for every fixed $z \in \mathbb{Z}$ and the function $z \to f(z,\omega)$ is continuous for almost every fixed $\omega \in \Omega$.

Lemma 4

Consider a Polish space $\mathbb{Z}$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $Z : \Omega \rightrightarrows \mathbb{Z}$ be a $\mathcal{F}$-measurable set-valued mapping with closed values. Let $\mathfrak{M}$ be a linear space of measurable functions $\mathfrak{z} : \Omega \to \mathbb{Z}$ and $\mathfrak{M}_Z$ the set of measurable functions with image in Z i.e., $\mathfrak{M}_Z := \{\mathfrak{z} \in \mathfrak{M} : \mathfrak{z}(\omega) \in Z(\omega) \subset \mathbb{Z},$ for a.e. $\omega \in \Omega\}$. Let $f : \mathbb{Z} \times \Omega \to \bar{\mathbb{R}}$ be a Carathédory function. Suppose that either $\mathbb{E}\left[\left[\inf_{z\in Z(\omega)} f(z,\omega)\right]_+\right] < \infty$ or $\mathbb{E}\left[\left[-\inf_{z\in Z(\omega)} f(z,\omega)\right]_+\right] < \infty$. Then

$$\mathbb{E}\left[\inf_{z\in Z(\omega)} f(z,\omega)\right] = \inf_{\mathfrak{z}\in\mathfrak{M}_Z} \mathbb{E}\left[F_{\mathfrak{z}}\right], \tag{9}$$

where $F_{\mathfrak{z}}(\omega) := f(\mathfrak{z}(\omega),\omega)$.

Interchangeability in the expected utility case

Lemma 5

Let $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space with sigma algebra* $\mathcal{F}$ *and probability measure* $\mathbb{P}$*. Let* $\eta : \Omega \to \mathbb{R}$ *be a random variable representing reward and* ξ *be a random vector representing state. For fixed* $\tau \in \mathbb{R}^d$*, let*

$$\mathcal{U}(\tau) \subseteq \mathscr{U} := \{u \in \mathcal{L}^p(\mathbb{R} \to \mathbb{R}) \mid u \text{ bounded and continuous real-valued}\},$$

and

$$\mathfrak{M}_{\mathcal{U}} := \{\mathfrak{u} \in \mathcal{L}^p(\mathbb{R} \times \mathbb{R}^d \to \mathbb{R}) \mid \mathfrak{u}(\cdot, \tau) \in \mathcal{U}(\tau), \text{ for any } \tau \in \mathbb{R}^d\},$$

where $\mathcal{L}^p(\mathbb{R} \times \mathbb{R}^d \to \mathbb{R})$ *denotes the set of all state-dependent Lebesgue integrable utility functions* $\mathfrak{u}(\cdot, \cdot)$*. If* $\mathfrak{M}_{\mathcal{U}} \neq \emptyset$*, then*

$$\inf_{\mathfrak{u} \in \mathfrak{M}_{\mathcal{U}}} \mathbb{E}\left[\mathfrak{u}(\eta, \xi)\right] = \mathbb{E}\left[\inf_{u \in \mathcal{U}(\xi)} \mathbb{E}\left[u(\eta) \mid \mathcal{F}_\xi\right]\right], \tag{10}$$

where $\mathcal{F}_\xi$ *is the minimal sub-sigma algebra of* $\mathcal{F}$ *to which* ξ *is adapted.*

the model

Proposition 1

Let $\mathcal{U}$ be defined as in Definition 1. Then rectangularity (7) holds

By Proposition 1, problem (6) can be rewritten as

$$\begin{aligned} \max_{\boldsymbol{x}_{[1,T]}} \quad & \inf_{u_1 \in \mathcal{U}_1} \mathbb{E}\Bigg[u_1(h_1(x_1, \xi_1)) + \inf_{u_2 \in \mathcal{U}_2(\xi_{[1]})} \mathbb{E}_{|\mathcal{F}_1} \Big[u_2(h_2(\boldsymbol{x}_2(\xi_1), \xi_2)) + \cdots \\ & + \inf_{u_T \in \mathcal{U}_T(\xi_{[T-1]})} \mathbb{E}_{|\mathcal{F}_{T-1}} \left[u_T(h_T(x_T(\xi_{[T-1]}), \xi_T)) \right] \cdots \Big] \Bigg] \\ \text{s.t.} \quad & x_1 \in \mathscr{X}_1, \ \boldsymbol{x}_t(\xi_{[t-1]}) \in \mathscr{X}_t\left(\boldsymbol{x}_{[t-1]}(\xi_{[t-2]}), \xi_{[t-1]} \right), \ t = 2, \ldots, T. \end{aligned} \tag{11}$$

Here, $U_1 = U_1(\xi_{[0]})$ relies only on $\xi_{[0]}$ and thus is deterministic.

Time consistency

Theorem 6

Let $\mathcal{U}_t(\xi_{[t-1]})$, $t = 2, \cdots, T$, and $\mathcal{U}$ be defined as defined in Definition 1. Assume: (a) for $t = 2, \cdots, T$, the utility functions in $\mathcal{U}_t(\xi_{[t-1]})$ are Lipschitz continuous with modulus being bounded by $\kappa(\xi_{[t-1]})$ and $\mathcal{U}_t(\xi_{[t-1]})$ is a compact set for any $\xi_{[t-1]}$; (b) the reward function $h_t : \mathbb{R}^{n_t} \times \mathbb{R}^{d_t} \to \mathbb{R}$ is continuous and Lipschitz continuous in x_t with modulus σ_t where $\mathbb{E}_{\mathcal{F}_{t-1}}[\sigma_t] < +\infty$, for $t = 1, \cdots, T$; (c) for $t = 2, \cdots, T$, the feasible set $\mathscr{X}_t(x_{[t-1]}, \xi_{[t-1]})$ is compact for any fixed $x_{[t-1]}$ and $\xi_{[t-1]}$ and as set-value mapping of $x_{[t-1]}$, $\mathscr{X}_t(\cdot, \xi_{[t-1]})$ is Lipschitz continuous. Then the (MS-PRO-SD) problem has the following dynamic programming reformulation:

$$V_t\left(x_{[t-1]}, \xi_{[t-1]}\right) = \max_{x_t \in \mathscr{X}_t\left(x_{[t-1]}, \xi_{[t-1]}\right)} \inf_{u_t \in \mathcal{U}_t(\xi_{[t-1]})} \mathbb{E}_{|\mathcal{F}_{t-1}}\left[u_t\left(h_t(x_t, \xi_t)\right) + V_{t+1}\left(x_{[t]}, \xi_{[t]}\right)\right]$$

for $t = 1, \ldots, T$, where $V_{T+1}(\cdot, \cdot) := 0$, and V_1 coincides with the optimal value of problem (MS-PRO-SD). The optimal policy of (MS-PRO-SD) is time consistent.

Pairwise comparisons

Let $\mathscr{U}$ be the set of continuous normalized non-decreasing utility functions in $\mathcal{L}^p([a,b])$ with $u(a)=0,\ u(b)=1$, and $\mathscr{U}^c$ a subset where the utility functions are concave. ($u(a)=0$, $u(b)=1$ for $t=1,\cdots,T$)

$$\mathcal{U}_t^P(\xi_{[t-1]}) := \left\{ u \in \mathscr{U}^c \;\middle|\; \begin{array}{c} z_k(\xi_{[t-1]})\mathbb{E}\left[u\left(W_k\right) \mid \xi_{[t-1]}\right] \geq z_k(\xi_{[t-1]})\mathbb{E}\left[u\left(Y_k\right) \mid \xi_{[t-1]}\right], \\ \text{for } k=1,\ldots,K, \\ \mathrm{Lip}(u) \leq L(\xi_{[t-1]}) \end{array} \right\}$$

- where $\{(W_k, Y_k), k=1,\cdots,K\}$ is a set of prospects for pairwise comparison.
- Here $z_k(\xi_{[t-1]}) \in \{+1,-1,0\}$ is used to indicate the choice of the decision-maker at stage t.
- Armbruster and Delage show that a static PRO problem with pairwise comparison ambiguity set $\mathcal{U}_t^P(\xi_{[t-1]})$ can be reformulated as a linear programming problem, given that the support of W_k and W_k are finite.

Pseudo-metric

Let $\mathscr{G}$ be a set of measurable functions defined over $[a,b]$. For $u, v \in \mathscr{U}$, define the semi-distance between u and v by

$$\mathrm{dl}_{\mathscr{G}}(u,v) := \sup_{g\in\mathscr{G}} \left| \int_a^b g(z)du(z) - \int_a^b g(z)dv(z) \right|$$

Some special cases:

- Kantorovich metric (denoted by $\mathrm{dl}_K(u,v)$)

$$\mathscr{G} = \mathscr{G}_L := \{g : [a,b] \to \mathbb{R} \mid \ g \text{ is Lipschitz continuous with modulus 1}\}$$

- uniform Kolmogorov metric

$$\mathscr{G} = \mathscr{G}_I = \left\{g := \mathbb{1}_{(a,z]}(\cdot) \mid \text{where } \mathbb{1}_{(a,z]}(s) := 1 \text{ if } s \in (a,z] \text{ and } 0 \text{ otherwise}\right\}.$$

ζ-ball

Definition 7 (Static ζ-ball of utility functions)

Let $\mathscr{U}$ be the set of all continuous, non-decreasing utility functions defined over interval $[a, b]$, $u(a) = 0$, $u(b) = 1$ for all $u \in \mathscr{U}$. For a fixed $\tilde{u} \in \mathscr{U}$, the ζ-ball of utility functions in $\mathscr{U}$ centered at $\tilde{u}$ with radius r under metric $\mathsf{dl}_{\mathscr{G}}$ is defined as:

$$\mathbb{B}(\tilde{u}, r) := \{u \in \mathscr{U} \mid \mathsf{dl}_{\mathscr{G}}(u, \tilde{u}) \leq r\}. \tag{13}$$

ambiguity set of a sequence of state-dependent utility functions

Definition 8

Consider the ambiguity set in (5). For given nominal state-dependent utility function $\tilde{\mathfrak{u}}_t(\cdot, \xi_{[t-1]}) \in \mathscr{U}$, define for all $\xi_{[t-1]}$,

$$\mathcal{U}_t^{\mathbb{B}}(\xi_{[t-1]}) := \left\{ u \in \mathscr{U}^c \,\middle|\, \begin{array}{l} u \in \mathbb{B}(\tilde{\mathfrak{u}}_t(\cdot, \xi_{[t-1]}), r_t(\xi_{[t-1]})), \\ \mathrm{Lip}(u) \leq L(\xi_{[t-1]}) \end{array} \right\}. \tag{14}$$

Measurablility and Rectangularity

Proposition 2

Let $\mathcal{U}_t^{\mathbb{B}}(\xi_{[t-1]})$ and $\mathcal{U}_t^{P}(\xi_{[t-1]})$ be defined as in (12) and (14). Then the following assertions hold.

- (i) *For each fixed ω, $\mathcal{U}_t^{\mathbb{B}}(\xi_{[t-1]}(\omega))$ and $\mathcal{U}_t^{P}(\xi_{[t-1]}(\omega))$ are compact sets.*
- (ii) *If $\tilde{\mathrm{u}}_t(\cdot, \xi_{[t-1]}), r_t(\xi_{[t-1]})$ and $L(\xi_{[t-1]})$ are continuous in $\xi_{[t-1]}$, then $\mathcal{U}_t^{\mathbb{B}}(\xi_{[t-1]}(\cdot))$ and $\mathcal{U}_t^{P}(\xi_{[t-1]}(\cdot)$ are $\mathcal{F}_{t-1}$-measurable.*
- (iii) *The ambiguity $\mathcal{U}$ constructed by $\mathcal{U}_t^{\mathbb{B}}(\xi_{[t-1]})$ ($\mathcal{U}_t^{P}(\xi_{[t-1]})$) in the form of (5) satisfies the rectangularity (the conditions in Definition 1).*

Proposition 3

Let $u, v \in \mathscr{U}$ and $r_1, r_2 \in \mathbb{R}_+$. u^* is the true utility and u_{ref} is a nominal utility. Then

$$\mathbb{H}(\mathbb{B}(u, r_1), \mathbb{B}(v, r_2); \mathsf{dl}_{\mathscr{G}}) \leq \mathsf{dl}_{\mathscr{G}}(u, v) + |r_2 - r_1|.$$
$$\mathbb{H}(u^*, \mathbb{B}(u_{ref}, r); \mathsf{dl}_{\mathscr{G}}) \leq \mathsf{dl}_{\mathscr{G}}(u^*, u_{ref}) + r.$$

Dynamic programming formulation

Multistage PRO model with the ambiguity set defined via (11) and (14) as follows:

$$\begin{aligned} \max_{\boldsymbol{x}_{[1,T]}} \quad & \inf_{u_1 \in \mathcal{U}_1^{\mathbb{B}}} \mathbb{E}\Bigg[u_1(h_1(x_1,\xi_1)) + \inf_{u_2 \in \mathcal{U}_2^{\mathbb{B}}(\xi_{[1]})} \mathbb{E}_{|\mathcal{F}_1}\Big[u_2(h_2(\boldsymbol{x}_2(\xi_1),\xi_2)) + \cdots \\ & + \inf_{u_T \in \mathcal{U}_T^{\mathbb{B}}(\xi_{[T-1]})} \mathbb{E}_{|\mathcal{F}_{T-1}}\left[u_T(h_T(x_T(\xi_{[T-1]}),\xi_T))\right] \cdots \Big]\Bigg] \\ \text{s.t.} \quad & x_1 \in \mathscr{X}_1, \boldsymbol{x}_t(\xi_{[t-1]}) \in \mathscr{X}_t\left(\boldsymbol{x}_{[t-1]}(\xi_{[t-1]}),\xi_{[t-1]}\right), t = 2,\ldots,T. \end{aligned} \tag{15}$$

Here, $\mathbb{B}(\tilde{\mathrm{u}}_1, r_1)$ in $\mathcal{U}_1^{\mathbb{B}}$ relies only on deterministic nominal utility $\tilde{\mathrm{u}}_1$ and radius r_1. By Theorem 6, (15) can be computed by the following dynamic programming equation,

$$\begin{aligned} & V_t\left(x_{[t-1]},\xi_{[t-1]}\right) \\ = & \max_{x_t \in \mathscr{X}_t\left(x_{[t-1]},\xi_{[t-1]}\right)} \inf_{u_t \in \mathcal{U}_t^{\mathbb{B}}(\xi_{[t-1]})} \mathbb{E}_{|\mathcal{F}_{t-1}}\left[u_t\left(h_t(x_t,\xi_t)\right) + V_{t+1}\left(x_{[t]},\xi_{[t]}\right)\right]. \end{aligned} \tag{16}$$

Piecewise linear approximation

Let $y_1 < \cdots < y_N$ be an ordered sequence of points in $[a, b]$ and $Y := \{y_1, \cdots, y_N\}$ with $y_1 = a$ and $y_N = b$. Let $\mathscr{U}_N$ be a class of continuous, non-decreasing, piecewise linear functions defined over the interval $[y_1, y_N]$ with breakpoints on Y. For a given $v \in \mathscr{U}_N$, let

$$\mathbb{B}_N(v, r) := \{u \in \mathscr{U}_N \mid \mathsf{dl}_{\mathscr{G}}(u, v) \le r\} \tag{17}$$

and

$$\mathcal{U}_t^{\mathbb{B}_N}(\xi_{[t-1]}) := \left\{ u \in \mathscr{U}^c \,\middle|\, \begin{array}{l} u \in \mathbb{B}_N(\tilde{\mathfrak{u}}_t(\cdot, \xi_{[t-1]}), r_t(\xi_{[t-1]})) \\ \mathrm{Lip}(u) \le L(\xi_{[t-1]}) \end{array} \right\},$$

for a given nominal utility function $\tilde{\mathfrak{u}}_t(\cdot, \xi_{[t-1]}) \in \mathscr{U}_N$.

Error bound

Theorem 9 (Error bound)

Let $V_t\left(x_{[t-1]},\xi_{[t-1]}\right)$ *and* $\tilde{V}_t\left(x_{[t-1]},\xi_{[t-1]}\right)$ *be the optimal value function of the true problem and piecewise linear approximation problem . Let* $\{\tilde{\mathrm{u}}_t(\cdot,\xi_{[t-1]})\}$ *be a sequence of nominal utility functions and* $\{\tilde{\mathrm{u}}_t^N(\cdot,\xi_{[t-1]})\}$ *its piecewise linear approximations. Let*

$$\beta_N(\xi_{[t-1]}) := \max_{i=2,\cdots,N}(y_i - y_{i-1}),$$

where the breakpoints are chosen according to historical data $\xi_{[t-1]}$*. Assume that* $\tilde{\mathrm{u}}_t(\cdot,\xi_{[t-1]})$ *is Lipschitz continuous with modulus* $L(\xi_{[t-1]})$*.*

$$\left|V_t\left(x_{[t-1]},\xi_{[t-1]}\right) - \tilde{V}_t\left(x_{[t-1]},\xi_{[t-1]}\right)\right| \le \sum_{s=t}^{T} 6\mathbb{E}\left[\max(2, L(\xi_{[s-1]}))\beta_N(\xi_{[s-1]}) \mid \mathcal{F}_{t-1}\right]$$

for $t=1,\ldots,T$*. When* $\beta_N(\xi_{[s-1]})$ *and* $L(\xi_{[s-1]})$ *are independent of states,*

$$\left|V_t\left(x_{[t-1]},\xi_{[t-1]}\right) - \tilde{V}_t\left(x_{[t-1]},\xi_{[t-1]}\right)\right| \le 6(T-t+1)\max(2,L)\beta_N.$$

Kantorovich ball model

Theorem 10

Consider

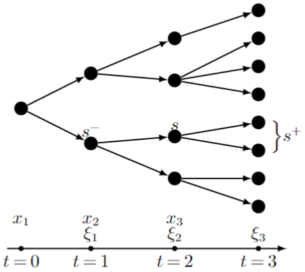
$$\mathcal{U}_t^K(\xi_{[t-1]}) := \left\{ u \in \mathscr{U}^c \;\middle|\; \begin{array}{l} u \in \mathbb{B}_K(\tilde{\mathrm{u}}_t^N(\xi_{[t-1]}), r_t(\xi_{[t-1]})) \\ \mathrm{Lip}(u) \leq L(\xi_{[t-1]}) \end{array} \right\}, \quad (18)$$

for all $\xi_{[t-1]}$. Suppose that the optimal value function at period $t+1$ is $\widetilde{V}_{t+1}\left(x_{[t]}, \xi_{[t]}\right)$. Given historical data $\xi_{[t-1]}$ and historical decision $x_{[t-1]}$, ξ_t is discretely distributed with S scenarios $\xi_t^1, \ldots, \xi_t^S$ and appearing probability $\mathbb{P}(\xi_t = \xi_t^i | \xi_{[t-1]})$, $i = 1, \ldots, S$, then the optimal decision x_t at stage t can be derived by solving the a deterministic programming problem, where the optimal value is $\widetilde{V}_t\left(x_{[t-1]}, \xi_{[t-1]}\right)$.

Reformulation of dynamic equation

$$
\begin{aligned}
\max \quad & \theta_{N-1} + \sum_{i=1}^{S} \left(\mu_{i,N} + \mathbb{P}(\xi_t = \xi_t^i | \xi_{[t-1]}) \widetilde{V}_{t+1} \left(x_{[t]}, [\xi_{[t-1]}, \xi_t^i] \right) \right) - L(\xi_{[t-1]}) \sum_{j=1}^{N-1} \eta_j \\
& - \tilde{L}(\xi_{[t-1]}) \sum_{j=1}^{N-2} (\tau_j + \sigma_j) (y_{j+2} - y_j) - \sum_{j=2}^{N} \tilde{\beta}_j w_j - r_t(\xi_{[t-1]}) \varsigma \\
\text{s.t.} & \sum_{j=1}^{N} y_j \mu_{i,j} \le \mathbb{P}(\xi_t = \xi_t^i | \xi_{[t-1]}) h_t(x_t, \xi_t^i), \; i = 1, \ldots, S, \\
& \mathbb{P}(\xi_t = \xi_t^i | \xi_{[t-1]}) - \sum_{j=1}^{N} \mu_{i,j} = 0, \; i = 1, \ldots, S, \\
& \theta_{j-1} y_{j-1} - \theta_{j-1} y_j + v_{j-2} (y_{j-1} - y_{j-2}) + w_j + \eta_{j-1} + \tau_{j-1} - \tau_{j-2} + \sigma_{j-2} - \sigma_{j-1} \ge 0, \\
& \qquad j = 3, \cdots, N-1, \\
& \theta_1 y_1 - \theta_1 y_2 + w_2 + \eta_1 + \tau_1 - \sigma_1 \ge 0 \\
& \theta_{N-1} y_{N-1} - \theta_{N-1} y_N + v_{N-2} (y_{N-1} - y_{N-2}) + w_N + \eta_{N-1} - \tau_{N-2} + \sigma_{N-2} \ge 0, \\
& \theta_{j-1} - \theta_j + \sum_{i=1}^{S} \mu_{i,j} - v_{j-1} + v_j = 0, \; j = 2, \cdots, N-2 \\
& \theta_{N-2} - \theta_{N-1} + \sum_{i=1}^{S} \mu_{i,N-1} - v_{N-2} = 0, \\
& w_j \le z_{j-1}(y_j - y_{j-1}) + \frac{1}{2}(y_j - y_{j-1})^2 \varsigma, \; j = 2, \cdots, N, \\
& -w_j \le -z_{j-1}(y_j - y_{j-1}) + \frac{1}{2}(y_j - y_{j-1})^2 \varsigma, \; j = 2, \cdots, N, \\
& w_j \le z_j(y_j - y_{j-1}) + \frac{1}{2}(y_j - y_{j-1})^2 \varsigma, \; j = 2, \cdots, N, \\
& -w_j \le -z_j(y_j - y_{j-1}) + \frac{1}{2}(y_j - y_{j-1})^2 \varsigma, \; j = 2, \cdots, N, \\
& x_t \in \mathscr{X}_t \left(x_{[t-1]}, \xi_{[t-1]} \right), \; \theta \in \mathbb{R}^{N-1}, v \in \mathbb{R}_+^{N-2}, \eta \in \mathbb{R}_+^{N-1}, \tau \in \mathbb{R}_+^{N-2}, \sigma \in \mathbb{R}_+^{N-2}, \\
& \mu \in \mathbb{R}_+^{S \times N}, \varsigma \in \mathbb{R}_+, w \in \mathbb{R}_+^{N-1}, z \in \mathbb{R}_+^{N},
\end{aligned} \tag{19}
$$

Solution approach: scenario tree



Given the scenario tree structure of $\{\xi_t\}$, the (MS-PRO) problem (6) can be reformulated as the following min-max programming problem,

$$\begin{array}{ll} \max\limits_{\{x(s), s\in S^-\}} & \sum\limits_{s\in S^-} p_s \inf\limits_{u_s\in\mathcal{U}(s)} \left(\sum\limits_{i\in s^+} \frac{p_i}{p_s} u_s\left(h_{t(i)}\left(x(s), \xi(i)\right)\right)\right) \\ \text{s.t.} & x(1) \in \mathcal{X}_1, x(s) \in \mathcal{X}_{t(s)}\left(x[s^-], \xi[s]\right), \ \forall s \in S^- \setminus \{1\}, \end{array} \tag{20}$$

Reformulation of Kantorovich ball model

Given the scenario tree structure of $\{\xi_t\}$ and a series of Kantorovich ball based ambiguity sets $\mathcal{U}^K(s) = \mathcal{U}^K_{t(s)}(\xi[s])$ on each node $s \in S^-$ of the scenario tree, MS-PRO-SD can be reformulated as

$$
\begin{aligned}
\max\ & \sum_{s\in S^-} p_s\Big(\theta_{N-1}(s) + \sum_{i\in s^+}\mu_{i,N} - L(s)\sum_{j=1}^{N-1}\eta_j(s) - \sum_{j=2}^{N}\tilde{\beta}_j(s)w_j(s) - r(s)\varsigma(s)\Big)\\
\text{s.t. } & \sum_{j=1}^{N} y_j\mu_{i,j} \le \frac{p_i}{p_{i^-}} h_{t(i)}(x(i^-),\xi(i)),\ i\in S\setminus\{1\},\\
& \frac{p_i}{p_{i^-}} - \sum_{j=1}^{N}\mu_{i,j} = 0,\ i\in S\setminus\{1\},\\
& \theta_{j-1}(s)y_{j-1} - \theta_{j-1}(s)y_j + v_{j-2}(s)(y_{j-1}-y_{j-2}) + w_j(s) + \eta_{j-1}(s) \ge 0,\ j=3,\cdots,N-1,\ s\in S^-,\\
& \theta_1(s)y_1 - \theta_1(s)y_2 + w_2(s) + \eta_1(s) \ge 0,\ s\in S^-,\\
& \theta_{N-1}(s)y_{N-1} - \theta_{N-1}(s)y_N + v_{N-2}(s)(y_{N-1}-y_{N-2}) + w_N(s) + \eta_{N-1}(s) \ge 0,\ s\in S^-,\\
& \theta_{j-1}(s) - \theta_j(s) + \sum_{i\in s^+}\mu_{i,j} - v_{j-1}(s) + v_j(s) = 0,\ j=2,\cdots,N-2,\ s\in S^-,\\
& \theta_{N-2}(s) - \theta_{N-1}(s) + \sum_{i\in s^+}\mu_{i,N-1} - v_{N-2}(s) = 0,\ s\in S^-,\\
& w_j(s) \le z_{j-1}(s)(y_j-y_{j-1}) + \frac{1}{2}(y_j-y_{j-1})^2\varsigma(s),\ j=2,\cdots,N,\ s\in S^-,\\
& -w_j(s) \le -z_{j-1}(s)(y_j-y_{j-1}) + \frac{1}{2}(y_j-y_{j-1})^2\varsigma(s),\ j=2,\cdots,N,\ s\in S^-,\\
& w_j(s) \le z_j(s)(y_j-y_{j-1}) + \frac{1}{2}(y_j-y_{j-1})^2\varsigma(s),\ j=2,\cdots,N,\ s\in S^-,\\
& -w_j(s) \le -z_j(s)(y_j-y_{j-1}) + \frac{1}{2}(y_j-y_{j-1})^2\varsigma(s),\ j=2,\cdots,N,\ s\in S^-,\\
& x(1)\in\mathscr{X}_1,\ x(s)\in\mathscr{X}_{t(s)}(x[s^-],\xi[s]),\ s\in S^-\setminus\{1\},\\
& \theta(s)\in\mathbb{R}^{N-1},\ v(s)\in\mathbb{R}^{N-2}_+,\ \eta(s)\in\mathbb{R}^{N-1}_+,\ s\in S^-,\\
& \varsigma(s)\in\mathbb{R}_+,\ w(s)\in\mathbb{R}^{N-1},\ z(s)\in\mathbb{R}^N,\ s\in S^-,\ \mu(s)\in\mathbb{R}^N_+,\ s\in S\setminus\{1\}.
\end{aligned}
\tag{36}
$$

Reformulation of pairwise comparisons model

Given the scenario tree structure of $\{\xi_t\}$ and pairwise comparisons ambiguity sets $\mathcal{U}^P(s) = \mathcal{U}^P_{t(s)}(\xi[s])$, MS-PRO-SD is equivalent to

$$
\begin{aligned}
\max\ & \sum_{s\in S^-} p_s\Big(\theta_{N-1}(s) + \sum_{i\in s+}\mu_{i,N} - L(s)\sum_{j=1}^{N-1}\eta_j(s) + \sum_{k=1}^{K} z_k(s)\left(\mathbb{P}[Y_k=y_N] - \mathbb{P}[W_k=y_N]\right)\lambda_k(s)\Big)\\
\text{s.t.}\ & \sum_{j=1}^{N} y_j\mu_{i,j} \le \frac{p_i}{p_{i^-}} h_{t(i)}(x(i^-),\xi(i)),\ i\in S\setminus\{1\},\\
& \frac{p_i}{p_{i^-}} - \sum_{j=1}^{N}\mu_{i,j} = 0,\ i\in S\setminus\{1\},\\
& \theta_{j-1}(s)y_{j-1} - \theta_{j-1}(s)y_j + v_{j-2}(s)\left(y_{j-1}-y_{j-2}\right) + \eta_{j-1}(s) \ge 0,\ j=3,\cdots,N-1,\ s\in S^-,\\
& \theta_1(s)y_1 - \theta_1(s)y_2 + \eta_1(s)\ge 0,\ s\in S^-,\\
& \theta_{N-1}(s)y_{N-1} - \theta_{N-1}(s)y_N + v_{N-2}(s)\left(y_{N-1}-y_{N-2}\right) + \eta_{N-1}(s)\ge 0,\ s\in S^-,\\
& \theta_{j-1}(s) - \theta_j(s) + \sum_{i\in s+}\mu_{i,j} - v_{j-1}(s) + v_j(s)\\
& \quad + \sum_{k=1}^{K} z_k(s)\left(\mathbb{P}[Y_k=y_j] - \mathbb{P}[W_k=y_j]\right)\lambda_k(s) = 0,\ j=2,\cdots,N-2,\ s\in S^-,\\
& \theta_{N-2}(s) - \theta_{N-1}(s) + \sum_{i\in s+}\mu_{i,N-1} - v_{N-2}(s)\\
& \quad + \sum_{k=1}^{K} z_k(s)\left(\mathbb{P}[Y_k=y_{N-1}] - \mathbb{P}[W_k=y_{N-1}]\right)\lambda_k(s) = 0,\ s\in S^-,\\
& x(1)\in\mathscr{X}_1, x(s)\in\mathscr{X}_{t(s)}\left(x[s^-],\xi[s]\right),\ s\in S^-\setminus\{1\},\\
& \theta(s)\in\mathbb{R}^{N-1}, v(s)\in\mathbb{R}^{N-2}_+, \eta(s)\in\mathbb{R}^{N-1}_+, \lambda(s)\in\mathbb{R}^K_+,\ s\in S^-, \mu(s)\in\mathbb{R}^N_+, s\in S\setminus\{1\}.
\end{aligned}
\tag{37}
$$

Numerical tests

multistage investment-consumption problem

$$
\begin{aligned}
\max_{x,q} \quad & \mathbb{E}\left[\mathfrak{u}_1(g_1(q_1)d_1, h_0) + \mathfrak{u}_2(g_2(q_2)d_2, h_{[1]}) + \cdots + \mathfrak{u}_T(g_T(q_T)d_T, h_{[T-1]})\right] && \text{(21a)}\\
\text{s.t.} \quad & e^\top x_1 = w_0 - q_1 p_0,\ x_1 \in \mathbb{R}^n_+,\ q_1 \in \mathbb{R}_+, && \text{(21b)}\\
& e^\top x_t = (e+r_{t-1})^\top x_{t-1} - q_t p_{t-1}, x_t(\cdot) \in \mathbb{R}^n_+, q_t(\cdot) \in \mathbb{R}_+, t=2,\ldots,T-1, && \text{(21c)}\\
& (e + r_{T-1})^\top x_{T-1} = q_T p_{T-1},\ q_T(\cdot) \in \mathbb{R}_+. && \text{(21d)}
\end{aligned}
$$

PRO counterpart

$$
\begin{aligned}
\max_{x,q} \quad & \inf_{\vec{u} \in \mathcal{U}} \mathbb{E}\left[\mathfrak{u}_1(g_1(q_1)d_1, h_0) + \mathfrak{u}_2(g_2(q_2)d_2, h_{[1]}) + \cdots + \mathfrak{u}_T(g_T(q_T)d_T, h_{[T-1]})\right] && \text{(22a)}\\
\text{s.t.} \quad & \text{(21b)} - \text{(21d)} && \text{(22b)}
\end{aligned}
$$

- MSP-True: Multistage utility maximization (21) with true utility
- MSP-PLN: Multistage utility maximization (21) with piecewise linear nominal utility
- MS-PRO-Kan: Multistage preference robust problem (22) with Kantorovich ball $\mathcal{U}_t^K(\xi_{[t-1]})$
- MS-PRO-PC: Multistage preference robust problem (22) with pairwise comparison set $\mathcal{U}_t^P(\xi_{[t-1]})$ with K random generated questionnaires and answers at each node.

Numerical results

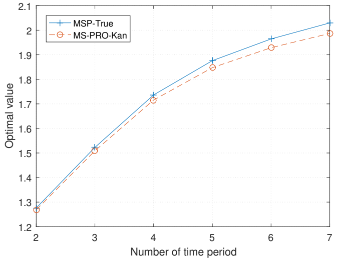


Figure: Optimal values of MSP-True and MS-PRO-Kan with increasing number of time period ($R = 0.001$, $N = 40$)

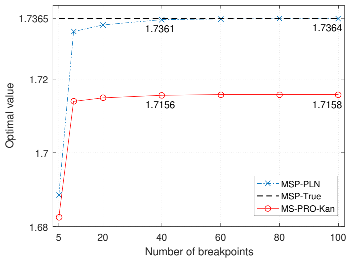


Figure: Optimal values of MSP-True, MSP-PLN and MS-PRO-Kan with increasing number of breakpoints ($T = 4$, $R = 0.001$)

Numerical results

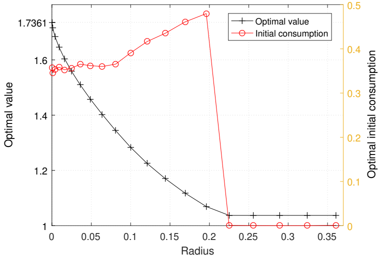


Figure: Optimal values and initial consumption of MS-PRO-Kan with increasing radius of Kantorovich Ball ($T = 4$, $N = 40$)

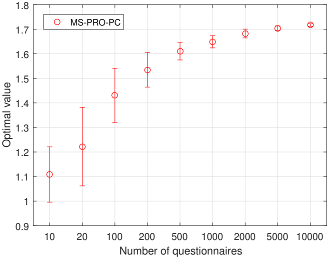


Figure: Confidence intervals of optimal values of MS-PRO-PC with increasing number of questionnaires (mean±std. of 50 tests, $T = 4$)

Conclusions

Summary:

- Extend PRO to dynamic case
- Inter-exchangeability and rectangularity in functional space
- ζ-ball approach
- Scenario tree approach

Current work on:

- Dynamic programming approach (Nested Benders and SDDP)
- Preference (ambiguity of preference) learning / dynamic elicitation

Thank you!

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