Multivariate robust second-order stochastic dominance and resulting risk-averse optimization

Jia Liu

Xi'an Jiaotong University, China

TEL:086-82663661, E-mail: jialiu@xjtu.edu.cn

Joint work with Zhiping Chen and Yu Mei (Xi'an Jiaotong University)

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Introduction

Let $(\Omega, \mathscr{F}, P_0)$ denote an abstract probability space. Denote $\mathscr{L}_p = \mathscr{L}_p(\Omega, \mathscr{F}, P_0; R)$ $(p \geq 1)$,

Definition (FSD)

$X \in \mathscr{L}_p$ dominates $Y \in \mathscr{L}_p$ in the first order, denoted $X \succeq_{(1)} Y$, if

$$P\{X \leq \eta\} \leq P\{Y \leq \eta\}, \quad \forall \eta \in R$$

We define expected shortfall function $F_2(X;\eta) = \int_{-\infty}^{\eta} F(X;\alpha)d\alpha = \mathbb{E}_{P_0}[(\eta - X)_+]$.

Definition (SSD)

$X \in \mathscr{L}_p$ dominates $Y \in \mathscr{L}_p$ in the second order, denoted $X \succeq_{(2)} Y$, if

$$F_2(X;\eta) \leq F_2(Y;\eta), \ \forall \eta \in R$$

Second-order stochastic dominance is particularly popular in industry since it models risk-averse preferences.

Introduction

Proposition

- $X \succeq_{(1)} Y$ *iff* $\mathbb{E}_{P_0}[u(X)] \geq \mathbb{E}_{P_0}[u(Y)]$ *for all* $u \in \mathscr{U}_1$*, here* $\mathscr{U}_1$ *denotes the set of all nondecreasing functions* $u : R \to R$*.*
- $X \succeq_{(2)} Y$ *iff* $\mathbb{E}_{P_0}[u(X)] \geq \mathbb{E}_{P_0}[u(Y)]$ *for all* $u \in \mathscr{U}_2$*, here* $\mathscr{U}_2$ *denotes the set of all concave and nondecreasing functions* $u : R \to R$*.*

- Dentcheva and Ruszczyński (2003) first considered optimization problem with SSD and derived the optimality conditions.
- Dentcheva and Ruszczyński (2006) developed duality relations and solved the dual problem by utilizing the piecewise linear structure of the dual functional
- Luedtke (2008) get new linear formulations for SSD with finite distributed benchmark
- Drapkin, Gollmer, Gotzes, Schultz, et al. (2011a,2011b) study cases where the random variables are induced by mixed-integer linear recourse

Introduction

Solution methods

- Sampling approachesa are the most popular solution method (see, Dentcheva and Ruszczyński, 2003, Liu, Sun and Xu, 2016)
- Cut plane methods are the most efficient solution algorithm (see, e.g., Rudolf and Ruszczyński, 2003; Homem-de-Mello and Mehrotra, 2009; Sun, Xu, et al., 2013).

Strong application background in finance

- e.g., portfolio selection applications (Dentcheva and Ruszczyński, 2006, Meskarian, Fliege and Xu 2014; Chen, Zhuang, L., 2019)

Our focus:

- Multivariate extensions: compare random vectors
- Distributionally robust counterparts: ambiguous distribution

Introduction

Multivariate extensions:
compare random vectors in $\mathscr{L}_p^m = \mathscr{L}_p(\Omega, \mathscr{F}, P_0; R^m)$,

- Define by order relationship between the expected multivariate utility functions (Müller and Stoyan, 2002; Armbruster and Luedtke, 2015).
  - Random vectors X, Y such that $\mathbb{E}_{P_0}[u(X)] \geq \mathbb{E}_{P_0}[u(Y)]$ for all $u \in \mathscr{U}_2$, here $\mathscr{U}_2$ denotes the set of all concave and nondecreasing functions $u\colon R^n \to R$.
- Introduce a scalarization function and model as a univariate SD (Dentcheva and Ruszczyński, 2010; Noyan and Rudolf, 2013, 2018)
  - $$\theta(c, X) \succeq_{(2)} \theta(c, Y), \ \forall c \in C.$$

Multivariate SSD

Definition (Multivariate SSD)

Random vector $X \in \mathscr{L}_p^m$ dominates $Y \in \mathscr{L}_p^m$ in the second order with respect to the scalarization function θ and a set C, denoted as $X \succeq_{(2)}^{\theta,C} Y$, if

$$\theta(c, X) \succeq_{(2)} \theta(c, Y), \ \forall c \in C. \tag{1}$$

where θ is the min-biaffine scalarization function, $c \in C \subset R^m$ plays the role of a scalarization vector.

- Linear scalarization function $\theta(c, x) = a^T(c)x + b(c)$ (Dentcheva and Ruszczyński, 2010, together with $C = R_+^m$)
- Min-biaffine scalarization function $\theta(c, x) = \min_{1 \le t \le T}\{a_t^T(c)x + b_t(c)\}$ (Noyan and Rudolf, 2018)

Robust SSD

Ambiguity of the distribution

- Fully distributional information is hardly known in practice
- Estimated distribution is usually imprecise

$\Rightarrow$ Find solution feasible for all possible distribution (Distributionally robust technique)

Definition (Robust SSD)

A random variable $X \in \mathscr{L}_p$ *dominates robustly* a random variable $Y \in \mathscr{L}_p$ *in the second-order* with respect to a set of probability measures Q if

$$\mathbb{E}_P[u(X)] \geq \mathbb{E}_P[u(Y)], \quad \forall u \in \mathscr{U}, \ \forall P \in Q. \tag{2}$$

Here $\mathscr{U}$ is the set of concave and nondecreasing utility functions defined above. Denote by $X \succeq_{(2)}^{Q} Y$.

Multivariate robust SSD

- Dentcheva and Ruszczyński (2010) proposed the notion of robust second-order stochastic dominance, investigated the optimization problem with this kind of constraints and derived the corresponding conditions of optimality under different cases.
- Guo, Xu and Zhang (2017) studied the efficient solution method for the problems with robust stochastic dominance constraints.
- Chen and Jiang (2018) studied stability Analysis of Optimization Problems with k-th order distributionally robust dominance constraints induced by full random recourse

Multivariate robust SSD

Uncertainty set is key point in robust optimization

- Box uncertainty (Natarajan et al., 2010)
- Ellipsoidal uncertainty (Ermoliev et al., 1985)
- Known first two order moments (El Ghaoui et al., 2003; Natarajan, Sim 2011; Chen, He, Zhang, 2010)
- Imprecise first two order moments (Delage and Ye, 2010; Cheng and Lisser, 2014)
- Mixture distribution uncertainty (Zhu and Fukushima, 2009)
- Probabilistic distance based uncertainty (Wasserstein distance, Pflug and Wozabal, 2012,2014; Phi-divergence, Ben-Tal et al. 2013, Guan and Jiang, 2017; K-L distance, Hu and Hong, 2014)

Assumption (Assumption 1)

Q is convex, closed, and bounded.

Multivariate robust SSD

Definition (Multivariate robust SSD)

A random vector $X \in \mathcal{L}_p^m$ *dominates robustly* a random vector $Y \in \mathcal{L}_p^m$ *in the second-order* with respect to a set of probability measures Q if

$$\mathbb{E}_P[u(\theta(c, X))] \geq \mathbb{E}_P[u(\theta(c, Y))], \quad \forall c \in C, \forall u \in \mathscr{U}, \ \forall P \in Q. \tag{3}$$

Denoted shortly by $X \succeq_{(2)}^{\theta,C,Q} Y$.

- If Q is a singleton set, $\theta(c, x) = c^T x$ and $C = R_+^m \Longrightarrow$ linearly multivariate SSD
- If $m = 1$ and $\theta(c, x) = x \Longrightarrow$ robust SSD
- If $m = 1$, Q is a singleton set and $\theta(c, x) = x \Longrightarrow$ classical SSD

Multivariate robust SSD

Our motivation: multivariate version of robust SSD

- Multivariate extensions: compare random vectors
- Distributionally robust counterparts: ambiguous distribution

Our contributions

- Study mathematical properties for min-biaffine scalarization
- Analyze their optimality conditions
- Examine the approximation scheme with stability results

Mathematical properties

We adopt min-biaffine scalarization function (Noyan and Rudolf, 2018OR) $\theta(c,x) = \min_{1\le t\le T}\{a_t^T(c)x + b_t(c)\}$

Assumption (Assumption 2)

For any fixed $c \in C$, $\theta(c,\cdot)$ is nondecreasing in the sense that $x \geq^{sep} y$ entails $\theta(c,x) \geq \theta(c,y)$.

Holds automatically for portfolio selection optimizations

Lemma (pth integrability)

For fixed $c \in C$, $X \in \mathscr{L}_p^m$ implies $\theta(c,X) \in \mathscr{L}_p$.

Lemma (Lipschitz continuity)

For any fixed $c \in C$, there exists a constant $C_0(c) := \max_{1\le t\le T} \|a_t(c)\|_1$ such that for any $X, Y \in \mathscr{L}_p^m$, we have

$$\|\theta(c,X) - \theta(c,Y)\|_p \leq C_0(c)\|X - Y\|_p.$$

Relations to utility functions

Theorem

For $X, Y \in \mathscr{L}_p^m$, the following conditions are equivalent:

1. $X \succeq_{(2)}^{\theta,C,Q} Y$;
2. $\theta(c,X) \succeq_{(2)}^{Q} \theta(c,Y), \forall c \in C$;
3. $\mathbb{E}_P[\varphi(X)] \geq \mathbb{E}_P[\varphi(Y)], \forall \varphi \in \Phi, \forall P \in Q$, *where*

$$\Phi = \Big\{ \int_C [Q(c)](\theta(c,x))\mu(dc) : \mu \in \mathcal{M}_+(C), Q : C \to \mathscr{U}, \text{ such that } (c,x) \to [Q(c)](\theta(c,x)) \text{ is Lebesgue measurable on } C \times R^m \Big\};$$

4. $\mathbb{E}_P[(\eta - \theta(c,X))_+] \leq \mathbb{E}_P[(\eta - \theta(c,Y))_+], \forall \eta \in R, \forall c \in C, \forall P \in Q$;
5. $CVaR_{\alpha,P}(\theta(c,X)) \geq CVaR_{\alpha,P}(\theta(c,Y)), \forall c \in C, \forall P \in Q, \forall \alpha \in (0,1]$, *where* $CVaR_{\alpha,P}(X) = \sup_\eta\{\eta - \frac{1}{\alpha}\mathbb{E}_P[(\eta - X)_+]\}$.

Reformulation

We know from above theorem that $X \succeq_{(2)}^{\theta,\tilde{C},Q} Y$ is equivalent to

$$\sup_{P\in Q} \mathbb{E}_P[(\eta - \theta(c,X))_+ - (\eta - \theta(c,Y))_+] \leq 0, \ \forall (c,\eta) \in \tilde{C} \times R.$$

We introduce a functional $\sigma : \mathscr{L}_p \to \bar{R}$ defined as

$$\sigma(V) = \sup_{P\in Q} \mathbb{E}_P[V],$$

We define $\rho_{c,\eta} : \mathscr{L}_p^m \to \bar{R}$ as

$$\rho_{c,\eta}(X) = \sigma[(\eta - \theta(c,X))_+ - (\eta - \theta(c,Y))_+]$$

$X \succeq_{(2)}^{\theta,\tilde{C},Q} Y$ is equivalent to

$$\rho_{c,\eta}(X) \leq 0, \ \forall (c,\eta) \in \tilde{C} \times R.$$

Lipschitz continuity of $\sigma(\cdot)$

Lemma

If the set $\mathcal{Q}$ is convex, closed and bounded, then $\sigma(\cdot)$ is convex and subdifferentiable everywhere. Moreover for any $V \in \mathscr{L}_p$, we have $\partial\sigma(V) = \{P \in \mathcal{Q} : \mathbb{E}_P[V] = \sigma(V)\}$, and $\sigma(\cdot)$ is Lipschitz continuous on $\mathscr{L}_p$ with modulus $B := \sup_{P_1 \in \mathcal{Q}} \sup_{P_2 \in \mathcal{Q}} \left\| \frac{dP_1}{dP_2} \right\|_q$.

Convexity and Lipschitz continuous of $\rho_{c,\eta}(\cdot)$

Proposition

Given Assumptions 1 and 2, $\rho_{c,\eta}(\cdot)$ has the following properties:

1. $\rho_{c,\eta}(\cdot)$ *is convex;*
2. $\rho_{c,\eta}(\cdot)$ *is nonincreasing in the sense that*

$$X_1(\omega) \succeq^{sep} X_2(\omega),\ \forall \omega \in \Omega \Rightarrow \rho_{c,\eta}(X_1) \le \rho_{c,\eta}(X_2);$$

3. $\rho_{c,\eta}(\cdot)$ *is Lipschitz continuous with modulus $B \cdot C_0(c)$.*

Subdifferentiability of $\rho_{c,\eta}(\cdot)$

Proposition (subdifferential)

Given Assumptions 1 and 2, for any $(c,\eta) \in \tilde{C} \times R$*, the functional* $\rho_{c,\eta}(\cdot)$ *is continuous and subdiffenretiable on* $\mathscr{L}_p^m$*, and its subdifferential at a point* $X \in \mathscr{L}_p^m$ *is*

$$\begin{aligned}\partial\rho_{c,\eta}(X) = \Big\{Q \in \mathscr{L}_q^m : \exists P_{c,\eta} \in \mathcal{A}_{c,\eta}(X), \exists\lambda(\omega) \in D_{c,\eta}(X,\omega) \text{ such that} \\ Q = \int_\Omega \lambda(\omega) dP_{c,\eta}(\omega)\Big\} \\ = D_{c,\eta}(X,\cdot) \circ \mathcal{A}_{c,\eta}(X).\end{aligned} \tag{4}$$

Subdifferentiability of $\rho_{c,\eta}(\cdot)$

where

$$D_{c,\eta}(X,\omega) = \begin{cases} \text{conv}\{-a_{i_1}(c), \cdots, -a_{i_l}(c)\}, \\ \quad \text{if } \theta_c(X(\omega)) < \eta \text{ and } \{i_1, \cdots, i_l\} = \text{argmin}_t\{a_t^T(c)X(\omega) + b_t(c)\}, \\ \quad l = 1, \cdots, T, \\ \text{conv}\{\mathbf{0}, -a_{i_1}(c), \cdots, -a_{i_l}(c)\}, \\ \quad \text{if } \theta_c(X(\omega)) = \eta \text{ and } \{i_1, \cdots, i_l\} = \text{argmin}_t\{a_t^T(c)X(\omega) + b_t(c)\}, \\ \quad l = 1, \cdots, T, \\ \{\mathbf{0}\}, \text{ if } \theta_c(X(\omega)) > \eta. \end{cases}$$

$$\mathcal{A}_{c,\eta}(X) = \partial\sigma[(\eta - \theta_c(X))_+ - (\eta - \theta_c(Y))_+] : \mathcal{L}_p^m \to 2^{\mathcal{L}_q},\ X \in \mathcal{L}_p^m.$$

$\theta_c(\cdot) := \theta(c, \cdot)$. 2^A represents the power set of a set A.

Optimization problems with multivariate robust SSD

Consider an optimization problem with a multivariate robust SSD constraint:

$$\min_{z \in Z_0} \quad \phi(H(z,\xi)) \tag{5}$$

$$\text{s.t.} \quad G(z,\xi) \succeq_{(2)}^{\theta,C,Q} Y(\xi), \tag{6}$$

Assumption (Assumption 3)

1. *The uncertainty is exogenous, i.e.,* $\xi \in \mathscr{L}_p^l$ *does not depend on decision* z.
2. Z_0 *is a nonempty, convex and compact subset of a Banach space* $\mathscr{Z}$
3. *For almost all* $\omega \in \Omega$, $z \mapsto [H(z,\xi(\omega))]$ *and* $z \mapsto [G(z,\xi(\omega))]$ *are continuous and concave mappings, here* G *is concave in the sense that:* $G(\lambda z_1 + (1-\lambda)z_2, \xi(\omega)) \succeq^{sep} \lambda G(z_1,\xi(\omega)) + (1-\lambda)G(z_2,\xi(\omega))$ *for any* $\lambda \in [0,1]$;
4. $\phi(\cdot)$ *is a continuous, nonincreasing and convex functional.* $Y(\cdot)$ *is continuous.*

Relaxation to a compact set

Let $\mathfrak{M}: \tilde{C} \rightrightarrows R$ be a multifunction with a nonempty compact graph. In what follows, we only consider (c, η) in the set graph$(\mathfrak{M})$, i.e., we consider the following relaxation problem of problem (5)-(6):

$$\min_{z \in Z_0} \quad \phi(H(z, \xi)) \tag{7}$$

$$\text{s.t.} \quad \rho_{c,\eta}(G(z, \xi)) \leq 0, \ \forall (c, \eta) \in \text{graph}(\mathfrak{M}) \subset \tilde{C} \times R. \tag{8}$$

The reason for this relaxation is to satisfy the Slater constraint qualification.

Assumption (Assumption 4: uniformity)

There exists a point $\tilde{z} \in Z_0$ such that

$$\max_{P \in Q} \max_{(c,\eta) \in \text{graph}(\mathfrak{M})} \mathbb{E}_P[(\eta - \theta(c, G(\tilde{z}, \xi)))_+ - (\eta - \theta(c, Y(\xi)))_+] < 0.$$

Optimality condition

Theorem (Optimality condition)

Given Assumptions 1-4. If $\hat{z}$ is an optimal solution to problem (7)-(8), then there exist measures $\hat{S} \in \partial\phi[H(\hat{z},\xi)]$, $P_{c,\eta} \in \mathcal{A}_{c,\eta}(G(\hat{z},\xi))$, a measurable selection $\lambda_{c,\eta} \in D_{c,\eta}(G(\hat{z},\xi),\omega), (c,\eta) \in \text{graph}(\mathfrak{M}), \omega \in \Omega$, and a measure $\hat{\nu} \in \mathcal{M}_+(\text{graph}(\mathfrak{M}))$, such that $\hat{z}$ is an optimal solution to the problem

$$\min_{z\in Z_0}\left\{\int_\Omega H(z,\xi)\hat{S}(d\omega) + \int_{\text{graph}(\mathfrak{M})}\int_\Omega \lambda_{c,\eta}(\omega)\cdot G(z,\xi)dP_{c,\eta}(\omega)d\hat{\nu}\right\} \tag{9}$$

and the following complementary condition is satisfied

$$\int_{\text{graph}(\mathfrak{M})} \mathbb{E}_{P_{c,\eta}}[(\eta-\theta(c,G(\hat{z},\xi)))_+]d\hat{\nu} = \int_{\text{graph}(\mathfrak{M})} \mathbb{E}_{P_{c,\eta}}[(\eta-\theta(c,Y(\xi)))_+]d\hat{\nu}. \tag{10}$$

Conversely, if for some $\hat{S} \in \partial\phi[H(\hat{z},\xi)], P_{c,\eta} \in \mathcal{A}_{c,\eta}(G(\hat{z},\xi)), \lambda_{c,\eta}(\omega) \in D_{c,\eta}(G(\hat{z},\xi),\omega)$ and $\hat{\nu} \in \mathcal{M}_+(\text{graph}(\mathfrak{M}))$, the optimal solution to problem (9) satisfies (10) and (8), then $\hat{z}$ is an optimal solution to problem (7)-(8).

Discretization and stability analysis: A case study for moment-based uncertainty set

Recall the optimization problem with a multivariate robust SSD constraint:

$$\begin{aligned} \min_{z \in Z_0} \quad & \phi(H(z,\xi)) \\ \text{s.t.} \quad & \mathbb{E}_P\left[\left(\eta - \theta(c, G(z,\xi))\right)_+ - \left(\eta - \theta(c, Y(\xi))\right)_+\right] \le 0,\ \forall (c,\eta) \in \text{graph}(\mathfrak{M}),\ \forall P \in \mathcal{Q}. \end{aligned} \tag{11}$$

with moment-based uncertainty set

$$\mathcal{Q} = \{P \in \mathscr{P} : \mathbb{E}_P[f(\xi)] \le 0\}, \tag{12}$$

$\mathscr{P}$ to denote the set of all probability measures on $(\Xi, \mathscr{B})$, where Ξ is the support set of Ω, assumed to be compact, $\mathscr{B}$ is the Borel sigma algebra on Ξ. $f : \Xi \to R^a$ is a continuous vector-valued functional and a is a positive integer.

Discrete approximation

Consider a discrete approximation to the set of probability distributions as

$$\mathcal{Q}_N := \left\{ P \in \mathscr{P} : f_P(\cdot) = \sum_{i=1}^{N} p_i \delta_{\xi^i}(\cdot), \sum_{i=1}^{N} p_i f(\xi^i) \le 0, \sum_{i=1}^{N} p_i = 1, p_i \ge 0, i = 1, \dots, N \right\},$$

where $f_P(\cdot)$ is the probability mass functions of measure P.

- Obviously $\mathcal{Q}_N \subset \mathcal{Q}$.

We can now construct an approximation to problem (11) as follows:

$$\begin{array}{lll} \min\limits_{z \in Z_0} & \phi(H(z,\xi)) & \\ \text{s.t.} & \sup\limits_{(c,\eta) \in \text{graph}(\mathfrak{M})} \sup\limits_{P \in \mathcal{Q}_N} \mathbb{E}_P[h(z,c,\eta,\xi)] \le 0, & \end{array} \tag{13}$$

where $h(z,c,\eta,\xi) = (\eta - \theta(c, G(z,\xi)))_+ - (\eta - \theta(c, Y(\xi)))$.

Stability?

Define

$$v_N(z) := \sup_{(c,\eta)\in \text{graph}(\mathfrak{M})} \sup_{P\in Q_N} \mathbb{E}_P[h(z,c,\eta,\xi)],$$

$$v(z) := \sup_{(c,\eta)\in \text{graph}(\mathfrak{M})} \sup_{P\in Q} \mathbb{E}_P[h(z,c,\eta,\xi)].$$

- Obviously, $v_N(z) \le v(z)$ as $Q_N \subset Q$.

Convergence of v_N when $N \to \infty$? Properties of $v(z)$?

- Feasible regions:
  $\mathcal{F} := \{z \in Z_0 : v(z) \le 0\}$ and $\mathcal{F}_N := \{z \in Z_0 : v_N(z) \le 0\}$.
- Optimal solution set:
  $\vartheta := \min\{\phi(H(z,\xi)) : z \in \mathcal{F}\}$ and $S := \{z \in \mathcal{F} : \vartheta = \phi(H(z,\xi))\}$.
  $\vartheta_N := \min\{\phi(H(z,\xi)) : z \in \mathcal{F}_N\}$ and $S_N := \{z \in \mathcal{F}_N : \vartheta_N = \phi(H(z,\xi))\}$.

Convergence of $\mathcal{F}_N$ and ϑ_N when $N \to \infty$?

Assumption

Assumption (Assumption 5)

1. *There exists a probability measure* $P^* \in \mathscr{P}$ *such that* $\mathbb{E}_{P^*}[f(\xi)] < 0$;
2. *The sequence* $\{\xi^i\}_{i \in N} \subset \Xi$ *satisfies that for any* $\epsilon > 0$ *and* $\xi \in \Xi$, *there exists an index* $N' \in \{1, \cdots, N\}$ *such that* $\|\xi - \xi^{N'}\| \leq \epsilon$;
3. *For each* $\xi \in \Xi$, *every component of* $G(z, \xi)$, *i.e.,*$G_i(z, \xi), i = 1, \cdots, m$, *is Lipschitz continuous with the Lipschitz modulus being* $\kappa(\xi)$, *i.e.,* $|G_i(z_1, \xi) - G_i(z_2, \xi)| \leq \kappa_i(\xi)\|z_1 - z_2\|_2$, *and* $\kappa := \sup_{\xi \in \Xi} \sum_{i=1}^m \kappa_i(\xi)$ *is finite;*
4. $C_0 := \sup_{c \in \tilde{C}} C_0(c)$ *is finite, where* $C_0(c) = \max_{1 \leq t \leq T} \|a_t(c)\|_1$;
5. $A_0 := \sup_{x \in \mathbb{G}} A_0(x)$ *is finite, where* $\mathbb{G} := G(Z_0 \times \Xi) \cup Y(\Xi)$ *and* $A_0(x) = \max_{1 \leq t \leq T} \|d_t(x)\|_2$.

Qualitative stability results

Lemma

For fixed x, $|\theta(c_1, x) - \theta(c_2, x)| \leq A_0(x) \cdot \|c_1 - c_2\|_2$.

Theorem

Given Assumptions 1-5 and $\mathcal{F}$ *is nonempty,*

1. $\lim_{N\to\infty} \mathbb{H}(\mathcal{F}_N, \mathcal{F}) = 0$;
2. $\lim_{N\to\infty} \vartheta_N = \vartheta$ *(converges uniformly);*
3. $\lim_{N\to\infty} \mathbb{D}(S_N, S) = 0$.

$\mathbb{H}(A, B) := \max\{\mathbb{D}(A, B), \mathbb{D}(A, B)\}$ is the Hausdorff distance between A and B, where $\mathbb{D}(A, B) := \sup_{x\in A} \inf_{y\in B}(x, y)$ is the deviation of A from B

Proposition

$v(\cdot)$ *is Lipschitz continuous on* Z_0 *with the Lipschitz modulus being* $C_0\kappa$.

Conclusions

- We study multivariate robust SSD
- Use min-biaffine scalarization
- Analyze optimality conditions
- Examine stability of discrete approximation

Further work

- Multivariate robust SSD is still very hard problem. Many conditions. Efficiently solution method?
- How about date-driven uncertainty set?

Thank you!