α-concave stochastic dominance

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Joint work with Yu Mei and Zhiping Chen (Xi'an Jiaotong University)

Outline

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- α-concave stochastic dominance
- $\frac{1}{k}$-concave stochastic dominance
- Bounded risk-aversion/loving
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Introduction

Definition 1 (Stochastic dominance)

For two random variables $X, Y \in L^{k-1}(\Omega, \mathcal{F}, P; \mathbb{R})$, we say that X is dominated by Y in the kth-order ($k \in \mathbb{Z}_+$) if

$$\mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)], \quad \forall\, u \in \mathcal{U}_k.$$

Here,

$$\mathcal{U}_k = \{u \in C^k \mid (-1)^{n+1} u^{(n)}(x) \geq 0,\ n = 1, \ldots, k, \},$$

where $u^{(n)}(x) = \mathrm{d}(u^{(n-1)}(x))/\mathrm{d}x$, $n = 1, \ldots, k$, $u^{(0)} := u$, C^k is the space of all kth-order continuously differentiable functions. The kth-order SD relationship is denoted by $X \preceq_{(k)} Y$ for short.

- $X \succeq_{(1)} Y$ iff $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$ for all $u \in \mathscr{U}_1$, here $\mathscr{U}_1$ denotes the set of all nondecreasing functions $u\colon R \to R$.
- $X \succeq_{(2)} Y$ iff $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$ for all $u \in \mathscr{U}_2$, here $\mathscr{U}_2$ denotes the set of all concave and nondecreasing functions $u\colon R \to R$.

Introduction

Basic propositions of stochastic dominance:

Proposition 1 (FSD)

X dominates Y in the first order, if and only if

$$P\{X \le \eta\} \le P\{Y \le \eta\}, \quad \forall \eta \in \mathbb{R}.$$

We define expected shortfall function $F_1(W;\eta) := P(W \le \eta)$, $F_2(X;\eta) = \int_{-\infty}^{\eta} F_1(X;\alpha)d\alpha = \mathbb{E}[(\eta - X)_+]$.

Proposition 2 (SSD)

X dominates Y in the second order, if and only if

$$F_2(X;\eta) \le F_2(Y;\eta), \; \forall \eta \in \mathbb{R}.$$

Second-order stochastic dominance is particularly popular in industry since it models risk-averse preferences.

- Bawa, Vijay S. Stochastic Dominance: A Research Bibliography. Management Science, 1982, 28(6): 698–712.

Introduction

Strong application background in finance: portfolio selection optimization with stochastic dominance constraints

- e.g., portfolio selection, index tracking applications (Dentcheva and Ruszczyński, 2006, Meskarian, Fliege and Xu 2014; Chen, Zhuang, L., 2019)

$$\begin{aligned} \max_{u} \quad & \mathbb{E}[r^\top u] \\ \text{s.t.} \quad & r^\top u \succeq_{(\beta,1)} y, \\ & e^\top u = x_0, \\ & u \geq 0. \end{aligned}$$

- Dentcheva D, Ruszczyński A. Portfolio optimization with stochastic dominance constraints. Journal of Banking & Finance, 2006, 30(2): 433-451.
- Meskarian, Rudabeh, Fliege, Jörg and Xu, Huifu (2014) Stochastic programming with multivariate second order stochastic dominance constraints with applications in portfolio optimization. Applied Mathematics & Optimization, 70 (1), 111-140.
- Chen Z, Zhuang X, Liu J. A Sustainability-Oriented Enhanced Indexation Model with Regime Switching and Cardinality Constraint. Sustainability. 2019 Jan;11(15):4055.

Introduction

Stochastic optimization with dominance constraints

- Dentcheva and Ruszczyński (2003) first considered optimization problem with SSD and derived the optimality conditions.
- Dentcheva and Ruszczyński (2010) developed duality relations and solved by piecewise linear structure of the dual functional
- Luedtke (2008) get linear formulations for SSD with finite distributed benchmark
- Gollmer, Neise, Schultz (2008) study FSD with mixed-integer linear recourse case

- Dentcheva D, Ruszczyński A. Optimization with stochastic dominance constraints. SIAM Journal on Optimization, 2003, 14(2):548-566.
- Dentcheva D, Ruszczyński A. Inverse cutting plane methods for optimization problems with second-order stochastic dominance constraints. Optimization, 2010b, 59(3): 323-338.
- Luedtke J. New formulations for optimization under stochastic dominance constraints. SIAM Journal on Optimization, 2008, 19(3): 1433-1450.
- Gollmer R, Neise F, Schultz R. Stochastic programs with first-order dominance constraints induced by mixed-integer linear recourse. SIAM Journal on Optimization, 2008, 19: 552-571.

Introduction: Extensions

Dynamic extension

$$\sum_{t=1}^{T} \rho_t x_t \succeq_{(2)} \sum_{t=1}^{T} \rho_t y_t, \ \forall \rho \in D$$

- Dentcheva and Ruszczyński (2010) first consider multistage stochastic dominance constraints
- SD constraints have recently been adopted to systematically describe the risk preference of the decision-maker in multi-stage models

- Dentcheva, D., Ruszczyński, A.: Stochastic dynamic optimization with discounted stochastic dominance constraints. SIAM Journal on Control and Optimization, 2010, 47(5), 2540–2556
- Consigli, G., Moriggia, V., Vitali, S.: Long-term individual financial planning under stochastic dominance constraints. Annals of Operations Research, 2019 292, 973–1000
- Mei Y., Chen Z., Liu J., Ji B., Multi-stage portfolio selection problem with dynamic stochastic dominance constraints, Journal of Global Optimization, 2021

Introduction: Extensions

Robust extension

$$\mathbb{E}_P[u(X)] \geq \mathbb{E}_P[u(Y)],\ \forall u \in \mathcal{U},\ \forall P \in \mathcal{Q}.$$

- Dentcheva and Ruszczyński (2010) first introduced the distributionally robust SD and established the optimality conditions
- Guo, Xu, and Zhang (2017) proposed a discrete approximation scheme for DR-SSD with moment-based ambiguity sets
- Mei, L. and Chen (2022) study DR-SSD with Wasserstain ball

- Dentcheva D, Ruszczyński A. Robust stochastic dominance and its application to risk-averse optimization. Mathematical Programming, 2010a, 123(1): 85-100.
- Guo S., Xu H., Zhang L., Probability approximation schemes for stochastic programs with distributionally robust second-order dominance constraints, Optim. Methods Softw., 2017, 32: 770–789.
- Mei Y., Liu J., Chen Z., Distributionally robust second-order stochastic dominance constrained optimization with Wasserstein ball, SIAM Journal on Optimization, 2022, 32(2):715-738

Introduction

Motivating examples:

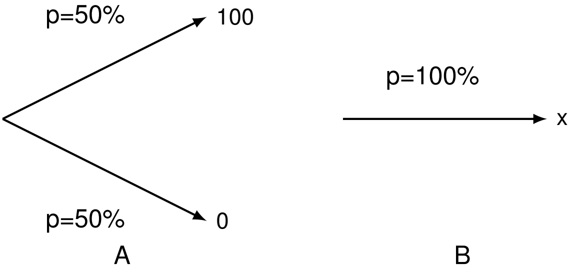

Figure: Two lotteries

- For high risk-aversion players, they all prefer B with even small x (for instance 25)
- For some risk-loving players, they all prefer A with even large x (for instance 70)

Introduction

Integer-order SD → continuous-order SD

- Fishburn (1980) adopted fractional integration to define a continuum

$$F^{\alpha}(x) \leq G^{\alpha}(x),\ \forall x \in [0, b].$$

$$F^{\alpha}(x) = \frac{1}{\Gamma(\alpha)} \int_{y=0}^{x} (x-y)^{\alpha-1} dF(y),\ \forall x \geq 0$$

- ε-Almost Stochastic Dominance $\left(F \succeq_1^{\text{almost } (\varepsilon)} G\right)$

$$\int_{S_1} [F(t) - G(t)]dt \leq \varepsilon \|F - G\|.$$

equivalent to $E_F[u(X)] \geq E_G[u(Y)], \forall u \in U_1^{\star}(\varepsilon)$

$$U_1^{\star}(\varepsilon) = \left\{ u \in U_1 : u'(x) \leq \inf\{u'(x)\} \left[\frac{1}{\varepsilon} - 1\right], \forall x \in [0,1] \right\}$$

- Fishburn PC. Continua of stochastic dominance relations for unbounded probability distributions. Journal of Mathematical Economics, 1980, 7(3): 271-285
- Leshno M, Levy H. Preferred by "all" and preferred by "most" decision makers: Almost stochastic dominance. Management Science, 2002, 48(8): 1074-1085.

Introduction

- Baucells and Heukamp (2006) studied stochastic dominance induced by cumulative prospect theory and found supporting evidence for loss aversion.

$$h(x) \leq h(0) \text{ for } a \leq x \leq 0, \text{ and } h(x) \geq h(0) \text{ for } 0 \leq x \leq b$$

$$h(x) \equiv \int_a^x [G(y) - F(y)]dy$$

- L., Chen, Consigli (2021) proposed interval stochastic dominance

$$\begin{cases} F_k(W;\eta) \leq F_k(Y;\eta), & \forall\, \eta \leq \beta, \\ F_{k+1}(W;\eta) \leq F_{k+1}(Y;\eta), & \forall\, \eta \geq \beta. \end{cases}$$

- Baucells M, Heukamp F. Stochastic dominance and cumulative prospect theory, Management Science, 2006, 52(9): 1409-1423
- Liu J., Chen Z., Consigli G., Interval-based stochastic dominance: theoretical framework and application to portfolio choices, Annals of Operations Research, 2021, 307: 329–361

Introduction

- Muller et al. [15] defined a continuum of SD between FSD and SSD, by considering all investors who are mostly risk-averse but cannot assert that they would dislike any risk.

$$0 \leq \gamma \left(\frac{u(x_4) - u(x_3)}{x_4 - x_3} \right) \leq \frac{u(x_2) - u(x_1)}{x_2 - x_1}$$

for all $x_1 < x_2 < x_3 < x_4$.

$$\int_{-\infty}^{t} (G(x) - F(x))_+ \mathrm{d}x \leq \gamma \int_{-\infty}^{t} (F(x) - G(x))_+ \mathrm{d}x, \ \forall t \in \mathbb{R}$$

Introduction

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$$0 \leq \gamma\left(\frac{u\left(x_4\right)-u\left(x_3\right)}{x_4-x_3}\right) \leq \frac{u\left(x_2\right)-u\left(x_1\right)}{x_2-x_1}$$

for all $x_1 < x_2 < x_3 < x_4$.

$$\int_{-\infty}^{t}(G(x)-F(x))_{+}\mathrm{d}x \leq \gamma \int_{-\infty}^{t}(F(x)-G(x))_{+}\mathrm{d}x, \;\; \forall t \in \mathbb{R}$$

- Light and Perlroth [11] proposed an SD concept consisting of all utility function

$$(u(b)-u(x))^{\frac{1}{\alpha}} \text{ is convex}$$

$\rightarrow$ fail to derive tractable reformulation

- Muller A, Scarsini M, Tsetlin I, Winkler RL. Between first- and second-order stochastic dominance. Management Science, 2016, 63(9): 2933-2947
- Light B, Perlroth A. The family of alpha, [a,b] stochastic orders: Risk vs. expected value. Journal of Mathematical Economics, 2021, 96: 102520

α-concavity

Definition 2 (α-concavity)

A nonnegative function $u(x)$ defined on a convex set $\Xi \subseteq \mathbb{R}^n$ is said to be α-concave, where $\alpha \in [-\infty, +\infty]$, if for all $x, y \in \Xi$ and all $\lambda \in [0, 1]$ the following inequality holds:

$$u(\lambda x + (1-\lambda)y) \geq m_\alpha(u(x), u(y), \lambda),$$

where $m_\alpha : \mathbb{R}_+ \times \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ is defined as

$$m_\alpha(a, b, \lambda) = 0 \text{ if } ab = 0,$$

and if $a > 0, b > 0, 0 \leq \lambda \leq 1$, then

$$m_\alpha(a, b, \lambda) = \begin{cases} a^\lambda b^{1-\lambda} & \text{if } \alpha = 0, \\ \max\{a, b\} & \text{if } \alpha = \infty, \\ \min\{a, b\} & \text{if } \alpha = -\infty, \\ (\lambda a^\alpha + (1-\lambda) b^\alpha)^{1/\alpha} & \text{otherwise.} \end{cases}$$

α-concavity (Con'd)

Lemma 3 (SDR09)

$u(x)$ is α-concave ($\alpha > 0$) if and only if $u(x)^\alpha$ is concave; $u(x)$ is concave if and only if $u(x)^\alpha$ is $\frac{1}{\alpha}$-concave ($\alpha > 0$).

- In the case of $\alpha = 0$, the function u is called logarithmically concave or log-concave because $\log(u)$ is a concave function.
- In the case of $\alpha = 1$, the function u becomes the usual concave function.
- In the case of $\alpha = -\infty$, the function u is quasi-concave.

A. Shapiro, D. Dentcheva and A. Ruszczyński. *Lecture on Stochastic Programming*. MPS-SIAM, 2009.

α-concavity (Con'd)

$m_\alpha(a, b, \lambda)$ is non-decreasing with respect to α. Thus, we have

Lemma 4 (SDR09)

α-concavity entails β-concavity for all $\beta \leq \alpha$.

Lemma 5 (Theorem 4.19 of SDR09)

If $u(x)$ is α-concave and $g(x)$ is β-concave, here $\alpha, \beta \geq 1$, then $h(x) = u(x) + g(x)$ is γ-concave with $\gamma = \min\{\alpha, \beta\}$.

Lemma 6 (Theorem 4.23 of SDR09)

If $u_i(x)$, $i = 1, \ldots, m$, are α_i-concave and α_i are such that $\sum_{i=1}^m \alpha_i^{-1} > 0$, then $g(x) = \prod_{i=1}^m u_i(x_i)$ is γ-concave with $\gamma = \left(\sum_{i=1}^m \alpha_i^{-1}\right)^{-1}$.

α-concave stochastic dominance

Definition 7 (α-concave stochastic dominance, α-concave SD)

For two bounded random variables $X, Y \in L^{k-1}(\Omega, \mathcal{F}, P; [a,b])$, we say that X is α-concave dominated by Y, ($\alpha \in [-\infty, +\infty]$), if

$$\mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)], \quad \forall\ u \in \tilde{\mathcal{U}}^{\alpha}. \tag{1}$$

Here,

$\tilde{\mathcal{U}}^{\alpha} = \{u \in C([a,b] \to \mathbb{R}_+) \mid u$ is monotonically increasing and α-concave$\}$.

- 1-concave SD is equivalent to SSD.
- $-\infty$-concave SD is equivalent to FSD.
- 0-concave SD is log-SD which is generated by the set of all log-concave utility functions

Introduction

Motivating examples:

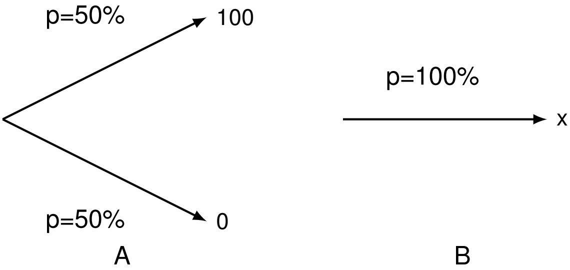

Figure: Two lotteries

$\mathbb{E}[u(A)] = \frac{1}{2}100^{\alpha} \leq \mathbb{E}[u(B)] = x^{\alpha}$ implies $x \geq (\frac{1}{2})^{\alpha}100$

- For some high risk-aversion players, they all prefer B with $x \geq 25$
- For all risk-aversion players, they all prefer B with $x \geq 50$
- For risk-aversion players together with some risk-loving players, they all prefer B with $x \geq 70$

$\frac{1}{k}$-concave SD

We choose $\alpha = \frac{1}{k}$ $(k = 1, 2, \ldots)$

- $\frac{1}{k}$-concave SD defines a stochastic ordering between SSD and log-SD (log-SD is between FSD and SSD)

Denote

$$F_k(W; \vec{\eta}, h) := \mathbb{E}\left[\prod_{t=1}^{h}(\eta_t - a - [\eta_t - W]_+)\right],$$

where $\vec{\eta} = [\eta_1, \ldots, \eta_h]$, $h \leq k$, $h \in \mathbb{Z}_+$. Then we have

Theorem 8

For two random variables $X, Y \in L^1(\Omega, \mathcal{F}, P; [a, b])$, X is $\frac{1}{k}$-concave dominated by Y $(k \in \mathbb{Z}_+)$ if and only if

$$F_k(X; \vec{\eta}, h) \leq F_k(Y; \vec{\eta}, h), \ \forall \ \vec{\eta} \in [a, b]^h, \ \forall h \leq k, h \in \mathbb{Z}_+. \tag{2}$$

Bounded risk-preference

Proposition 3

For any $\frac{1}{k}$-concave ($k \in \mathbb{Z}_+$) function $u(x)$ on $[0, 1]$ with $u(0) = 0$ and $u(1) = 1$, we have $u(x) \geq x^k$ for all $x \in [0, 1]$.

- power function x^k is the minimal $\frac{1}{k}$-concave utility function over $[0, 1]$.
- The $\frac{1}{k}$-concave SD involves all concave utility functions as well as some non-concave functions which are lower bounded by the power function x^k.
- The extreme case when $k \to \infty$ is the log-concavity which can be viewed as a kind of bounded risk-aversion/loving

Bounded risk-preference

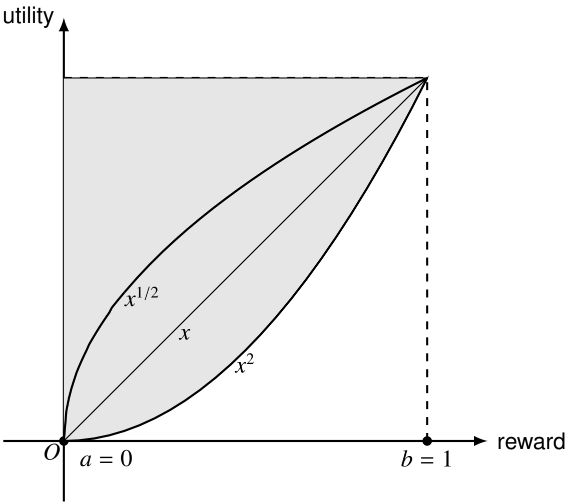

Figure: All power generators of $\frac{1}{2}$-concave SD, i.e., all power utility functions with order not larger than 2

Bounded risk-preference

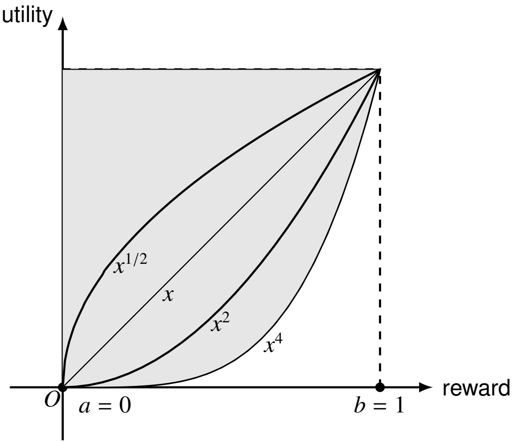

Figure: All power generators of $\frac{1}{4}$-concave SD

the non-concavity of a $\frac{1}{k}$-concave utility function is bounded.

Bounded risk-preference

Proposition 4

For a $\frac{1}{k}$-concave function $u(x)$, $u(x)$ is concave when $\frac{u''(x)}{u'(x)^2}u(x) \leq \frac{k-1}{k}$, and convex when $\frac{u''(x)}{u'(x)^2}u(x) \geq \frac{k-1}{k}$.

- Piecewise power functions are important S-shape utility functions which can characterize the loss/gain dependent risk attitude in prospect theory TvK92.
- We consider an example of piecewise quadratic S-shape utility functions on $[0, 1]$, with a reference point at 0.5, taken from HDB18.

$$u(x) = \begin{cases} 2x^2 & \text{for } x \in [0\,,\ 0.5] \\ 1 - 2(x-1)^2 & \text{for } x \in [0.5\,,\ 1]. \end{cases} \tag{3}$$

S-shape utility function

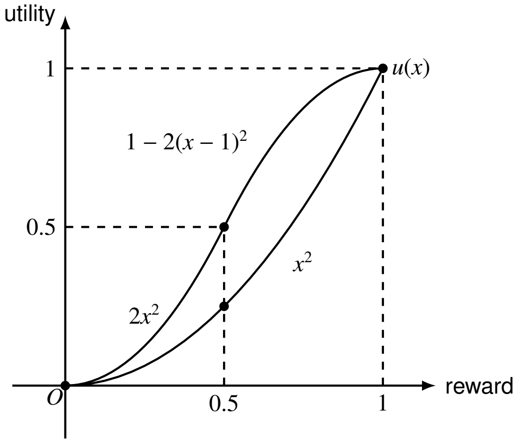

Figure: An $\frac{1}{2}$-concave S-shape utility function and the minimal $\frac{1}{2}$-concave utility function

- Hens T, De Giorgi EG, Bachmann KK. Behavioral Finance for Private Banking: From the Art of Advice

Bounded risk-preference

- $u(x)$ is concave above the reference point but convex below the reference point, i.e., the decision-maker may be irrational (risk-seeking) below the reference point, which is consistent with the prospect theory.
- We would argue that even in the downside part, the irrationality, i.e., the level of risk-seeking, is still limited.
- Mathematically, we require that the level of convexity (non-concavity) is bounded by the $\frac{1}{2}$-concavity.
- By extending the order of power functions in this example from 2 to k, we can deduce that the piecewise k-th order power S-shape utility functions are consistent with the $\frac{1}{k}$-concave SD.
- That is the bounded risk-preference which the $\frac{1}{k}$-concave SD implies.

Conclusions

Summary:

- We proposes a new SD concept which spans a continuous spectrum of the SD relationship between integer-order SDs
- We study the reformulation and examples for the case between SSD and log-SD.

Further works:

- (Portfolio) optimization with $\frac{1}{k}$-concave SD constraints
- reformulations of α-concave SD when $\alpha > 1$ and $\alpha < 0$
- multivariate version of α-concave SD which can cover some important utility functions like the Cobb-Douglas utility function

Thank you!