Non-integer order stochastic dominance

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Introduction

Definition 1 (Stochastic dominance)

For two random variables $X, Y \in L^{k-1}(\Omega, \mathcal{F}, P; \mathbb{R})$, we say that X is dominated by Y in the kth-order ($k \in \mathbb{Z}_+$) if

$$\mathbb{E}[u(X)] \le \mathbb{E}[u(Y)], \quad \forall\, u \in \mathcal{U}_k.$$

Here,

$$\mathcal{U}_k = \{u \in C^k \mid (-1)^{n+1} u^{(n)}(x) \ge 0,\ n = 1, \ldots, k, \},$$

where $u^{(n)}(x) = \mathrm{d}(u^{(n-1)}(x))/\mathrm{d}x$, $n = 1, \ldots, k$, $u^{(0)} := u$, C^k is the space of all kth-order continuously differentiable functions. The kth-order SD relationship is denoted by $X \preceq_{(k)} Y$ for short.

- $X \succeq_{(1)} Y$ iff $\mathbb{E}[u(X)] \ge \mathbb{E}[u(Y)]$ for all $u \in \mathscr{U}_1$, here $\mathscr{U}_1$ denotes the set of all nondecreasing functions $u\colon R \to R$.
- $X \succeq_{(2)} Y$ iff $\mathbb{E}[u(X)] \ge \mathbb{E}[u(Y)]$ for all $u \in \mathscr{U}_2$, here $\mathscr{U}_2$ denotes the set of all concave and nondecreasing functions $u\colon R \to R$.

Introduction

Basic propositions of stochastic dominance:

Proposition 1 (FSD)

X dominates Y in the first order, if and only if

$$P\{X \leq \eta\} \leq P\{Y \leq \eta\}, \quad \forall \eta \in \mathbb{R}.$$

We define expected shortfall function $F_1(W;\eta) := P(W \leq \eta)$,
$F_2(X;\eta) = \int_{-\infty}^{\eta} F_1(X;\alpha)d\alpha = \mathbb{E}[(\eta - X)_+]$.

Proposition 2 (SSD)

X dominates Y in the second order, if and only if

$$F_2(X;\eta) \leq F_2(Y;\eta), \forall \eta \in \mathbb{R}.$$

Second-order stochastic dominance is particularly popular in industry since it models risk-averse preferences.

- Bawa, Vijay S. Stochastic Dominance: A Research Bibliography. Management Science, 1982, 28(6): 698–712.

Introduction

Strong application background in finance: portfolio selection optimization with stochastic dominance constraints

- e.g., portfolio selection, index tracking applications (Dentcheva and Ruszczyński, 2006, Meskarian, Fliege and Xu 2014; Chen, Zhuang, L., 2019)

$$
\begin{aligned}
\max_{u} \quad & \mathbb{E}[r^\top u] \\
\text{s.t.} \quad & r^\top u \succeq_{(\beta,1)} y, \\
& e^\top u = x_0, \\
& u \geq 0.
\end{aligned}
$$

- Dentcheva D, Ruszczyński A. Portfolio optimization with stochastic dominance constraints. Journal of Banking & Finance, 2006, 30(2): 433-451.
- Meskarian, Rudabeh, Fliege, Jörg and Xu, Huifu (2014) Stochastic programming with multivariate second order stochastic dominance constraints with applications in portfolio optimization. Applied Mathematics & Optimization, 70 (1), 111-140.
- Chen Z, Zhuang X, Liu J. A Sustainability-Oriented Enhanced Indexation Model with Regime Switching and Cardinality Constraint. Sustainability. 2019 Jan;11(15):4055.

Introduction

Stochastic optimization with dominance constraints

- Dentcheva and Ruszczyński (2003) first considered optimization problem with SSD and derived the optimality conditions.
- Dentcheva and Ruszczyński (2010) developed duality relations and solved by piecewise linear structure of the dual functional
- Luedtke (2008) get linear formulations for SSD with finite distributed benchmark
- Gollmer, Neise, Schultz (2008) study FSD with mixed-integer linear recourse case

- Dentcheva D, Ruszczyński A. Optimization with stochastic dominance constraints. SIAM Journal on Optimization, 2003, 14(2):548-566.
- Dentcheva D, Ruszczyński A. Inverse cutting plane methods for optimization problems with second-order stochastic dominance constraints. Optimization, 2010b, 59(3): 323-338.
- Luedtke J. New formulations for optimization under stochastic dominance constraints. SIAM Journal on Optimization, 2008, 19(3): 1433-1450.
- Gollmer R, Neise F, Schultz R. Stochastic programs with first-order dominance constraints induced by mixed-integer linear recourse. SIAM Journal on Optimization, 2008, 19: 552-571.

Introduction: Extensions

Dynamic extension

$$\sum_{t=1}^{T} \rho_t x_t \succeq_{(2)} \sum_{t=1}^{T} \rho_t y_t, \ \forall \rho \in D$$

- Dentcheva and Ruszczyński (2010) first consider multistage stochastic dominance constraints
- SD constraints have recently been adopted to systematically describe the risk preference of the decision-maker in multi-stage models

- Dentcheva, D., Ruszczyński, A.: Stochastic dynamic optimization with discounted stochastic dominance constraints. SIAM Journal on Control and Optimization, 2010, 47(5), 2540–2556
- Consigli, G., Moriggia, V., Vitali, S.: Long-term individual financial planning under stochastic dominance constraints. Annals of Operations Research, 2019 292, 973–1000
- Mei Y., Chen Z., Liu J., Ji B., Multi-stage portfolio selection problem with dynamic stochastic dominance constraints, Journal of Global Optimization, 2021

Introduction: Extensions

Robust extension

$$\mathbb{E}_P[u(X)] \geq \mathbb{E}_P[u(Y)],\ \forall u \in \mathcal{U},\ \forall P \in \mathcal{Q}.$$

- Dentcheva and Ruszczyński (2010) first introduced the distributionally robust SD and established the optimality conditions
- Guo, Xu, and Zhang (2017) proposed a discrete approximation scheme for DR-SSD with moment-based ambiguity sets
- Mei, L. and Chen (2022) study DR-SSD with Wasserstain ball

- Dentcheva D, Ruszczyński A. Robust stochastic dominance and its application to risk-averse optimization. Mathematical Programming, 2010a, 123(1): 85-100.
- Guo S., Xu H., Zhang L., Probability approximation schemes for stochastic programs with distributionally robust second-order dominance constraints, Optim. Methods Softw., 2017, 32: 770–789.
- Mei Y., Liu J., Chen Z., Distributionally robust second-order stochastic dominance constrained optimization with Wasserstein ball, SIAM Journal on Optimization, 2022, 32(2):715-738

Introduction

Motivating examples:

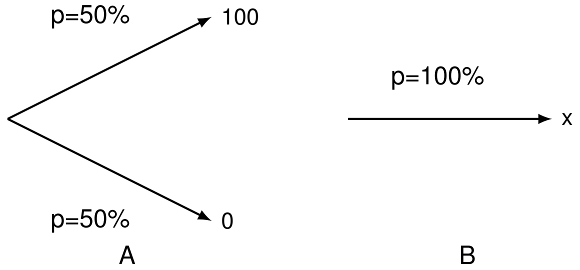

Figure: Two lotteries

- For high risk-aversion players, they all prefer B with even small x (for instance 25)
- For some risk-loving players, they all prefer A with even large x (for instance 70)

Introduction

Integer-order SD → continuous-order SD

- Fishburn (1980) adopted fractional integration to define a continuum

$$F^{\alpha}(x) \le G^{\alpha}(x), \ \forall x \in [0, b].$$

$$F^{\alpha}(x) = \frac{1}{\Gamma(\alpha)} \int_{y=0}^{x} (x-y)^{\alpha-1} dF(y), \ \forall x \ge 0$$

- ε-Almost Stochastic Dominance $\left(F \ge_1^{\text{almost } (\varepsilon)} G\right)$

$$\int_{S_1} [F(t) - G(t)]dt \le \varepsilon ||F - G||.$$

equivalent to $E_F[u(X)] \ge E_G[u(Y)], \forall u \in U_1^{\star}(\varepsilon)$

$$U_1^{\star}(\varepsilon) = \left\{ u \in U_1 : u'(x) \le \inf\{u'(x)\} \left[\frac{1}{\varepsilon} - 1 \right], \forall x \in [0, 1] \right\}$$

- Fishburn PC. Continua of stochastic dominance relations for unbounded probability distributions. Journal of Mathematical Economics, 1980, 7(3): 271-285
- Leshno M, Levy H. Preferred by "all" and preferred by "most" decision makers: Almost stochastic dominance. Management Science, 2002, 48(8): 1074-1085.

Introduction

- Baucells and Heukamp (2006) studied stochastic dominance induced by cumulative prospect theory and found supporting evidence for loss aversion.

$$h(x) \le h(0) \text{ for } a \le x \le 0, \text{ and } h(x) \ge h(0) \text{ for } 0 \le x \le b$$

$$h(x) \equiv \int_a^x [G(y) - F(y)]dy$$

- Baucells M, Heukamp F. Stochastic dominance and cumulative prospect theory, Management Science, 2006, 52(9): 1409-1423

Introduction

- Muller et al. [15] defined a continuum of SD between FSD and SSD, by considering all investors who are mostly risk-averse but cannot assert that they would dislike any risk.

$$0 \leq \gamma \left(\frac{u(x_4) - u(x_3)}{x_4 - x_3} \right) \leq \frac{u(x_2) - u(x_1)}{x_2 - x_1}$$

for all $x_1 < x_2 < x_3 < x_4$.

$$\int_{-\infty}^{t} (G(x) - F(x))_+ \mathrm{d}x \leq \gamma \int_{-\infty}^{t} (F(x) - G(x))_+ \mathrm{d}x, \ \forall t \in \mathbb{R}$$

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for all $x_1 < x_2 < x_3 < x_4$.

$$\int_{-\infty}^{t} (G(x) - F(x))_+ \mathrm{d}x \leq \gamma \int_{-\infty}^{t} (F(x) - G(x))_+ \mathrm{d}x, \ \forall t \in \mathbb{R}$$

- Light and Perlroth [11] proposed an SD concept consisting of all utility function

$$(u(b) - u(x))^{\frac{1}{\alpha}} \text{ is convex}$$

$\rightarrow$ fail to derive tractable reformulation

- Muller A, Scarsini M, Tsetlin I, Winkler RL. Between first- and second-order stochastic dominance. Management Science, 2016, 63(9): 2933-2947
- Light B, Perlroth A. The family of alpha, [a,b] stochastic orders: Risk vs. expected value. Journal of Mathematical Economics, 2021, 96: 102520

α-concavity

Definition 2 (α-concavity)

A nonnegative function $u(x)$ defined on a convex set $\Xi \subseteq \mathbb{R}^n$ is said to be α-concave, where $\alpha \in [-\infty, +\infty]$, if for all $x, y \in \Xi$ and all $\lambda \in [0, 1]$ the following inequality holds:

$$u(\lambda x + (1-\lambda)y) \geq m_\alpha(u(x), u(y), \lambda),$$

where $m_\alpha : \mathbb{R}_+ \times \mathbb{R}_+ \times [0, 1] \to \mathbb{R}$ is defined as

$$m_\alpha(a, b, \lambda) = 0 \text{ if } ab = 0,$$

and if $a > 0, b > 0, 0 \leq \lambda \leq 1$, then

$$m_\alpha(a, b, \lambda) = \begin{cases} a^\lambda b^{1-\lambda} & \text{if } \alpha = 0, \\ \max\{a, b\} & \text{if } \alpha = \infty, \\ \min\{a, b\} & \text{if } \alpha = -\infty, \\ (\lambda a^\alpha + (1-\lambda)b^\alpha)^{1/\alpha} & \text{otherwise.} \end{cases}$$

α-concavity (Con'd)

Lemma 3 (SDR09)

$u(x)$ is α-concave ($\alpha > 0$) if and only if $u(x)^\alpha$ is concave; $u(x)$ is concave if and only if $u(x)^\alpha$ is $\frac{1}{\alpha}$-concave ($\alpha > 0$).

- In the case of $\alpha = 0$, the function u is called logarithmically concave or log-concave because $\log(u)$ is a concave function.
- In the case of $\alpha = 1$, the function u becomes the usual concave function.
- In the case of $\alpha = -\infty$, the function u is quasi-concave.

A. Shapiro, D. Dentcheva and A. Ruszczyński. *Lecture on Stochastic Programming*. MPS-SIAM, 2009.

α-concavity (Con'd)

$m_\alpha(a, b, \lambda)$ is non-decreasing with respect to α. Thus, we have

Lemma 4 (SDR09)

α-concavity entails β-concavity for all $\beta \leq \alpha$.

Lemma 5 (Theorem 4.19 of SDR09)

If $u(x)$ is α-concave and $g(x)$ is β-concave, here $\alpha, \beta \geq 1$, then $h(x) = u(x) + g(x)$ is γ-concave with $\gamma = \min\{\alpha, \beta\}$.

Lemma 6 (Theorem 4.23 of SDR09)

If $u_i(x)$, $i = 1, \ldots, m$, are α_i-concave and α_i are such that $\sum_{i=1}^m \alpha_i^{-1} > 0$, then $g(x) = \prod_{i=1}^m u_i(x_i)$ is γ-concave with $\gamma = \left(\sum_{i=1}^m \alpha_i^{-1}\right)^{-1}$.

α-concave stochastic dominance

Definition 7 (α-concave stochastic dominance, α-concave SD)

For two bounded random variables $X, Y \in L^{k-1}(\Omega, \mathcal{F}, P; [a, b])$, we say that X is α-concave dominated by Y, $(\alpha \in [-\infty, +\infty])$, if

$$\mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)], \quad \forall\, u \in \tilde{\mathcal{U}}^{\alpha}. \tag{1}$$

Here,

$\tilde{\mathcal{U}}^{\alpha} = \{u \in C([a, b] \to \mathbb{R}_{+}) \mid u$ is monotonically increasing and α-concave$\}$.

- 1-concave SD is equivalent to SSD.
- $-\infty$-concave SD is equivalent to FSD.
- 0-concave SD is log-SD which is generated by the set of all log-concave utility functions

Introduction

Motivating examples:

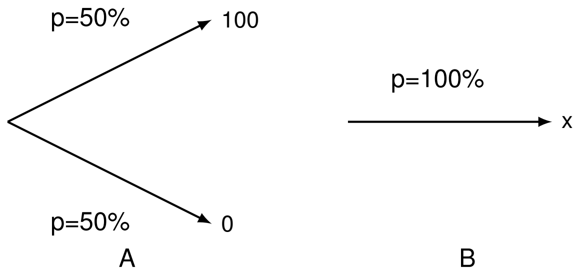

Figure: Two lotteries

$\mathbb{E}[u(A)] = \frac{1}{2}100^{\alpha} \leq \mathbb{E}[u(B)] = x^{\alpha}$ implies $x \geq (\frac{1}{2})^{\alpha}100$

- For some high risk-aversion players, they all prefer B with $x \geq 25$
- For all risk-aversion players, they all prefer B with $x \geq 50$
- For risk-aversion players together with some risk-loving players, they all prefer B with $x \geq 70$

$\frac{1}{k}$-concave SD

We choose $\alpha = \frac{1}{k}$ $(k = 1, 2, \ldots)$

- $\frac{1}{k}$-concave SD defines a stochastic ordering between SSD and log-SD (log-SD is between FSD and SSD)

Denote

$$F_k(W; \vec{\eta}, h) := \mathbb{E}\left[\prod_{t=1}^{h} (\eta_t - a - [\eta_t - W]_+)\right],$$

where $\vec{\eta} = [\eta_1, \ldots, \eta_h]$, $h \le k$, $h \in \mathbb{Z}_+$. Then we have

Theorem 8

For two random variables $X, Y \in L^1(\Omega, \mathcal{F}, P; [a, b])$, X is $\frac{1}{k}$-concave dominated by Y $(k \in \mathbb{Z}_+)$ if and only if

$$F_k(X; \vec{\eta}, h) \le F_k(Y; \vec{\eta}, h),\ \forall\ \vec{\eta} \in [a, b]^h,\ \forall h \le k, h \in \mathbb{Z}_+. \quad (2)$$

Bounded risk-preference

Proposition 3

For any $\frac{1}{k}$*-concave* ($k \in \mathbb{Z}_+$) *function* $u(x)$ *on* $[0, 1]$ *with* $u(0) = 0$ *and* $u(1) = 1$*, we have* $u(x) \geq x^k$ *for all* $x \in [0, 1]$.

- power function x^k is the minimal $\frac{1}{k}$-concave utility function over $[0, 1]$.
- The $\frac{1}{k}$-concave SD involves all concave utility functions as well as some non-concave functions which are lower bounded by the power function x^k.
- The extreme case when $k \to \infty$ is the log-concavity which can be viewed as a kind of bounded risk-aversion/loving

Bounded risk-preference

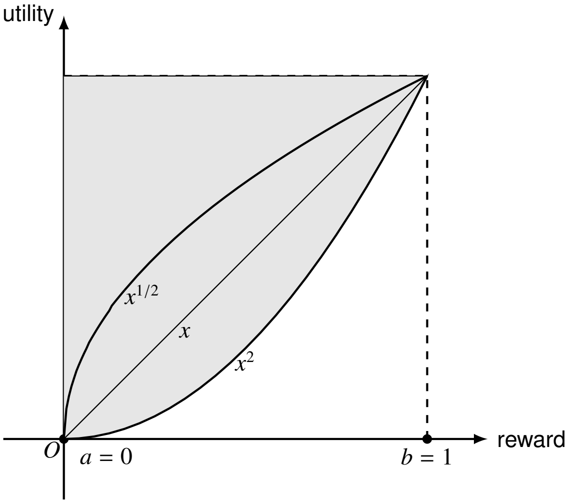

Figure: All power generators of $\frac{1}{2}$-concave SD, i.e., all power utility functions with order not larger than 2

Bounded risk-preference

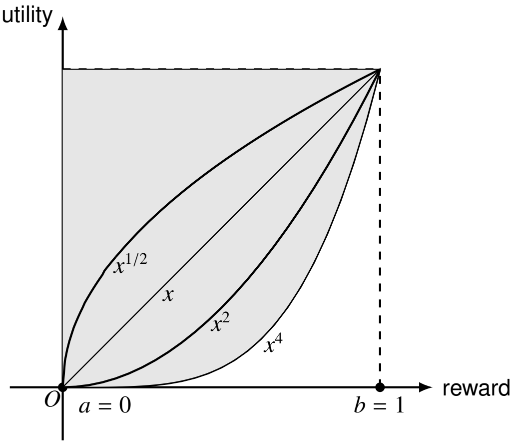

Figure: All power generators of $\frac{1}{4}$-concave SD

the non-concavity of a $\frac{1}{k}$-concave utility function is bounded.

Bounded risk-preference

Proposition 4

For a $\frac{1}{k}$-concave function $u(x)$, $u(x)$ is concave when $\frac{u''(x)}{u'(x)^2}u(x) \leq \frac{k-1}{k}$, and convex when $\frac{u''(x)}{u'(x)^2}u(x) \geq \frac{k-1}{k}$.

- Piecewise power functions are important S-shape utility functions which can characterize the loss/gain dependent risk attitude in prospect theory TvK92.
- We consider an example of piecewise quadratic S-shape utility functions on $[0, 1]$, with a reference point at 0.5, taken from HDB18.

$$u(x) = \begin{cases} 2x^2 & \text{for } x \in [0\,,\ 0.5] \\ 1 - 2(x-1)^2 & \text{for } x \in [0.5\,,\ 1]. \end{cases} \tag{3}$$

S-shape utility function

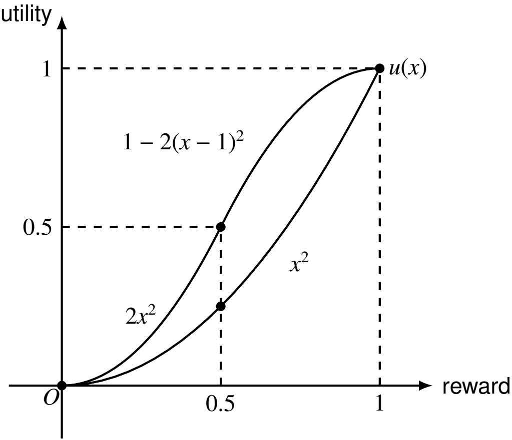

Figure: An $\frac{1}{2}$-concave S-shape utility function and the minimal $\frac{1}{2}$-concave utility function

- Hens T, De Giorgi EG, Bachmann KK. Behavioral Finance for Private Banking: From the Art of Advice

Bounded risk-preference

- $u(x)$ is concave above the reference point but convex below the reference point, i.e., the decision-maker may be irrational (risk-seeking) below the reference point, which is consistent with the prospect theory.
- We would argue that even in the downside part, the irrationality, i.e., the level of risk-seeking, is still limited.
- Mathematically, we require that the level of convexity (non-concavity) is bounded by the $\frac{1}{2}$-concavity.
- By extending the order of power functions in this example from 2 to k, we can deduce that the piecewise k-th order power S-shape utility functions are consistent with the $\frac{1}{k}$-concave SD.
- That is the bounded risk-preference which the $\frac{1}{k}$-concave SD implies.

Interval Stochastic Dominance (FSD,SSD,TSD,ISD-k)

Assume a random variable $W \in L^m(\Omega, \mathcal{F}, P; \mathbb{R})$ with distribution function $F_1(W;\eta) := P(W \leq \eta), \quad \forall\, \eta \in \mathbb{R}$. We define recursively a series of non-decreasing functions,

$$F_k(W;\eta) = \int_{-\infty}^{\eta} F_{k-1}(W;\xi)\, \mathrm{d}\xi, \quad \forall\, \eta \in \mathbb{R}, \quad k = 2, 3, \cdots, m+1.$$

Definition 9

Given two random variables $W, Y \in L^k(\Omega, \mathcal{F}, P; \mathbb{R})$, here $k \in \mathbb{N}$, we say that W intervally stochastically dominates Y to the kth-order if, for given $\beta \in \mathbb{R}$, we have:

$$\begin{cases} F_k(W;\eta) \leq F_k(Y;\eta), & \forall\, \eta \leq \beta, \\ F_{k+1}(W;\eta) \leq F_{k+1}(Y;\eta), & \forall\, \eta \geq \beta. \end{cases} \tag{4}$$

We denote this new dominance order by $W \succeq_{(\beta,k)} Y$. Moreover, we define the feasible set of W intervally dominating Y as

$$A_{(\beta,k)}(Y) := \{W \in L^k(\Omega, \mathcal{F}, P; \mathbb{R}) : W \succeq_{(\beta,k)} Y\}.$$

Motivating example

To clarify the implications of this order, consider the following example. Assume a security market with a market index Y and two portfolios, W and X, with the following return distributions:

- Y follows a uniform distribution on $[-1, 1]$;
- X follows a piecewise uniform distribution on $[-1, 1]$ with density

$$p(x) = \begin{cases} 1/8, & X \in [-1, -0.2], \\ 2, & X \in [-0.2, 0.1], \\ 1/3, & X \in [0.1, 1]; \end{cases}$$

- W follows a piecewise uniform distribution on $[-1, 1]$ with density

$$p(w) = \begin{cases} 1/22, & W \in [-1, 0.1], \\ 1.75, & W \in [0.1, 0.6], \\ 3/16, & W \in [0.6, 1]. \end{cases}$$

Motivation

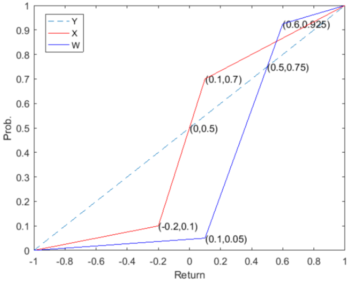

Figure: Cumulative distributions of X, Y and W and SD limitation

Here the distributions of X and W clearly differ and $X \not\succeq_{(1)} Y$, $W \not\succeq_{(1)} Y$; and $X \succeq_{(2)} Y$, $W \succeq_{(2)} Y$; but $X \not\succeq_{(0.5,1)} Y$, $W \succeq_{(0.5,1)} Y$. Interval SD with $\beta = 0.5$ can distinguish the performances of X and W over Y, while neither FSD nor SSD can.

A surely feasible ISD-1 problem

Definition 10

Given two random variables $X, Y \in L^k(\Omega, \mathcal{F}, P; \mathbb{R})$ satisfying $X \succeq_{(\beta,k)} Y$, we define the maximum dominance level $\beta_k(X, Y)$ as the largest possible β such that $X \succeq_{(\beta,k)} Y$ holds. That is,

$$\beta_k(X, Y) = \sup \{\beta \in \mathbb{R} \,|\, X \succeq_{(\beta,k)} Y\}.$$

In the example, we found that the maximum dominance level between X and Y was $\beta_k(X, Y) = 0$, while the maximum dominance level between W and Y was $\beta_k(W, Y) = 0.5$. By choosing a reference point between 0 and 0.5, the ISD constraint could distinguish between X and W over Y.

ISD and SD

Since kth-order SD implies $(k+1)$th-order SD and ISD-k lies between the two, we can easily establish the following relationship:

Proposition 5

For any $\beta \in \mathbb{R}$ we have

$$W \succeq_{(k)} Y \Rightarrow W \succeq_{(\beta,k)} Y \Rightarrow W \succeq_{(k+1)} Y,$$

and

$$A_k(Y) \subseteq A_{(\beta,k)}(Y) \subseteq A_{k+1}(Y)$$

Thus β can span from the kth- to $(k+1)$th-order SD.

SD within ISD ordering

Proposition 6

When $\beta \leq \inf_{y \in supp(Y)} y$, ISD-$k$ is equivalent to $(k+1)$th-order SD. When $\beta \to +\infty$, ISD-k is asymptotically equivalent to kth-order SD. For $k = 1$, when $\beta \geq \sup_{y \in supp(Y)} y$, ISD-$1$ is equivalent to FSD.

Proof.

Proof: First, when $\beta \leq \inf_{y \in supp(Y)} y$, we have $F_k(Y;\eta) = F_{k+1}(Y;\eta) = 0$, $\forall \eta \leq \beta$. It means that, in this case, $W \leq \eta$, for all $\eta \leq \beta$. This implies that the first constraint in (4) is equivalent to $F_{k+1}(W;\eta) \leq F_{k+1}(Y;\eta), \quad \forall\ \eta \leq \beta$. Together with the second constraint in (4), we can obtain the equivalence between ISD-k and kth-order SD.
Second, the equivalence in the limit when $\beta \to +\infty$ can be trivially recovered from (4).
Finally: when $\beta \geq \sup_{y \in supp(Y)} y$, $F_1(Y;\eta) = P(Y \leq \eta) = 1$ for any $\eta \geq \beta$. In this case the first inequality in (4) will imply the FSD constraint. □

ISD-1 and utility theory

From the definition of ISD-1 we have

$$\begin{cases} F_1(W;\eta) \le F_1(Y;\eta), & \forall\ \eta \le \beta, \\ F_2(W;\eta) \le F_2(Y;\eta), & \forall\ \eta \in \mathbb{R}. \end{cases} \quad (5)$$

The second constraint in (5) is SSD, which is equivalent to

$$E[u(W)] \ge E[u(Y)],\ \forall u \in \mathcal{U}_S,$$

where

$$\mathcal{U}_S = \{u : u \text{ is monotone increasing and concave on } \mathbb{R}\}.$$

ISD-1 and utility theory

The first constraint in (5) is equivalent to

$$\int_{x \le \beta} r(x) dP_W(x) \le \int_{y \le \beta} r(y) dP_Y(y), \ \forall r \in \mathcal{R}_F, \tag{6}$$

where $\mathcal{R}_F = \{r : r$ is monotone decreasing on $(-\infty, \beta]\}$. Furthermore (6) is equivalent to

$$E[u(W)] \ge E[u(Y)], \ \forall u \in \mathcal{U}_{F'}, \tag{7}$$

where
$\mathcal{U}_{F'} = \{u : u$ is monotone increasing on $(-\infty, \beta]$, and $u(x) = 0, \ \forall x > \beta\}$.
We call utility functions in $\mathcal{U}_{F'}$ downside utility functions. By introducing $\mathcal{U}_{FP} = \mathcal{U}_S \cup \mathcal{U}_{F'}$, we also have

Proposition 7

ISD-1 is equivalent to

$$E[u(W)] \ge E[u(Y)], \ \forall u \in \mathcal{U}_{FP}.$$

ISD-2 and risk measures

We can extend the previous reasoning to ISD-2 and discuss its relationship with risk measures. Again as SSD implies TSD, ISD-2 is equivalent to

$$\begin{cases} F_2(W;\eta) \le F_2(Y;\eta), & \forall\ \eta \le \beta, \\ F_3(W;\eta) \le F_3(Y;\eta), & \forall\ \eta \in \mathbb{R}. \end{cases} \tag{8}$$

The second constraint in (8) is just TSD.

Proposition 8

The constraint

$$F_2(W;\eta) \le F_2(Y;\eta), \quad \forall\ \eta \le \beta,$$

is equivalent to

$$\rho_{\alpha,\beta}(W) \ge \rho_{\alpha,\beta}(Y), \quad \forall\ \alpha \in [0,1),$$

where

$$\rho_{\alpha,\beta}(W) = \sup_{\eta \le \beta}\{\eta - \frac{1}{1-\alpha}\mathbb{E}[\eta - W]_+\},\ \alpha \in [0,1).$$

Interval Conditional Value-at-Risk

We find that ISD-2 is related to a new risk measure $\rho_{\alpha,\beta}(W)$: we call it *Invertal Conditional Value-at-Risk* (ICVaR). The only difference between ICVaR and CVaR is that the supreme is taken over $(-\infty, \beta]$ rather than over $\mathbb{R}$. We have:

Proposition 9

For $\beta \geq VaR_\alpha(W)$,

$$\rho_{\alpha,\beta}(W) = CVaR_\alpha(W);$$

while for $\beta \leq VaR_\alpha(W)$,

$$\rho_{\alpha,\beta}(W) = \beta - \frac{1}{1-\alpha}\mathbb{E}[\beta - W]_+.$$

Interval Conditional Value-at-Risk

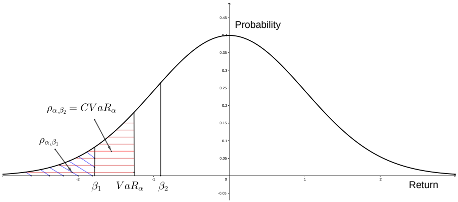

Figure: Interval CVaR and CVaR

Proposition 10

$\rho_{\alpha,\beta}(W)$ is monotone increasing and concave. Moreover,

$$\rho_{\alpha,\beta}(kW) = k\rho_{\alpha,\beta/k}(W), \ (k > 0),$$

$$\rho_{\alpha,\beta}(W + k) = \rho_{\alpha,\beta-k}(W) + k, \ (k \in \mathbb{R}).$$

We can find that, the property of $\rho_{\alpha,\beta}(W)$ is strongly related to β, just like ISD. When $\beta = -\infty$, $\rho_{\alpha,\beta}(W)$ is coherent and degenerates to $CVaR_\alpha(W)$.

ISD-1 and ISD-2 models

Consider an expected terminal wealth. Then, with β as reference point and ruling out short selling, we can formulate the problem as follows:

$$\max_u \quad \mathbb{E}[r^\top u] \tag{9}$$

$$\text{s.t.} \quad r^\top u \succeq_{(\beta,k)} y, \quad k = 1, 2 \tag{10}$$

$$e^\top u = x_0, \tag{11}$$

$$u \geq 0. \tag{12}$$

We denote problem (9) under (10)-(12) by *ISD-k*. We assume that the distribution of r is discrete with samples $r^1, \ldots, r^N$ and probabilities $q_j = \frac{1}{N}$, $j = 1, \ldots, N$.

Interval CVaR model

We consider the following mean-ICVaR model:

$$\max_{u} \quad \mathbb{E}[r^\top u] + \lambda \rho_{\alpha,\beta}(r^\top u) \tag{13}$$

$$\text{s.t.} \quad (11) - (12). \tag{14}$$

where

$$\rho_{\alpha,\beta}(W) = \sup_{\eta \leq \beta}\{\eta - \frac{1}{1-\alpha}\mathbb{E}[\eta - W]_+\}, \ \alpha \in [0, 1).$$

Interval CVaR model -2

When the distribution of r is discrete with samples $r^1, \ldots, r^N$, with probability q_j, $j = 1, \ldots, N$, we have the following reformulation of (13)-(14),

$$\begin{aligned} \max_{u,\eta,\theta} \quad & \lambda\eta + \sum_{j=1}^{N} q_j(u^\top r^j - \frac{\lambda}{1-\alpha}\theta_j) && (15) \\ \text{s.t.} \quad & \eta \leq \beta, && (16) \\ & \theta_j \geq \eta - u^\top r^j, \ j = 1, \ldots, N, && (17) \\ & \theta_j \geq 0, \ j = 1, \ldots, N, && (18) \\ & \text{(11)} - \text{(12)}, && (19) \end{aligned}$$

which is a linear programming problem.
Compared to the traditional mean-CVaR problem, an additional constraint on η, (16), is added in a mean-ICVaR problem.

Asset universe

Table: Statistics of return rates of S&P 500 and sub-indexes in 2007/1/8 - 2018/7/2, weekly data

	SP500	XLU	XLE	XLF	XLK	XLV	XLP	XLY	XLI	XLB
mean(%)	0.140	0.160	0.149	0.165	0.252	0.212	0.181	0.245	0.205	0.176
max(%)	8.13	7.32	17.76	32.52	10.56	8.26	5.79	18.33	13.99	15.17
min(%)	-14.59	-19.8	-25.18	-23.96	-14.65	-18.58	-13.33	-14.71	-15.38	-14.94
var($\times 10^4$)	5.218	5.347	12.378	19.385	6.726	4.950	3.108	8.576	8.490	10.239
skewness	-1.044	-1.388	-0.616	1.184	-0.431	-1.241	-0.934	0.160	-0.169	-0.287
kurtosis	8.17	12.53	9.36	18.23	6.25	12.68	8.91	9.48	6.77	6.35
sharpe ratio	0.061	0.069	0.042	0.037	0.097	0.095	0.103	0.084	0.070	0.055

Portfolio models comparison

We test the following models.

- Portfolio selection under first (FSD), second (SSD) and third (TSD) order SD constraints;
- ISD-1 and ISD-2 for $l = 1, 13, 26, 39, 51$ to span, over 52 weeks in a year, different reference points; and ISD-1 for l determined according to the bisection method;
- Mean-variance (MV), mean-CVaR ($CVaR$) and mean-interval CVaR ($ICVaR$) models.

Problem formulation and solution approach

Table: Settings of the tested models

Model	Formulation	Equiv. Model	settings	Solver
TSD	Bawa et al. (1985)*	convex QP	-	SDPT3**
ISD-2	authors approach	convex QP	$l = 1/13/26/39/51$****	SDPT3**
SSD	Luedtke (2008)*	LP	-	SDPT3**
ISD-1	authors approach	MILP	$l = 1/13/26/39/51$ max-l thrrough bisection m.	CPLEX***
FSD	Luedtke (2008)*	MILP		CPLEX***
*MV******	Markowitz (1952)*	convex QP	$\lambda = 0.2$	SDPT3**
*CVaR******	Rockafellar & Uryasev (2002)*	LP	$\alpha = 90\%, \lambda = 0.2$	SDPT3**
*ICVaR******	(15) s.t.(16-19)	LP	$\alpha = 90\%, \lambda = 0.2 , l = 5$	SDPT3**

*: The model is with transaction constraints (11-12).
**: SDPT3 solver is called by CVX v2.1 in MATLAB R2016a in a PC with 3.4GHz CPU and 16.0GB memory.
***: CPLEX v12.8 is run in MATLAB R2016a in a PC with 3.4GHz CPU and 16.0GB memory.
****: Corresponding β is the lth-smallest historical sample of S&P 500.
*****: All the three mean-risk models take the mean-λ·risk form.

Model validation

We intend to validate the introduced ISD paradigm against the market benchmark, the S&P500 index, by:

- Analysing the diversification properties of optimal portfolios generated by alternative ISD models relative to a set of current portfolio approaches. Similarly for the specification of the $ICVaR$ model against $CVaR$ and MV problems.
- Comparing a range of ISD-1 and ISD-2 optimal portfolios for different values of β (lth-quantile): aim of this comparison is to discriminate between ISD-1 or ISD-2 optimal allocations and analyse their consistency againt canonical optimal FSD or SSD portfolios.

Portfolio diversification

Table: Average values of Herfindahl-Hirschman Index (*HHI*), Shannon entropy (*entropy*) and proportions invested in risk-free asset, of out-of-sample weekly portfolios over 2008/1/7-2018/7/2

	TSD	*ISD*-2					*SSD*	*MV*	
		$l=1$	$l=13$	$l=26$	$l=39$	$l=51$			
*HHI**	0.723	0.721	0.701	0.685	0.685	0.700	0.704	0.980	
*Entropy**	0.441	0.444	0.482	0.511	0.512	0.483	0.476	0.030	
*Inf.Weeks****	5	5	5	5	5	19	20	0	
Inf.Rate	0.92%	0.92%	0.92%	0.92%	0.92%	3.48%	3.66%	0%	
*Feas.HHI***	0.720	0.719	0.698	0.682	0.682	0.689	0.692	0.980	
*Feas.Entropy***	0.445	0.449	0.486	0.516	0.516	0.500	0.494	0.030	
RF	14.21%	14.21%	14.85%	15.18%	15.29%	17.06%	17.27%	8.88%	
*Feas.RF***	13.42%	13.42%	14.06%	14.40%	14.50%	14.07%	14.12%	8.88%	
	ISD-1						*FSD*	$CVaR_{90\%}$	$ICVaR_{90\%,5}$
	$l=1$	$l=13$	$l=26$	$l=39$	$l=51$	max-l		$\lambda=0.2$	$\lambda=0.2, l=5$
*HHI**	0.712	0.624	0.695	0.792	0.956	0.559	0.973	0.866	0.697
*Entropy**	0.459	0.646	0.558	0.383	0.083	0.856	0.052	0.215	0.476
Inf.Weeks	20	41	174	330	499	20	525	0	0
Inf.Rate	3.66%	7.51%	31.87%	60.44%	91.39%	3.66%	96.15%	0%	0%
*Feas.HHI***	0.701	0.594	0.552	0.474	0.486	0.542	0.306	0.866	0.697
*Feas.Entropy***	0.477	0.698	0.819	0.967	0.966	0.888	1.363	0.215	0.476
RF	17.18%	22.81%	46.63%	66.71%	92.25%	35.63%	96.82%	61.90%	16.28%
*Feas.RF***	14.03%	16.55%	21.67%	15.86%	9.94%	33.18%	17.41%	61.90%	16.28%

*: *HHI* and *Entropy* give the average *HHI* and *Entropy* values of all optimal weekly portfolios over 2008/1/7-2018/7/2.
When a model is infeasible, the portfolio is all in risk-free asset and the *HHI* is 1 at that week.

**: *Feas.HHI*, *Feas.Entropy* and *Feas.RF* account the average *HHI*, *Entropy* and *RF* values when the model is feasible.

***: *Inf*. accounts when the solver CPLEX returns '-2: No feasible point was found' or SDPT3 returns 'Failed' or 'Infeasible'.

Model comparison

	ISD-3.0(TSD)	ISD-2.9808	ISD-2.75	ISD-2.5	ISD-2.25	ISD-2.0192	ISD-2.0(SSD)	ISD-1.9808	ISD-1.75	ISD-1.5	ISD-1.25	ISD-1.0192	ISD-1.min	ISD-1.0(FSD)
ISD-3.0(TSD)	1.00	0.96	0.59	0.55	0.55	0.56	0.56	0.60	0.30	0.16	0.06	0.06	0.03	0.07
ISD-2.9808	0.96	1.00	0.59	0.55	0.55	0.55	0.55	0.60	0.30	0.16	0.06	0.06	0.03	0.07
ISD-2.75	0.59	0.59	1.00	0.62	0.62	0.62	0.62	0.65	0.29	0.16	0.06	0.06	0.03	0.07
ISD-2.5	0.55	0.55	0.62	1.00	0.71	0.68	0.69	0.74	0.28	0.16	0.06	0.06	0.04	0.07
ISD-2.25	0.55	0.55	0.62	0.71	1.00	0.75	0.75	0.79	0.28	0.16	0.06	0.06	0.04	0.07
ISD-2.0192	0.56	0.55	0.62	0.68	0.75	1.00	0.78	0.82	0.31	0.18	0.08	0.08	0.06	0.09
ISD-2.0(SSD)	0.56	0.55	0.62	0.69	0.75	0.78	1.00	0.84	0.31	0.18	0.08	0.08	0.06	0.09
ISD-1.9808	0.60	0.60	0.65	0.74	0.79	0.82	0.84	1.00	0.33	0.19	0.08	0.08	0.07	0.09
ISD-1.75	0.30	0.30	0.29	0.28	0.28	0.31	0.31	0.33	1.00	0.36	0.18	0.12	0.09	0.13
ISD-1.5	0.16	0.16	0.16	0.16	0.16	0.18	0.18	0.19	0.36	1.00	0.45	0.33	0.17	0.33
ISD-1.25	0.06	0.06	0.06	0.06	0.06	0.08	0.08	0.08	0.18	0.45	1.00	0.59	0.25	0.59
ISD-1.0192	0.06	0.06	0.06	0.06	0.06	0.08	0.08	0.08	0.12	0.33	0.59	1.00	0.24	0.95
ISD-1.min	0.03	0.03	0.03	0.04	0.04	0.06	0.06	0.07	0.09	0.17	0.25	0.24	1.00	0.21
ISD-1.0(FSD)	0.07	0.07	0.07	0.07	0.07	0.09	0.09	0.09	0.13	0.33	0.59	0.95	0.21	1.00

Figure: Correlations between different models (number of weeks in which the optimal portfolios of two models are the same)

In-sample evidences

We summarise the following relevant evidences from Table 3, Fig. 4 and Table **??**:

- The FSD-constrained problem is hardly feasible thus not a viable model. Similar evidence for the ISD-1 model when $l = 51, 39, 26$ with infeasibility rates that make these models impractical.
- The model with max-l selection proves to overcome the problem and among all optimal portfolios has also the associated higher diversification on average.
- ISD-1 for $l = 1$, SSD and ISD-2 for $l = 51$ show very similar statistics and relatively good diversification properties.
- For increasing l: ISD-2 and ISD-1 optimal portfolios do actually span respectively from TSD to SSD and from SSD to FSD optimal portfolios.
- Among MV, CVaR and ICVaR optimal portfolios, all solved to optimality, the last one with the higher diversification while MV optimal portfolios often lead to concentrated *corner* solutions.
- Optimal TSD portfolios can be regarded as equivalent to ISD-2 optimal portfolios for $l = 1$. Same with respect to SSD-constrained portfolios considering ISD-2 for $l = 51$ or ISD-1 for $l = 1$.
- ISD-2 optimal portfolios for $l \in \{13, 26, 39\}$ show very similar composition with negligible differences.
- ISD-1 optimal portfolios provide for increasing l on average different optimal portfolios when solved to optimality. The ISD-1 average optimal portfolios with max-l have their own specific structure.
- Mean-risk models on average produce different optimal portfolios.

Out-of-sample results

Table: Out-of-sample weekly returns statistics over 2008/1/7-2018/7/2

Model	Mean (%)	Std. (%)	Sharpe (%)	$CVaR_{95\%}$ (%)	$CVaR_{90\%}$ (%)	$E(ER)_+$ (%)	$E(ER)_-$ (%)
S&P500	0.166	2.300	0.072	-5.833	-4.455	0	0
TSD	0.138	1.952	0.071	-4.876	-3.719	0.850	-0.877
ISD-2 ($l = 13$)	0.137	1.917	0.072	-4.799	-3.662	0.842	-0.870
SSD	0.136	1.860	0.073	-4.590	-3.511	0.844	-0.873
ISD-1 ($l = 13$)	0.103	1.742	0.059	-4.378	-3.344	0.788	-0.851
ISD-1 (max-l)	0.172	1.693	0.101	-4.038	-3.085	0.793	-0.787
$CVaR_{90\%}$ ($\lambda = 0.2$)	0.021	1.107	0.019	-3.197	-2.270	0.748	-0.892
$ICVaR_{90\%}$ ($\lambda = 0.2, l = 5$)	0.148	1.772	0.083	-4.450	-3.381	0.844	-0.862
mixed ISD-1-2-ICVaR	0.169	1.566	0.108	-3.724	-2.926	0.828	-0.825

2008-2018 out-of-sample return distributions

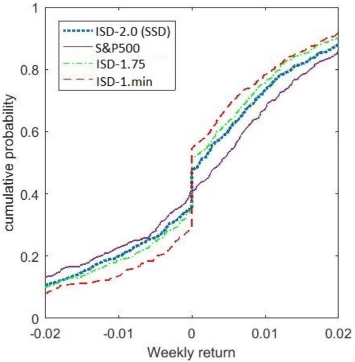

Figure: *ISD*-1 ($l = 13$ and max-l) and *SSD* models

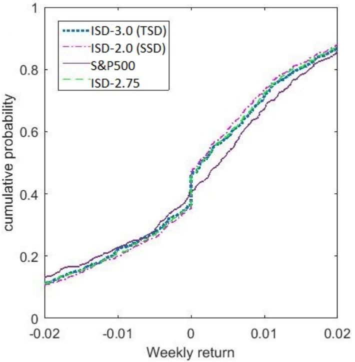

Figure: *SSD*, *ISD*-2 ($l = 13$) and *TSD* models

Summary evidences

- Compared to the S&P500 benchmarks all the proposed portfolio models exhibit a good downside control as can be observed from the ex-post returns' standard deviations, CVaR $(90\%, 95\%)$ and mean excess returns below the benchmarks.
- The ISD-1 (max-l) model generates consistently the highest and the CVaR model the least risk-adjusted returns – return per unit volatility (Sharpe ratio).
- Relative to the minimum CVaR (90%) portfolio the minimum ICVaR $(90\%, 0.2)$ portfolios produces consistently a positive average excess return and thus a higher risk-adjusted return.
- ISD-2, TSD and SSD optimal portfolios produce comparable market performances in terms of tail risk control and risk-adjusted returns.
- The introduction of a volatility-based early-warning-signal to discriminate between markets' volatility regimes and associated ISD-based portfolio models can further enhance market performance and downside control.

Conclusion

- The main contribution of this work is related to the extension of the canonical order over probability measures induced by stochastic dominance (SD).
- For $k = 1$ we have shown that ISD-1 allows the order relationship to hold for all monotone increasing utilities which on the left part of the support may be concave or convex and surely concave beyond the ISD reference point.
- For $k = 2$, ISD-2 was shown to lead to a new risk measure, the Interval Conditional Value-at-Risk, whose relationship with the VaR and the CVaR has been analysed.
- For the case of discrete random variables, a computationally efficient reformulation of the ISD-k constraints leading to convex programs that can be solved efficiently for the cases of 1^{st} and 2^{nd} ISD spanning for varying reference points from first to third SD degrees.
- As application domain, we have considered a portfolio selection problem with ISD-k constraints, for $k = 1, 2$ and a risk-reward trade-off model based on expected return and ICVaR as risk measure. The associated portfolio models do contribute to already established portfolio optimization models.

Conclusions

Summary:

- We proposes two new SD concepts which span a continuous spectrum of the SD relationship between integer-order SDs
- We study the reformulation and examples for the case between SSD and log-SD.

Further works:

- (Portfolio) optimization with $\frac{1}{k}$-concave SD constraints
- reformulations of α-concave SD when $\alpha > 1$ and $\alpha < 0$
- dynamic extension of Interval SD
- multivariate version of α-concave SD which can cover some important utility functions like the Cobb-Douglas utility function

Thank you!