A Sustainability-Oriented Enhanced Indexation Model with Regime Switching and Cardinality Constraint

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Joint work with Zhiping Chen & Xinkai Zhuang (Xi'an Jiaotong University)

(Shanghai, JORSC-FERM, 2019.8.29)

Index tracking

- Index tracking problems aim to replicate the holdings and performance of a designated index, which is a popular form of passive fund management.
- A classic index tracking model (L_2-norm):

$$\begin{aligned} \min_{x} \quad & \mathbb{E}(r^\top x - r_I)^2 \\ \text{s.t.} \quad & e_n^\top x = 1, \end{aligned}$$

- Index tracking model penalties deviations of portfolio return both above and below the benchmark index. Others: L_1-norm...

Enhanced indexation

- Enhanced indexation model penalties deviations only below the benchmark index: a positive portfolio management
- An classic enhanced indexation model:

$$\begin{aligned} \max_{x} \quad & \mathbb{E}[r^\top x - r_I] \\ \text{s.t.} \quad & \mathbb{E}[(r^\top x - r_I)_-] \le \alpha, \\ & e_n^\top x = 1. \end{aligned}$$

- Tracking error measures: mean-absolute deviation; chance constraints; conditional Value-at-Risk; stochastic dominance.

Enhanced indexation(Con'd)

- Dose and Cincotti (2005PASMA) desire a relatively high excess return within a reduced tracking error by adopting historical look-back approach.
- Konno and Hatagi (2005JIMO) use a mean-absolute deviation objective in an index-plus-alpha model and get the linearly constrained concave minimization.
- Goel et al. (2018JCAM) use mixed conditional Value-at-Risk measure to track the index.
- Roman et al. (2013EJOR) study the second-order stochastic dominance based models and demonstrate an empirical study in enhanced indexation problem.
- Xu et al. (2018JGO) present a sparse enhanced indexation model with chance constraints and cardinality constraints.
- Canakgoz and Beasley (2009EJOR) formulate the enhanced indexation problem as mixed-integer linear programming.

Our motivation

Motivation: sustainability

- Stable performance in long-run
  - Model adaptive to market environment
  - Non-fixed target
  - Regime switching
- Reduce transaction complexity
  - Limit size of portfolio set
  - Cardinality constraints: L_0 norm

Regime switching

Regime switching (Hamilton, 1989): nonlinear dynamic change of the market environment

- observation switching
- Markov switching

We assume that the regime switching is Markovian, with J finite states, and stationary.

- current regime is s_0; regime forthcoming in next period is s_f.
- $Q_{s^i s^j} := Q\{s_f = s^j : s_0 = s^i\}$: transition probability from regime s^i in the current period to regime s^j in the next period.

$$Q = \begin{bmatrix} Q_{s^1 s^1} & Q_{s^1 s^2} & \cdots & Q_{s^1 s^J} \\ Q_{s^2 s^1} & Q_{s^2 s^2} & \cdots & Q_{s^2 s^J} \\ \cdots & \cdots & \cdots & \cdots \\ Q_{s^J s^1} & Q_{s^J s^2} & \cdots & Q_{s^J s^J} \end{bmatrix}.$$

Enhanced indexation model

- Regime-based cardinality constrained enhanced indexation model

$$
\text{(RCEI)} \quad \begin{array}{ll} \max\limits_{x} & \mathbb{E}[R^\top x - R_I(s_f)] \\ \text{s.t.} & \mathbb{E}[(R^\top x - R_I(s_f))_- : s_f = s^j] \leq \alpha(s^j),\ j = 1, 2, \ldots, J, \\ & e_N^\top x = 1, \\ & \|x\|_0 \leq K(s_0), \end{array}
$$

- $R_I(s_f)$: return rate of the market index whose distribution relies on the forthcoming market regime s_f.
- Cardinality upper bound $K(s_0)$: rely on current market regime s_0.

Sample reformulation

- S: set of T historical observations of R, $\{r_1, r_2, \ldots, r_T\}$,
- $s(t)$: market regime to which the t-th observation belongs.
- S_j: observations under the j-th market regime, $S = \bigcup_{1,2,\ldots,J} S_j$

$$
\begin{aligned}
\max_{x} \quad & \sum_{j=1}^{J} \left(Q(s_0, s^j) \frac{1}{T_j} \sum_{t \in S_j} (r_t^\top x - r_I(s(t))) \right), \\
\text{s.t.} \quad & \frac{1}{T_j} \sum_{t \in S_j} \left[(r_t^\top x - r_I(s(t)))_- \right] \le \alpha(s^j), \ j = 1, 2, \ldots, J, \\
& e_N^\top x = 1, \\
& \|x\|_0 \le K(s_0).
\end{aligned}
$$

Further reformulation

- Further reformulation

$$
\begin{aligned}
\max_{x,y} \quad & \sum_{j=1}^{J} \left(Q(s_0, s^j) \frac{1}{T_j} \sum_{t \in S_j} r_t^\top x \right), \\
\text{s.t.} \quad & \frac{1}{T_j} \sum_{t \in S_j} y_t \leq \alpha(s^j), \ j = 1, 2, \ldots, J, \\
& y_t \geq r_I(s(t)) - r_t^\top x, \ t = 1, 2, \ldots, T, \\
& y_t \geq 0, \ t = 1, 2, \ldots, T, \\
& e_N^\top x = 1, \\
& \|x\|_0 \leq K(s_0).
\end{aligned}
$$

- Linear programming with an L_0 norm constraint

Solution method

Current approaches

- Reformulated by MIP (Canakgoz and Beasley, 2003EJOR; Guastaroba and Speranza, 2012EJOR)
- Relax L_0 norm into L_1-norm (Bruckstein, 2009SIAMrew), L_p-norm (Chen et al. 2013) or $L_{\frac{1}{2}}$-norm (Zhao et al., 2019JORS)
- Nonmonotone projected gradient method (Xu, Lu, Xu, 2015OMS)
- Heuristic method (Chang, et al. 2000COR)
- DC approximation (Gulpinar et al. 2010OPTIM)
- Cutting plance (Roman et al. 2013EJOR)

Our approach

- Block structure: ADMM (not converge for more than 3 blocks, Chen, He, Ye, Yuan, 2016MP) ⇒ Proximal ADMM (He, Yuan, Chen, Ye, Wang, Zhang, Xu, Hong, Luo et al.)
- Motivated by GY Li & TK Pong, 2015SIAMOPT (Non-constrained; smooth part + non-smooth part); Chen et al. 2017;
- Extended to constrained problem (relaxation with partial penalty) with smooth part + non-smooth part (L_0-norm)

Solution method

- Rewrite (RCEI)

$$
\begin{aligned}
\max_{x,w} \quad & \sum_{j=1}^{J} \left(Q(s_0, s^j) \frac{1}{T_j} \sum_{t \in S_j} r_t^\top A w \right), \\
\text{s.t.} \quad & \frac{1}{T_j} \sum_{t \in S_j} B_t w \le \alpha(s^j), \; j = 1, 2, \ldots, J, \\
& B_t w \ge r_I(s(t)) - r_t^\top A w, \; t = 1, 2, \ldots, T, \\
& B w \ge 0, \\
& e_N^\top A w = 1, \\
& A w - x = 0, \\
& \|x\|_0 \le K(s_0).
\end{aligned}
$$

Solution method

- Rewrite (RCEI) shortly

$$
\begin{aligned}
\min_{x,w} \quad & c^\top w + \delta_C(x), \\
\text{s.t.} \quad & \bar{A}w \le \bar{b}, \\
& \bar{C}w = \bar{d}, \\
& Aw - x = 0.
\end{aligned}
$$

Solution method

- partial penalty

$$\begin{aligned}\min_{x,w,u,v} \quad & c^\top w + \mu(\|u_+\|^2 + \|v\|^2) + \delta_C(x), \\ \text{s.t.} \quad & \bar{A}w - u = \bar{b}, \\ & \bar{C}w - v = \bar{d}, \\ & Aw - x = 0,\end{aligned}$$

- augmented Lagrangian function

$$\mathcal{L}_\mu(x, w, u, v, \lambda; \beta) = c^\top w + \mu(\|u_+\|^2 + \|v\|^2) + \delta_C(x)$$
$$+\lambda^T\left(\mathcal{A}w + \mathcal{B}\begin{bmatrix}u\\v\end{bmatrix} + \mathcal{D}x - b\right) + \frac{\beta}{2}\left\|\mathcal{A}w + \mathcal{B}\begin{bmatrix}u\\v\end{bmatrix} + \mathcal{D}x - b\right\|^2,$$

Solution method

- **Step 1.** Given an initial point $(x^0, w^0) \in \mathbb{R}^N \times \mathbb{R}^{N+T}$, the outer tolerance $\epsilon_O > 0$, $\gamma > 1$. Let $k = 1$, $\mu^k > 0$;
- **Step 2.** Solve the enhanced indexation subproblem (7) by the proximal ADMM algorithm:
  - **Step 2.1.** Given $\mu = \mu^k$, the inner tolerance $\epsilon_I > 0$, $\sigma > 0$. Let $\beta > 0$, $(x^0, w^0) = (x^{k-1}, w^{k-1})$, $u^0 = \bar{A}w^0 - \bar{b}$, $v^0 = \bar{C}w^0 - \bar{d}$, $\lambda^0 \in \mathbb{R}^{J+2T+N+1}$, $i = 0$;
  - **Step 2.2.** Perform the $(i+1)$-th iteration as follows:

$$\begin{cases} x^{i+1} & = \arg\min_{x} \mathcal{L}_\mu(x, w^i, u^i, v^i, \lambda^i; \beta), \\ w^{i+1} & = \arg\min_{w} \{\mathcal{L}_\mu(x^{i+1}, w, u^i, v^i, \lambda^i; \beta) + \frac{\sigma}{2}\|w - w^i\|^2\}, \\ \begin{bmatrix} u^{i+1} \\ v^{i+1} \end{bmatrix} & = \arg\min_{u,v} \mathcal{L}_\mu(x^{i+1}, w^{i+1}, u, v, \lambda^i; \beta), \\ \lambda^{i+1} & = \lambda^i + \beta(\mathcal{A}w^{i+1} + \mathcal{B}\begin{bmatrix} u^{i+1} \\ v^{i+1} \end{bmatrix} + \mathcal{D}z^{i+1} - b), \end{cases}$$

  - **Step 2.3.** If the inner stopping criterion

$$\max\{\mu, \beta, 1\}(\|w^{i+1} - w^i\| + \|(\lambda^{i+1} - \lambda^i)/\beta\|) \le \epsilon_I$$

    is satisfied, stop the proximal ADMM algorithm and go to Step 3; otherwise, let $\beta = \gamma\beta$, $i = i + 1$ and go to Step 2.2;
- **Step 3.** If the outer stopping criterion

$$\|\mathcal{A}w^k + \mathcal{B}\begin{bmatrix} u^k \\ v^k \end{bmatrix} + \mathcal{D}x^k - b\| \le \epsilon_O$$

  is satisfied, stop the algorithm and return the approximate optimal solution (x^k, w^k); otherwise, let $\mu^{k+1} = \gamma\mu^k$ and go to Step 2 with $k = k + 1$.

Numerical evidence

Table 1. Test settings.

	Constituent Stocks	Benchmark	K	In-Sample Weeks	Out-of-Sample Period (Weeks)
Test 1	100	FTSE 100	5, 10, 20	50	2004.12.23–2018.12.27 (732)
Test 2	500	S&P 500	5, 10, 20	50	2007.12.24–2018.12.31 (576)

- Rolling forward out-of-sample tests with moving window
- Adopt method in Liu and Chen (2018EJOR) to determine market regimes based on the average values of the market index over a time window.

Numerical evidence

Transaction probability matrix of regime (assume to be stationary)

$$Q_{\text{FTSE}} = \begin{bmatrix} 0.9023 & 0.0951 & 0.0026 \\ 0.2482 & 0.5674 & 0.1844 \\ 0.0237 & 0.1361 & 0.8402 \end{bmatrix}, \quad Q_{\text{S\&P}} = \begin{bmatrix} 0.9342 & 0.0608 & 0.0051 \\ 0.2933 & 0.5467 & 0.1600 \\ 0.0310 & 0.0775 & 0.8915 \end{bmatrix}$$

Stable to stay in the bull or bear regime; but instable to stay in consolidation regime

Table 2. Mean and standard deviation of out-of-sample return rates under different market regimes.

Index	Mean (s^1)	Mean (s^2)	Mean (s^3)	Mean	Std (s^1)	Std (s^2)	Std (s^3)	Std
FTSE 100	0.0036	0.0010	−0.0063	0.0005	0.0187	0.0195	0.0318	0.0224
S&P 500	0.0033	0.0014	−0.0071	0.0009	0.0190	0.0240	0.0396	0.0254

Noticeably different among three market regimes

Numerical evidence

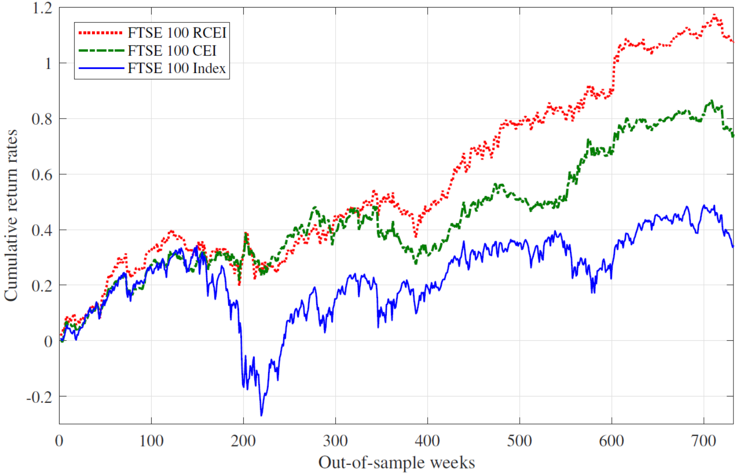

Figure 1. Out-of-sample performances of RCEI and CEI models in FTSE 100.

Numerical evidence

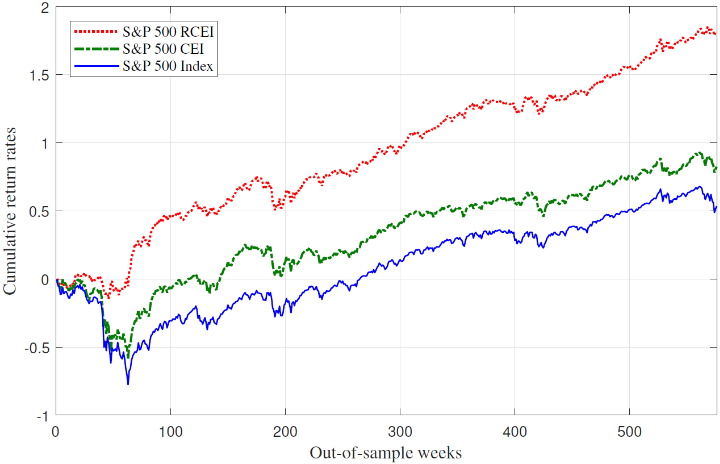

Figure 2. Out-of-sample performances of RCEI and CEI models in S&P 500.

Numerical evidence

Table 3. Statistics of out-of-sample portfolio return rates of RCEI and CEI models.

Index	Model	Mean	Std	SR	MDD	TE_+	TE_-	Time(s)
FTSE 100	RCEI	0.0015	0.0158	0.0931	0.2522	0.0067	0.0057	1.07
	CEI	0.0010	0.0143	0.0709	0.5918	0.0063	0.0057	1.10
	Index	0.0005	0.0224	0.0209	0.7717	-	-	-
S&P 500	RCEI	0.0031	0.0220	0.1425	0.1986	0.0060	0.0038	18.79
	CEI	0.0014	0.0258	0.0553	0.2104	0.0041	0.0036	18.94
	Index	0.0009	0.0254	0.0364	0.6117	-	-	-

- Regime switching helps, especially in maximum drawdown.

Numerical evidence

Table 5. Statistics of out-of-sample return rates of RCEI and REI models.

Index	Model	Mean	Std	SR	TE_+	TE_-	Stocks	Time(s)
FTSE 100	RCEI	0.0015	0.0158	0.0931	0.0067	0.0057	9.36	1.07
	REI	0.0017	0.0173	0.0974	0.0071	0.0061	88.75	0.06
S&P 500	RCEI	0.0031	0.0220	0.1425	0.0060	0.0038	9.67	18.79
	REI	0.0036	0.0282	0.1277	0.0104	0.0083	383.17	0.83

- Adding cardinality constraint significantly reduce the number of really invested stocks without losing too much performance.

Conclusions

- Propose enhanced indexation model with regime switching and cardinality constraint
- Solution method by the partial penalty proximal ADMM
- Numerical evidences examine the necessariness of regime switching and efficiency of cardinality constraint

Further work

- Static $\Rightarrow$ dynamic
- Other tracking error measures

Thank you!