Solution to problems of Chapter 4

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Relation between the units:

1 ft=0.3048m
1lb=0.454 kg
1lb/ft^2=47.89N/m^2=47.89 Pa
1^{o}R=5/9K

4.1 Consider an oblique shock wave with wave angle equal to 35^o. Upstream of the wave $p_1 = 2000lb/ft^2$, $T_1 = 520^oR$, and $V_1 = 3355ft/s$. Calculate p_2, T_2, V_2, and the flow deflection angle.

Solution:

The Mach number in front of the oblique shock is

$$M_1 = \frac{V_1}{a_1} = \frac{V_1}{\sqrt{\gamma RT_1}} = 3.0$$

For M_1=3.0 and $\beta = 35^o$, from the $\theta - \beta - M_1$ relation, we know the flow deflection angle $\theta = 18^o(\beta = 35.47^o)$.
For $M_{n1} = M_1 \sin\beta = 1.72$.
From the normal shock relation

$$M_{n2}^2 = \frac{2 + M_{n1}^2(\gamma - 1)}{2\gamma M_{n1}^2 - (\gamma - 1)}$$

we have M_{n2}=0.6355 and $p_2/p_1 = 3.285$ and $T_2/T_1 = 1.473$.
Thus

$$M_2 = \frac{M_{n2}}{\sin(\beta - \theta)} = \frac{0.6355}{\sin(35^o - 18^o)} = 2.17$$

$$p_2 = 3.285p_1 = 3.285 \times 2000 \times 47.89Pa = 3.15 \times 10^5 Pa$$

$$T_2 = 1.473T1 = 1.473 \times 520 \times \frac{5}{9} = 425.5K$$

$$V_2 = M_2 \times a_2 = M_2 \times \sqrt{\gamma RT_2} = 897.3m/s$$

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4.2 Consider a wedge with a half angle of 10^o flying at Mach 2. Calculate the ratio of total pressure across the shock wave emanating from the leading edge of the wedge.

Solution:

The total pressure across the oblique shock is determined by the Mach numbers in front and behind the shock. From $\theta - \beta - M_1$ relation, we have β=39.32^o for $M = 2$ and $\theta = 10^o$. $Mn_1 = M_1 \sin\beta = 1.267$.
From the normal shock relation, we have $M_{n2} = 0.8071$ (for $M_{n1} = 1.26$). Thus $M_2 = M_{n2}/\sin(\beta - \theta) = 1.65$.

$$\frac{p_{02}}{p_{01}} = \frac{p_{02}}{p_2}\frac{p_2}{p_1}\frac{p_1}{p_{01}} = (1 + \frac{\gamma-1}{2}M_2^2)^{\gamma/(\gamma-1)}(1 + \frac{2\gamma}{\gamma+1}(M_{n1}^2 - 1))(1 + \frac{\gamma-1}{2}M_1^2)^{-\gamma/(\gamma-1)} = 0.9864$$

4.3 Calculate the maximum surface pressure (in units of Newton per square meter) that can be achieved on the forward face of a wedge flying at Mach 3 at standard sea level conditions ($p_1 = 1.01 \times 10^5$ N/m^2) with an attached shock wave.

Solution:

From the $\theta - \beta - M_1$ relation, it can be seen that for a given Mach number, there is a maximum wave deflection angle β beyond which the shock will be detached.
The larger the wave deflection angle β, the larger the pressure behind the oblique shock.
For $M = 3$, the maximum wave angle is $\beta = 65.24^o$ and $p_2/p_1 = 8.492$. The half angle of the wedge is then $\theta = 34.07^o$.
Thus the maximum pressure is $p_2 = 8.492 \times p_1 = 8.492$ atm.

4.4 In the flow past a compression corner, the upstream Mach number and pressure are 3.5 and 1 atm, respectively. Downstream of the corner, the pressure is 5.48 atm. Calculate the deflection angle of the corner.

Solution:

The pressure ratio across the oblique shock wave is $p_2/p_1 = 5.48$. From the normal shock relation, we know $M_{n1} = M_1 \sin\beta = 2.2$.
Thus $\sin\beta = M_{n1}/M_1 = 2.2/2.5 = 0.88 \Rightarrow \beta = 38.95^o$.
From $\theta - \beta - M_1$ relation, we know $\theta = 23.63^o$.

4.5 Consider a 20^o half angle wedge in a supersonic flow at Mach 3 at standard sea level ($p_1 = 2116 lb/ft^2 = 1 atm$ and $T_1 = 519^o R = 288K$). Calculate the wave angle, and the surface pressure, temperature, and Mach number.

Solution:

From the $\theta - \beta - M_1$ relation for $M_1 = 3$ and $\theta = 20^o$, we know the wave angle $\beta = 37.76^o$ and $M_2 = 1.994$.
Thus $M_{n1} = M_1 \sin\beta = 1.84$.
From normal shock relation, we know $T_2/T_1 = 1.562$ and $p_2/p_1 = 3.783 \Rightarrow p_2 = 3.783$ atm and $T_2 = 449.9$K

4.6 A supersonic stream at $M_1 = 3.6$ flows past a compression comer with a deflection angle of 20^o. The incident shock wave is reflected from an opposite wall which is parallel to the upstream supersonic flow, as sketched in Fig. 4.18. Calculate the angle of the reflected shock relative to the straight wall.

Solution:

From the $\theta-\beta-M_1$ relation, we know the wave angle for M_1= 3.6 and $\theta = 20^o$ is $\beta = 34.11^o$ and $M_2 = 2.40$.
For the reflected wave with compression angle of θ and $M_2 = 2.40$, we know the angle between the reflected wave and the incident wave is $\beta_2 = 44.34^o$.
Thus the angle between the reflected wave and the straight wall is $\beta_2 - \theta = 24.32^o$.

4.7 An incident shock wave with wave angle= 30^o impinges on a straight wall. If the upstream properties are $M_1 = 2.8, p_1 = 1$ atm, and $T_1 = 300$ K, calculate the pressure, temperature, Mach number, and total pressure downstream of the reflected wave.

Solution:

$M_{n1} = M_1 \sin\beta_1 = 2.8 \times \sin 30^o = 1.4$.

$$M_{n2}^2 = \frac{2 + M_{n1}^2(\gamma-1)}{2\gamma M_{n1}^2 - (\gamma-1)} = 0.5472 \Rightarrow M_{n2} = 0.74$$

$$M_2 = \frac{M_{n2}}{\sin(\beta_1 - \theta_1)} = \frac{0.74}{\sin 18^o} = 2.4$$

From the $\theta-\beta-M_1$ relation, we know

$$\theta_1 = 12^o$$

; From the normal shock relation, we know

$$p_2/p_1 = 2.12, \quad T_2/T_1 = 1.255, \quad and \quad p_{02}/p_{01} = 0.9582$$

Now calculate the flow properties of the reflected wave:
The incoming flow condition of the reflected wave is $M_2 = 2.4$ and $\theta_2 = 12^o$.
From the $\theta-\beta-M_1$ relation, we know

$$\beta_2 = 35.01^o$$

Thus $M_{2n}^{'} = M_2 \sin\beta_2 = 2.4 \times \sin 35.01^o = 1.38$.
From normal shock relation, we have $M_{n3} = 0.75 \Rightarrow M_3 = M_{n3}/\sin(\beta_2 - \theta_2) = 0.72/\sin(23^o) = 1.92$,

$$p_3/p_2 = 2.055, \quad T_3/T_2 = 1.242, \quad p_{03}/p_{02} = 0.9630$$

$$p_3 = \frac{p_3}{p_2}\frac{p_2}{p_1}p_1 = 2.055 \times 2.12 \times 1 = 4.36atm$$

$$T_3 = \frac{T_3}{T_2}\frac{T_2}{T_1}T_1 = 1.242 \times 1.255 \times 300 = 467.6K$$

$$p_{03} = \frac{p_{03}}{p_{02}}\frac{p_{02}}{p_{01}}p_{01} = 0.9630 \times 0.9582 \times 27.14 = 25.04atm \quad (p_{01} = 27.14p = 27.14atm).$$

4.8 Consider a streamline with the properties $M_1 = 4.0$ and $p_1 = 1$ atm.Consider also the following two different shock structures encountered by such a streamline: (a) a single normal shock wave; and (b) an oblique shock wave with $\beta = 40^o$, followed by a normal shock. Calculate and compare the total pressure behind the shock structure of each (a) and (b) above. From this comparison, can you deduce a general principal concerning the efficiency of a single normal shock in relation to an oblique shock plus normal shock in decelerating a supersonic flow to subsonic speeds (which, for example, is the purpose of an inlet of a convectional jet engine).

Solution:

(a) For a single normal shock with M=4.0, the total pressure before the normal shock is $p_{01} = 151.8p_1 = 151.8atm$

The total pressure ratio is $p_{02}/p_{01} = 0.1388 \Rightarrow p_{02} = 0.1388 \times p_{01} = 21.07atm$

(b) For a oblique shock followed by a normal shock:

First consider the oblique shock with $\beta = 40^o$.

From $\theta - \beta - M_1$ relation, we know $\theta = 26^o(\beta = 39.74^o)$. $M_{n1} = M_1 \sin\beta = 4 \times \sin 40^o = 2.5712$. From normal shock relation with $M = 2.57$, we know $p_{02}/p_{01} = 0.4793$ and $M_{n2} = 0.5064$.

Thus $M_2 = M_{n2}/\sin(\beta - \theta) = 0.5064/\sin(40^o - 26^o) = 2.093$.

For a normal shock with M=2.093, we know $p_{03}/p_{02} = 0.6742(M = 2.1)$.

Thus

$$\frac{p_{03}}{p_{01}} = \frac{p_{03}}{p_{02}}\frac{p_{02}}{p_{01}} = 0.6742 \times 0.4793 = 0.3231 \Rightarrow p_{03} = 0.3132 \times 151.8 = 49.07atm$$

It is seen that the total pressure in method in (b) is 2.33 times larger than that in method (a), suggesting that method (b) is more efficient in keeping the total pressure. As it is always desire to keep the total pressure loss small, method (b) should be used in actual jet engine.

4.9 Consider the intersection of two shocks of opposite families, as sketched in Fig. 4.23. For $M_1 = 3$, $p_1 = 1atm$, $\theta_2 = 20^o$, and $\theta_3 = 15^o$, calculate the pressure in region 4 and $4'$, and the flow direction Φ behind the refracted shocks.

Solution:

4.10 Consider the flow past a 30^o expansion corner, as sketched in Fig. 4.32.The upstream conditions are $M_1 = 2, p_1 = 3$ atm, and $T_1 = 400$ K, calculate the following downstream conditions: M_2, p_2, T_2, T_{02}, and p_{02}.

Solution:

$$\nu(M_2) = \nu(M_1) + \theta_2 = \nu(2) + 30^o = 26.38^o + 30^o = 56.38^o \Rightarrow M_2 = 3.35(\nu(3.35) = 56.07)$$

$$p_2 = \frac{p_2}{p_{02}}\frac{p_{01}}{p_1}p_1 = 1/61.52 \times 7.824 \times 3 = 0.38atm$$

$$T_2 = \frac{T_2}{T_{02}}\frac{T_{01}}{T_1}p_1 = 1/3.244 \times 1.8 \times 400 = 221.9K$$

$$T_{02} = 3.244 T_2 = 719.8K \qquad p_{02} = 61.52 p_2 = 23.37 atm$$

4.11 For a given Prandtl-Meyer expansion, the upstream Mach number is 3 and the pressure ratio across the wave is $p_2/p_1 = 0.4$. Calculate the angle of the forward and the rearward Mach lines of the expansion fan relative to the free stream direction.

Solution:

The Mach angle is given by $\mu = \arcsin \frac{1}{M}$.

For the forward Mach wave with M=3, the Mach angle is $\mu_1 = \arcsin(1/3) = 19.47^o$.

For M=3, $p_{01}/p_1 = 36.73$.

$$p_2/p_1 = p_2/p_{02} \times p_{01}/p_1 \Rightarrow p_{02}/p_2 = 36.73/0.4 = 91.825 \Rightarrow M_2 = 3.63 \Rightarrow \mu_2 = \arcsin(1/3.63) = 15.99^o$$

4.12 Consider a supersonic flow with upstream Mach number of 4 and pressure of 1 atm. This flow is first expanded around an expansion corner with $\theta = 15^o$, and then compressed through a compression corner with equal angle $\theta = 15^o$, so that it is returned to its original upstream direction. Calculate the Mach number and pressure downstream of the compression corner.

Solution:

Consider the expansion wave first.

$$\nu(M_2) = \nu(M_1) + \theta = 65.78^o + 15^o = 80.78^o \Rightarrow M_2 = 5.4 (\nu(5.4) = 80.43^o)$$
$$\frac{p_2}{p_1} = \frac{p_2}{p_{02}} \frac{p_{02}}{p_{01}} \frac{p_{01}}{p_1} = 1/833.5 \times 1 \times 151.8 = 0.1821$$

Now consider the oblique shock with a compression angle $\theta = 15^o$.

From $\theta - \beta - M_1$ relation, we know for $M_2 = 5.4$ and $\theta = 15^o$, the wave deflection angle is $\beta = 23.5^o$.

$M_{n2} = M_2 \sin\beta = 2.15$. From the normal shock relation with $M_{n2} = 2.15$, we know

$$M_{n3} = 0.5540 \Rightarrow M_3 = M_{n3}/\sin(\beta - \theta_2) = 0.5540/\sin(23.5^o - 15^o) = 3.748$$

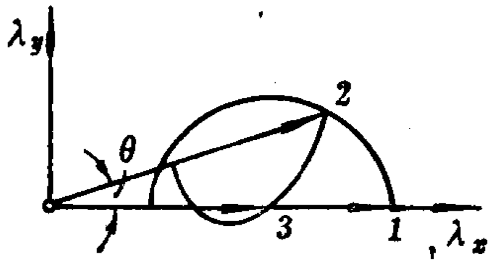
$$p_3/p_2 = 5.226$$

Thus the pressure downstream of the compression corner is

$$p_3 = \frac{p_3}{p_2}\frac{p_2}{p_1} p_1 = 5.266 \times 0.1821 \times 1 = 0.96 atm$$

4.13 Consider the incident and reflected shock waves as sketched in Fig. 4.17. Show by means of sketches how you will use shock polars to solve for the reflected wave properties.

Solution:
激波极线表示的激波反射如下图所示



图中 $\lambda_x = \frac{V_x}{a^*}$ 且 $\lambda_y = \frac{V_y}{a^*}$。箭头 1 的位置代表是来流速度大小。箭头 2 的位置代表经过气流折角为 θ 的斜激波以后的流动速度大小和方向。箭头 3 代表经过反射的斜激波以后的气流速度大小和相对于来流的气流方向。

4.14 Consider a supersonic flow past a compression corner with $\theta = 20^o$. The upstream properties are $M_1 = 3$ and $p_1 = 2116\text{lb}/\text{ft}^2$. A Pitot tube is inserted in the flow downstream of the corner. Calculate the value of pressure measured by the Pitot tube.

Solution:
A normal shock forms in front of the Pitot tube if the downstream Mach number is larger than 1. The pressure measured by the Pitot tube is the total pressure behind the normal shock.
First consider the flow properties behind the oblique shock:
From $\theta - \beta - M_1$ relation, we know $\beta = 37.76^o$ for $M_1 = 3$.Thus $Mn1 = M_1 \sin\beta = 1.84$.
From normal shock relation, we know

$$M_{n2}^2 = \frac{2 + M_{n1}^2(\gamma - 1)}{2\gamma M_{n1}^2 - (\gamma - 1)} \Rightarrow M_{n2} = 0.6078 \Rightarrow M_2 = \frac{M_{n2}}{\sin(\beta - \theta)} = 2.0 \quad and p_{02}/p_{01} = 0.7948$$

Now consider the normal shock in front of the Pitot tube with M=2.0:
The total pressure upstream and downstream of the normal shock is $p_{03}/p_{02} = 0.7209$. Thus the pressure measure by the Pitot tube p_{03} is

$$p_{03} = \frac{p_{03}}{p_{02}} \frac{p_{02}}{p_{01}} \frac{p_{01}}{p_1} p_1 = 0.7209 \times 0.7948 \times 36.73 = 21.04 atm.$$

4.20 The flow of a chemically reacting gas is sometimes approximated by the use of relations obtained assuming a calorically perfect gas, such as in this chapter, but using an “effective gamma,”a ratio of specific heats less than 1.4.

Consider the Mach 3 flow of chemically reacting air, where the flow is approximated by a ratio of specific heats equal to 1.2. If this gas flows over a compression corner with a deflection angle of 20 degrees, calculate the wave angle of the oblique shock. Compare this result with that for ordinary air with a ratio of specific heats equal to 1.4. What conclusion can you make about the general effect of a chemically reacting gas on wave angle?

Solution:

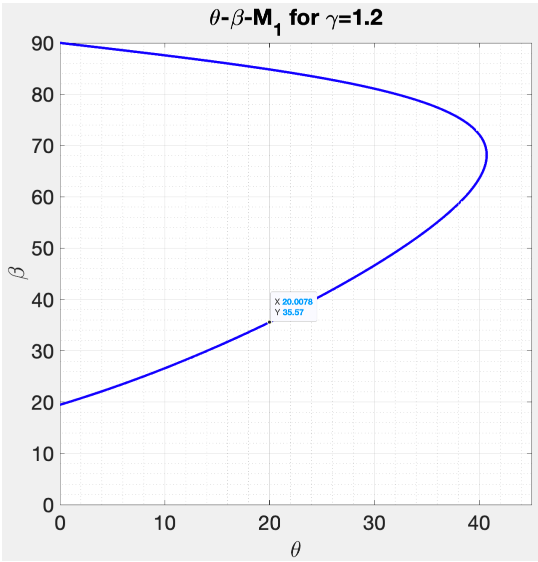
The $\theta - \beta - M_1$ relation reads:

$$\tan\theta = 2\cot\beta[\frac{M_1^2 \sin^2\beta - 1}{M_1^2(\gamma + cos(2\beta)) + 2}] \quad (1)$$

Substitute $M_1 = 3$, $\theta = 20^o$ and $\gamma = 1.2$ into equation (1), we have $\beta = 35.57^o$.

For $M_1 = 3$, $\theta = 20^o$ and $\gamma = 1.4$, we have $\beta = 37.67^o$.

Comparing the two cases, it is seen that chemically reacting gas with smaller γ reduces the wave deflection angle.



4.21For the two cases treated in Problem 4.20, calculate and compare the pressure ratio (shock strength) across the oblique shock wave. What can you conclude about the effect of a chemically reacting gas on shock strength?

Solution:

$$M_n 1 = M_1 \sin\beta$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}(M_{n1}^2 - 1)$$

For $\gamma = 1.2$ and $\beta = 35.57$ $p_2/p_1 = 3.59$

For $\gamma = 1.4$ and $\beta = 37.76$ $p_2/p_1 = 3.77$

It is seen that chemically reacting gas reduces the shock strength.