Solution to problems of Chapter 5

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5.1 A supersonic wind tunnel is designed to produce flow in the test section at Mach 2.4 at standard atmospheric conditions. Calculate:
(a). The exit-to-throat area ratio of the nozzle.
(b). Reservoir pressure and temperature.
Solution:
(a) The flow is isentropic. From the area-Mach number relation:

$$(\frac{A}{A^*})^2 = \frac{1}{M^2}[\frac{2}{\gamma+1}(1+\frac{\gamma-1}{2}M^2)]^{\frac{\gamma+1}{\gamma-1}} \tag{1}$$

Substitute $M = 2.4$ and $\gamma = 1.4$ into equation (1), we have the exit to throat ratio

$$\frac{Ae}{A_t} = \frac{A_e}{A^*} = 2.403$$

(b) The pressure and temperature at the exit is $p = 1atm$ and $T = 288K$ (Standard sea level condition). From the isentropic relation, we have

$$\frac{T_0}{T} = 1 + \frac{\gamma-1}{2}M^2 \quad and \quad \frac{p_0}{p} = (1+\frac{\gamma-1}{2}M^2)^{\frac{\gamma}{\gamma-1}} \tag{2}$$

Substitute $p = 1atm$, $T = 288K$ and $M = 2.4$ into equation (2), we have

$$p_0 = 14.62atm \quad and \quad T_0 = 619.8K$$

The reservoir pressure is 14.62 atm and its temperature is 619.8K.

5.2 The reservoir pressure of a supersonic wind tunnel is 10 atm. A Pitot tube inserted in the test section measures a pressure of 0.627 atm. Calculate the test section Mach number and area ratio.
Solution:
The pressure measured by the Pitot tube is the total pressure behind the normal shock in front of it, i.e., p_{02}=0.627 atm. The total pressure in front of the normal shock is the reservoir pressure, i.e., p_{01}=10 atm. Thus

$$\frac{p_{02}}{p_{01}} = \frac{0.627}{10} = 0.0627$$

From table A_2, when$p_{02}/p_{01} = 0.0627$, we have $M = 5$. For $M = 5$, $A_e/A_t = 25$.

5.3 The reservoir pressure of a supersonic wind tunnel is 5 atm. A static pressure probe is moved along the centreline of the nozzle, taking measurements at various stations. For these probe measurements, calculate the local Mach number and area ratio:
(a). 4.00 atm.
(b). 2.64 atm.
(c). 0.50 atm.

Solution:
The process is isentropic. Thus the relation between the static pressure and Mach number is

$$\frac{p_0}{p} = (1 + \frac{\gamma - 1}{2} M^2)^{\gamma/(\gamma-1)} \quad (1)$$

The relation between the Area ratio and the Mach number is

$$(\frac{A}{A^*})^2 = \frac{1}{M^2}[\frac{2}{\gamma+1}(1 + \frac{\gamma-1}{2})M^2]^{(\gamma+1)/(\gamma-1)} \quad (2)$$

For the three cases we have
(a) $p_0/p = 5/4$, substitute p_0/p and $\gamma = 1.4$ into equation (1), hence M=0.574;
Substitute$M$=0.574 into equation (2), hence $A/A^* = 1.221$
(b) $p_0/p = 5/2.64 = 1.984$, substitute p_0/p and $\gamma = 1.4$ into equation (1), hence M=1;
Substitute$M$=1 into equation (2), hence $A/A^* = 1$
(c) $p_0/p = 5/0.5 = 10$, substitute p_0/p and $\gamma = 1.4$ into equation (1), hence M=2.157;
Substitute$M$=2.157 into equation (2), hence $A/A^* = 1.930$

5.4 Consider the purely subsonic flow in a convergent–divergent duct. The inlet, throat, and exit area are 1.00 m^2, 0.70 m^2, and 0.85 m^2, respectively. If the inlet Mach number and pressure are 0.3 and 0.8×10^5 N/M^2, respectively, calculate:
(a). M and p at the throat.
(b). M and p at the exit.
Solution:
The flow is isentropic process. Thus

$$(\frac{A}{A^*})^2 = \frac{1}{M^2}[\frac{2}{\gamma+1}(1 + \frac{\gamma-1}{2})M^2]^{(\gamma+1)/(\gamma-1)} \quad (1)$$

For M=0.3 and A_e=1 m^2, we have A_e/A^*=2.035 $\Rightarrow A^* = A_e/2.035 = 0.491$ m^2.
The pressure at the inlet is the total pressure, i.e., $p_0/p_2 = (1 + \frac{(\gamma-1)M^2}{2})^{\gamma/(\gamma-1)}$.
Substitute $M = 0.3$ and $p_e = 0.8 \times 10^5$Pa into the above equation, we have $p_0 = 8.52 \times 10^4 \text{N/m}^2$.
(a) At the throat, $A_t/A^* = 0.7/0.491 = 1.453$.
Substitute $A_t/A^* = 1.453$ into equation (1), we have $M = 0.46$ and $p_0/p_t = 1.156$.
Thus, p_t= $8.52 \times 10^4/1.156 = 7.37 \times 10^4$ N/m^2.
(b) At the exit, $A_e/A^* = 0.85/0.491 = 1.731$.
Substitute $A_e/A^* = 1.731$ into equation (1), we have $M = 0.36$ and $p_0/p_e = 1.094$.
Thus, p_t= $8.52 \times 10^4/1.094 = 7.79 \times 10^4$ N/m^2.

5.5. Consider the subsonic flow through a divergent duct with area ratio $A_2/A_1 = 1.7$. If the inlet conditions are $T_1 = 300K$ and $u_1 = 250m/s$, and the pressure at the exit is $p_2 = 1atm$, calculate:
(a). Inlet pressure p_1.
(b). Exit velocity u_2.
Solution:
The Mach number at the inlet is

$$M_1 = \frac{u_1}{\sqrt{\gamma R T_1}} = \frac{250}{\sqrt{1.4 \times 287 \times 300}} = 0.73$$

For M=0.73, $A_1/A_* = 1.075$, $T_0/T_1 = 1.107$ and $p_0/p_1 = 1.426$,
thus

$$\frac{A_2}{A^*} = \frac{A_2}{A_1}\frac{A_1}{A^*} = 1.7 \times 1.075 = 1.828$$

Thus $M_2 = 0.34$, $T_0/T_2 = 1.023$ and $p_0/p_2 = 1.083$.

$$\frac{T_2}{T_1} = \frac{T_2}{T_0}\frac{T_0}{T_1} = \frac{1}{1.023} \times 1.107 = 1.082 \Rightarrow T_2 = 1.082T1 = 324.63K$$

$$u_2 = M_2 \times a_2 = M_2 \times \sqrt{\gamma R T_2} = 0.34 \times \sqrt{1.4 \times 287 \times 324.63} = 122.74m/s$$

$$\frac{p_2}{p_1} = \frac{p_2}{p_0}\frac{p_0}{p_1} = 1/1.083 \times 1.426 = 1.314 \Rightarrow p_1 = \frac{p_2}{1.314} = 0.761atm$$

5.6 The mass flow of a calorically perfect gas through a choked nozzle is given by

$$\dot{m} = \frac{p_0 A^*}{\sqrt{T_0}}\sqrt{\frac{\gamma}{R}(\frac{2}{\gamma+1})^{\frac{\gamma+1}{\gamma-1}}}$$

Derive this relation.

Solution:

The mass flow can be calculated at the throat, i.e., $\dot{m} = \rho^* u^* A^*$ and $M^* = 1$. From the isentropic relation with $M = 1$, we have

$$\frac{\rho_0}{\rho^*} = (1 + \frac{\gamma-1}{2})^{\frac{1}{\gamma-1}} \Rightarrow \rho^* = \frac{\rho_0}{((\gamma+1)/2)^{1/(\gamma-1)}}$$

$$\frac{T_0}{T*} = 1 + \frac{\gamma-1}{2} = \frac{\gamma+1}{2}$$

$$u^* = a^* = \sqrt{\gamma R T^*}$$

Consider the equation of state for idea gas, we have $p_0 = \rho_0 R T_0$. Combining the above three equations together with the equation of state, we have

$$\dot{m} = \frac{p_0 A^*}{\sqrt{T_0}}\sqrt{\frac{\gamma}{R}(\frac{2}{\gamma+1})^{\frac{\gamma+1}{\gamma-1}}}$$

5.7 When the reservoir pressure and temperature of a supersonic wind tunnel are 15 atm and 750 K, respectively, the mass flow is 1.5 kg/s. If the reservoir conditions are changed to $p_0 = 20$ atm and $T_0 = 600$ K, calculate the mass flow.

Solution: From problem 5.6, we can write $\dot{m} = c\frac{p_0}{\sqrt{T_0}}$ with c being a constant for a given tunnel. Let $\dot{m}_1$ be the mass flow for $p_{01} = 15$ atm and $T_{01} = 750$ K and $\dot{m}_2$ be the mass flow for $p_{02} = 20$ atm and $T_{02} = 600$ K. Hence,

$$\frac{\dot{m}_2}{\dot{m}_1} = \frac{p_{02}}{p_{01}}\sqrt{\frac{T_{01}}{T_{02}}} = \frac{20}{15} \times \sqrt{\frac{750}{600}} = 1.49 \Rightarrow \dot{m}_2 = \dot{m}_1 \times 1.49 = 2.24kg/s$$

5.8 A blunt-nosed aerodynamic model is mounted in the test section of a supersonic wind tunnel. If the tunnel reservoir pressure and temperature are 10 atm and 800°R, respectively, and the exit-to-throat area ratio is 25, calculate the pressure and temperature at the nose of the model.

Solution:

For supersonic flow, there will be a normal shock wave in front of the blunt nosed model. Thus, the pressure and temperature at the nose of the model is the total pressure and temperature after a normal shock wave, i.e., T_{02} and p_{02}.

For $A_e/A_t = 25$, we can calculate from the area-Mach-number relation that $M_e = 5$. From the normal shock relation, we know the Mach number behind the wave is $M_2 = 0.4152$. Thus

$$\frac{p_{02}}{p_{01}} = \frac{p_{02}}{p_2}\frac{p_2}{p_1}\frac{p_1}{p_{01}} = 0.06172 \Rightarrow p_{02} = 0.6172atm.$$

$$\frac{T_{02}}{T_{01}} = \frac{T_{02}}{T_2}\frac{T_2}{T_1}\frac{T_1}{T_{01}} = 1.035 \times 5.8 \times \frac{1}{6} = 1.005 \Rightarrow T_{02} = 1.005T_{01} = 446.6K$$

5.10 Consider a supersonic nozzle with a Pitot tube mounted at the exit. The reservoir pressure and temperature are 10 atm and 500 K, respectively. The pressure measured by the Pitot tube is 0.6172 atm. The throat area is 0.3m^2. Calculate:
(a). Exit Mach number M_e.
(b). Exit area A_e.
(c). Exit pressure and temperature p_e and T_e.
(d). Mass flow through the nozzle.
Solution:
(a) The pressure measured by the Pitot tube is the total pressure behind a normal shock wave in front of it. Thus we have $p_{02}/p_2 = 0.6172/10 = 0.06172$.
Substitute this value into the Rayleigh-Pitot relation, we have the Mach number at the exit $M_e = 5$.
(b) Substitute $M_e = 5$ into the area-Mach-number relation, we have $A_e/A^* = A_e/A_t = 25$.
Thus $A_e = 25 \times A_t = 0.3 \times 25$ =7.5 m^2.
(c) The process is isentropic. For $M_e = 5$, we have $p_0/p_e = 529.1 \Rightarrow p_e = 0.0189atm$ and $T_0/T_e = 6 \Rightarrow T_e = 83.33K$.
(d) The mass flow through the nozzle can be calculated at the exit, i.e.,

$$\dot{m} = \rho_e u_e A_e = \frac{p_e}{RT_e} M_e \sqrt{\gamma R T_e} A_e = 0.0054kg/s$$

5.14 In a supersonic nozzle flow, the exit-to-throat area ratio is 10, $p_0 = 10.00$ atm, and the back pressure $p_B = 0.04$ atm. Calculate the angle θ through which the flow is deflected immediately after leaving the edge (or lip) of the nozzle exit.
Solution:
From the area-Mach-number relation with $A_e/A_t = A_e/A^* = 10$, we know the Mach number at the exit is $M_e = 3.95$ and $p_0/p_e = 142$.
Thus $p_e = p_0/142 = 0.0704$ atm.
As $p_e > p_B$ there must be an expansion wave such that the pressure behind the expansion wave matches the back pressure.
The Mach number in front of the expansion wave is $M_1 = M_e = 3.95$.
The pressure ratio across the expansion wave is $p_0/p_B = 250$.
Thus the Mach number behind the expansion wave is $M = 4.4$
The deflection angle of the expansion wave is then $\theta = \nu(M_2) - \nu(M_1) = 70.71^o - 65.12^o = 5.59^o$
Here $\nu(M)$ is the Prandtl-Mayer function.

5.17 We wish to design a Mach 3 supersonic wind tunnel, with a static pressure and temperature in the test section of 0.1 atm and 400°R, respectively. Calculate:
(a). The exit-to-throat area ratio of the nozzle.
(b). The ratio of diffuser throat area to nozzle throat area.
(c). Reservoir pressure.

(d). Reservoir temperature.

Solution:

(a) From the area-Mach-number relation with $M_e = 3$, we know $A_e/A_t = A_e/A^* = 4.235$.

(b) There must exist a normal shock before the diffuser throat. The Mach number behind the normal shock with $M_e = 3$ is $M_2 = 0.4752$ It can be proved that

$$A_{t2}/A_{t1} = p_{01}/p_{02} = \frac{p_{01}}{p_1}\frac{p_1}{p_2}\frac{p_2}{p_{02}} = 3.056$$

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(c) For $M_e = 3$, $p_0/p_e = 36.73 \Rightarrow p_0 = 36.73p_e = 3.673$ atm.

(d) For $M_e = 3$, $T_0/T_e = 2.8 \Rightarrow T_0 = 2.8T_e = 621.6$ K.