气体动力学
Gas Dynamics

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Ch2 Integral Forms of the Conservation Equations for Inviscid flow

Mathematics up to the present day have been quite useless to us in regard to flying.

From the 14th Annual Report of the Aeronautical Society of Great Britain, 1879

Mathematical theories from the happy hunting grounds of pure mathematicians are found suitable to describe the airflow produced by aircraft with such excellent accuracy that they can be applied directly to airplane design.

Theodore von Karman, 1954

Outline

2.1 Philosophy (原理)

2.2 Approach (流体力学的研究方法)

2.3 Continuity equation (连续性方程)

2.4 Momentum equation (动量定理)

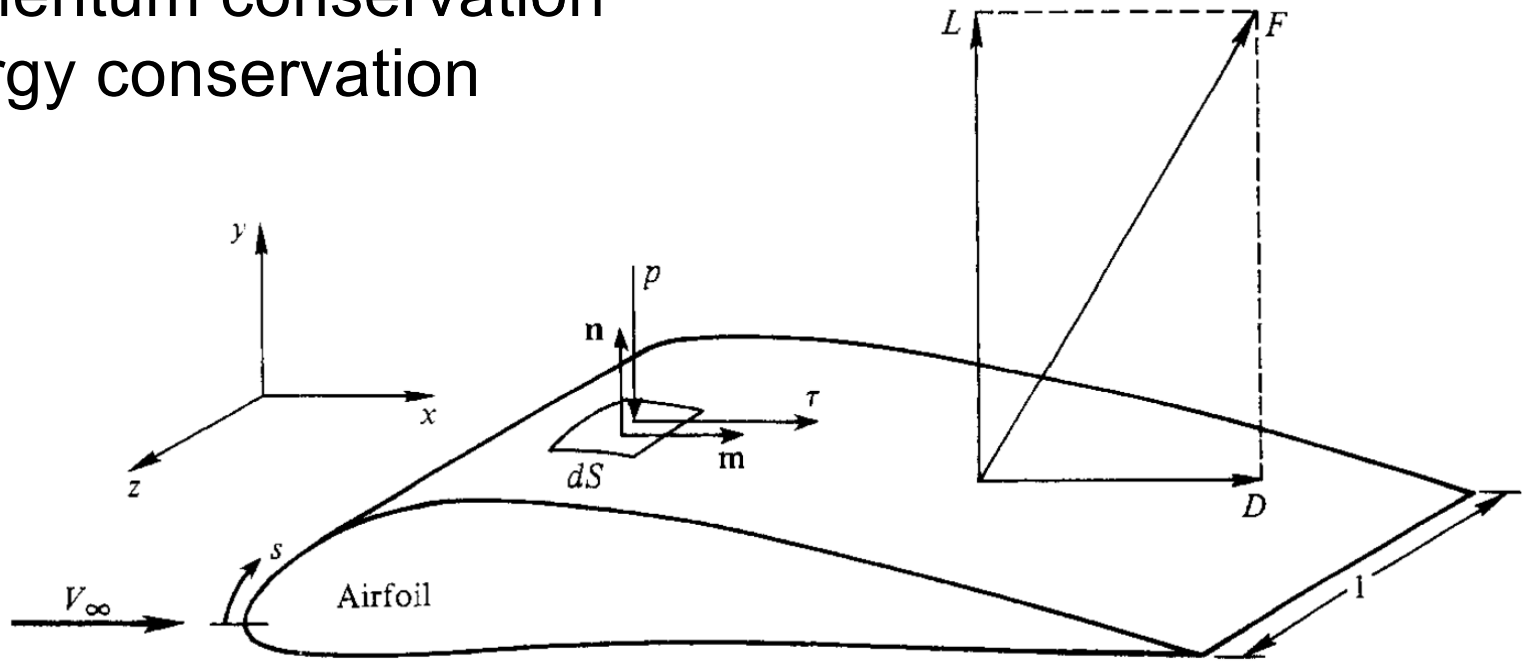
2.5 Energy equation (能量方程)

2.6 Propulsion force of an air jet engine
(动量定理的应用：喷气发动机的推力)

2.1 Philosophy (原理)

对于处在流场中的物体，只要知道流场每一个位置的特性（*p*, *ρ*, *T*, *V* ,*e* 等），原则上就可以求解出固体受到流体的作用力。（适用于可压缩和不可压缩流动）

Starting point： Fundamental Laws of Fluid Mechanics：
Mass conservation
Momentum conservation
Energy conservation



2.2 Approach (流体力学研究的方法)

- Choose the physical principles from the laws of nature:
  - Mass conservation
  - Newton's second law of motion
  - Energy conservation
- Apply the laws to a model of flow
- Obtain equations and solve them

(1) Finite control volume approach 有限控制体法
Integral forms of the governing equations

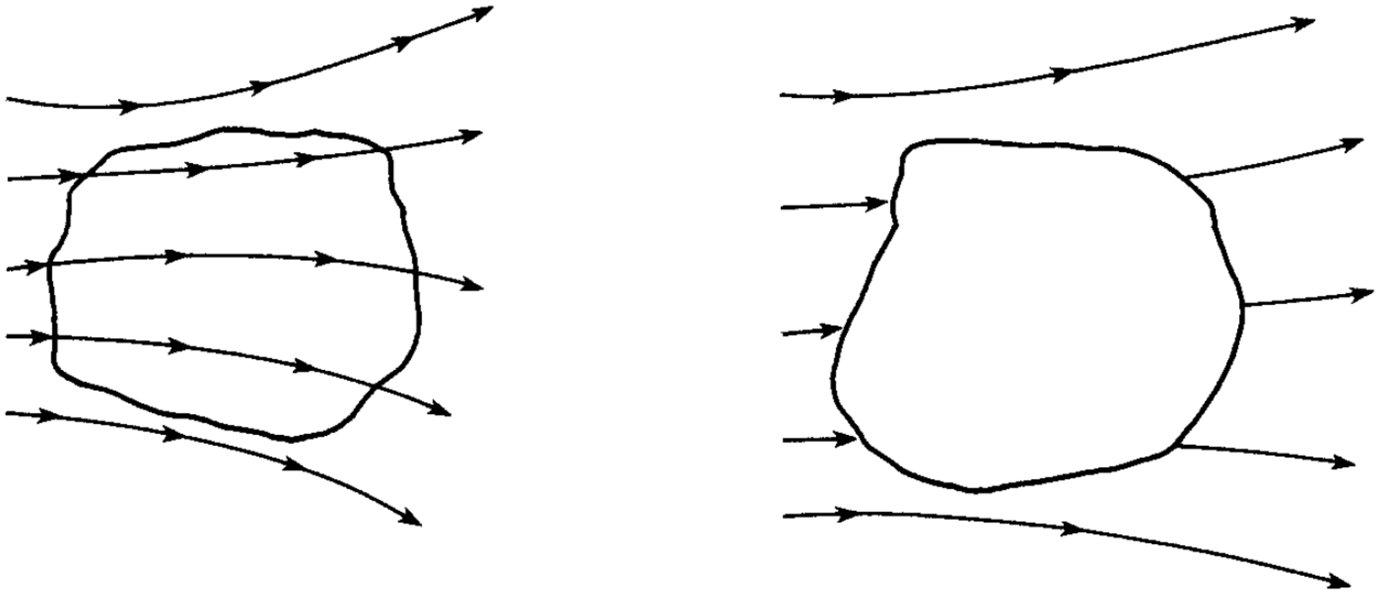
(2) Finite fluid element approach 有限流体微元法
Differential forms of the governing equations

(3) Statistical approach 统计方法、分子方法

2.2 Approach (流体力学研究的方法)

(1) Finite control volume approach 有限控制体法

Control volume with volume $\mathscr{V}$ and Bounding surface S



Finite control volume fixed in space with the fluid moving through it.

Finite control volume moving with the fluid such that the same fluid particles are always in the same control volume

Figure 2.2 | Finite control volume approach.

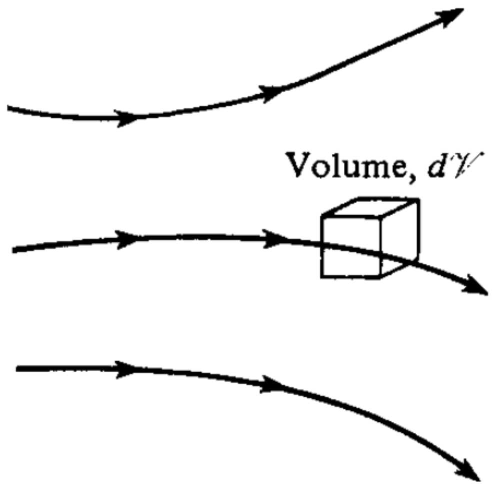
Eulerian description 欧拉描述

Langragian description 拉格朗日描述

2.2 Approach (流体力学研究的方法)

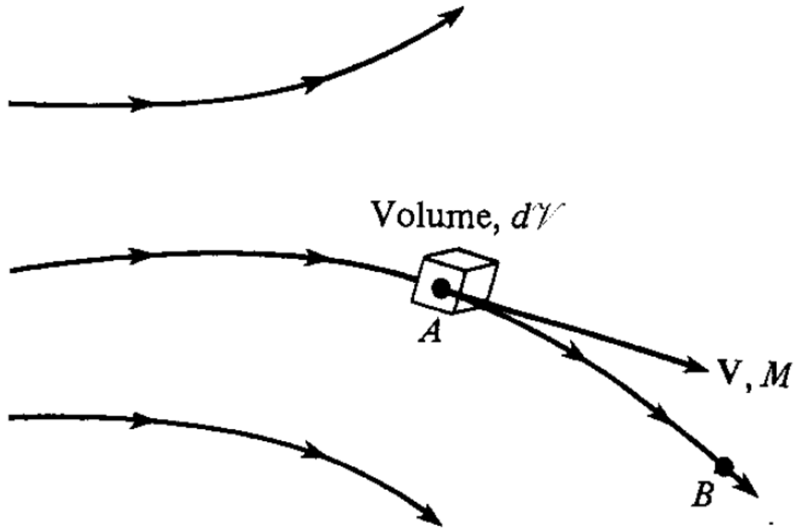
(2) Finite fluid element approach 有限流体微元法

Fluid element: large enough to have many molecules; small enough compared with the flow



Infinitesimal fluid element fixed in space with the fluid moving through it

Eulerian description 欧拉描述



Infinitesimal fluid element moving along a streamline with the velocity **V** equal to the flow velocity at each point

Langragian description 拉格朗日描述

2.2 Approach (流体力学研究的方法)

(3) Statistical approach 统计方法、分子方法

Annu. Rev. Fluid Mech. 1998. 30:329–64

LATTICE BOLTZMANN METHOD FOR FLUID FLOWS

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KEY WORDS: lattice Boltzmann method, mesoscopic approach, fluid flow simulation

2.3 Continuity equation

Mass can not be destroyed nor be created.

Mass flow out from the *dS* elementary bounding surface：

$$\dot{m} = \rho(V cos\theta)dS = \rho V_n dS = \rho V \cdot d\vec{S}$$

Mass flow into the control volume

$$-\iint_S \rho\vec{V} \cdot d\vec{S}$$

Mass change inside the control volume

$$\frac{\partial}{\partial t}\iiint_{\mathscr{V}} \rho d\mathscr{V}$$

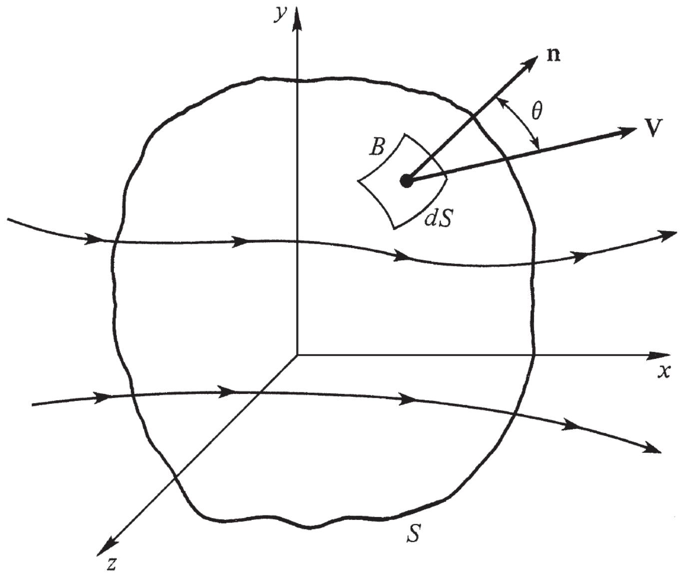


Figure 2.4 | Fixed control volume for derivation of the governing equations.

$$-\iint_S \rho\vec{V} \cdot d\vec{S} = \frac{\partial}{\partial t}\iiint_{\mathscr{V}} \rho d\mathscr{V}$$

$$\frac{\partial}{\partial t}\iiint_{\mathscr{V}} \rho d\mathscr{V} + \iint_S \rho\vec{V} \cdot d\vec{S} = 0 \qquad (2.1)$$

The net mass flow into the control volume must equal the rate of increase of mass inside the control volume.

2.4 Momentum Equation

The time rate of change of momentum of a body equals to the net force exerted on it.

$$\frac{d}{dt}(m\vec{V}) = \vec{F}$$

For constant m

$$\vec{F} = m\frac{d\vec{V}}{dt} = m\vec{a}$$

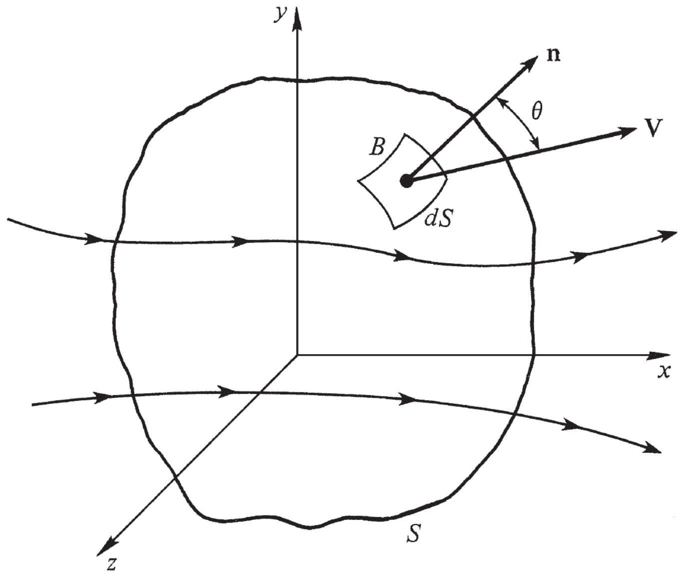


Figure 2.4 | Fixed control volume for derivation of the governing equations.

$\vec{f}$: Body force per unit mass

$$\text{Total body force} = \iiint_{\mathscr{V}} \rho \vec{f} d\mathscr{V}$$

p : Pressure per unit area

$$\text{Total surface force due to pressure} = -\iint_S p d\vec{S}$$

$$\text{Total force } \vec{F} = \iiint_{\mathscr{V}} \rho \vec{f} d\mathscr{V} - \iint_S p d\vec{S}$$

2.4 Momentum Equation

Momentum flux across boundary $\vec{A}_1 = \iint_S (\rho \vec{V} \cdot d\vec{S})\vec{V}$

Total momentum inside $\mathscr{V} = \iiint_{\mathscr{V}} \rho \vec{V} d\mathscr{V}$

$$\vec{A}_2 = \frac{\partial}{\partial t} \iiint_{\mathscr{V}} (\rho \vec{V}) d\mathscr{V} = \iiint_{\mathscr{V}} \frac{\partial}{\partial t} (\rho \vec{V}) d\mathscr{V}$$

$$\frac{d}{dt}(m\vec{V}) = \vec{A}_1 + \vec{A}_2 = \iint_S (\rho \vec{V} \cdot d\vec{S})\vec{V} + \iiint_{\mathscr{V}} \frac{\partial}{\partial t} (\rho \vec{V}) d\mathscr{V}$$

Newton's 2nd law: $\frac{d}{dt}(m\vec{V}) = \vec{F}$

$$\iint_S (\rho \vec{V} \cdot d\vec{S})\vec{V} + \iiint_{\mathscr{V}} \frac{\partial}{\partial t} (\rho \vec{V}) d\mathscr{V} = \iiint_{\mathscr{V}} \rho \vec{f} d\mathscr{V} - \iint_S p d\vec{S} + \vec{F}_{viscous}$$

$$\iint_S (\rho \vec{V} \cdot d\vec{S})\vec{V} + \iiint_{\mathscr{V}} \frac{\partial}{\partial t} (\rho \vec{V}) d\mathscr{V} = \iiint_{\mathscr{V}} \rho \vec{f} d\mathscr{V} - \iint_S p d\vec{S} \tag{2.2}$$

2.5 Energy Equation

Energy can be neither created or destroyed; It can only change in form.

$$B_1 + B_2 = B_3$$

$$de = \delta q + \delta w$$

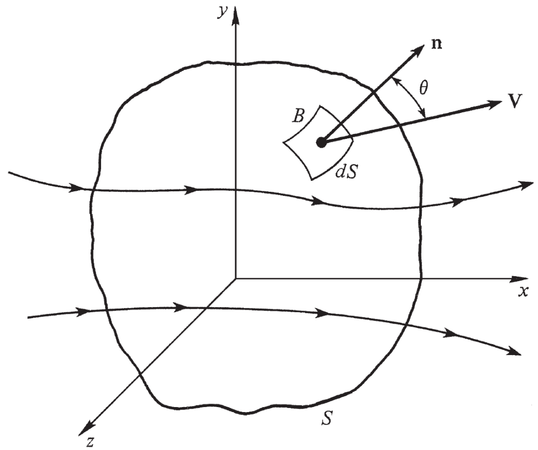


Figure 2.4 | Fixed control volume for derivation of the governing equations.

B_1: Energy transferred to the system

B_2: Rate of work done on the system by forces

- surface force p
- body force $\vec{f}$

B_3: Energy change rate

2.5 Energy Equation

Energy can be neither created or destroyed; It can only change in form.

$$B_1 + B_2 = B_3$$

B_1: Energy transferred to the system

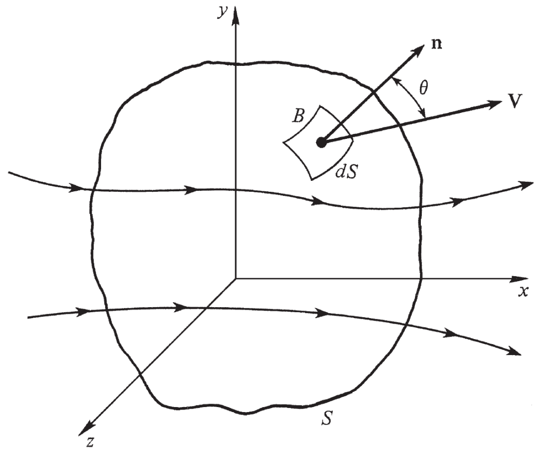


Figure 2.4 | Fixed control volume for derivation of the governing equations.

$\dot{q}$ Heat added per unit mass due to absorption of energy outside of the volume or radiation from the volume due to high temperature.

$$B_1 = \iiint_{\mathscr{V}} \dot{q} \rho d\mathscr{V}$$

2.5 Energy Equation

Energy can be neither created or destroyed; It can only change in form.

$$B_1 + B_2 = B_3$$

B_2: Rate of work done on the system by forces

- Surface force p
- Body force $\vec{f}$

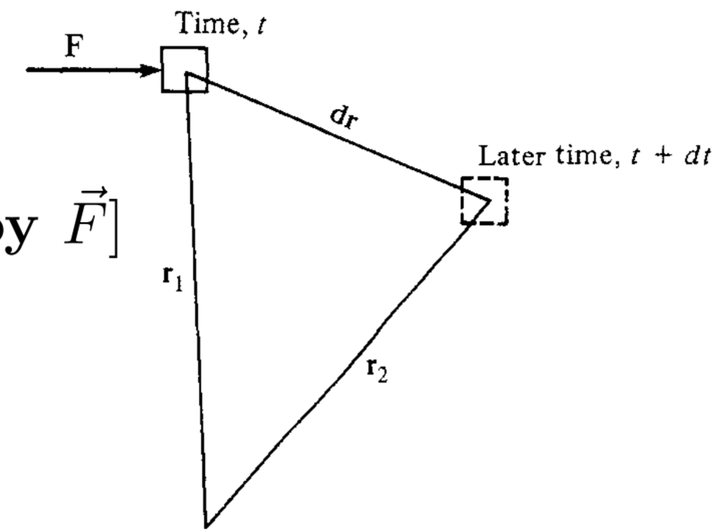


Figure 2.5 | Rate of doing work.

[**rate of doing work on a moving body by** $\vec{F}$]

$$= \vec{F} \cdot \vec{V}$$

2.5 Energy Equation

Energy can be neither created or destroyed; It can only change in form.

$$B_1 + B_2 = B_3$$

B_2 : rate of work done on the system by forces

Surface force (pressure) ; Body force $\vec{f}$

[rate of doing work on a moving body by pressure force on S]

$$=-\iint_S (pd\vec{S}) \cdot \vec{V}$$

[rate of doing work on a moving body by body force $\vec{f}$]

$$=\iiint_{\mathcal{V}} (\vec{f}\rho d\mathcal{V}) \cdot \vec{V}$$

2.5 Energy Equation

Energy can be neither created or destroyed; It can only change in form.

$$B_1 + B_2 = B_3$$

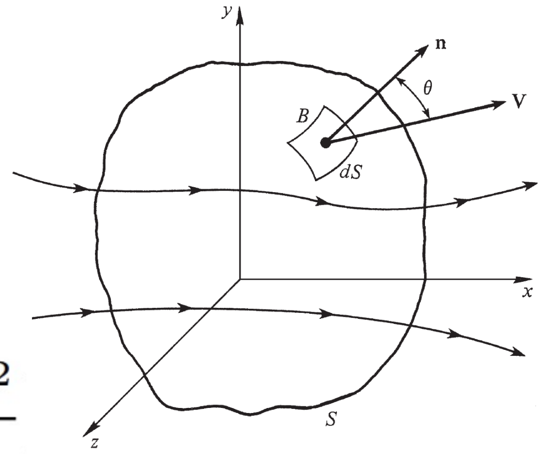


Figure 2.4 | Fixed control volume for derivation of the governing equations.

B_3: Energy change rate

Energy of volume with velocity V $e + \frac{V^2}{2}$

Energy flow in/out from the surface

$$\iint_S (e + \frac{V^2}{2})\rho d\vec{S} \cdot \vec{V}$$

Energy change rate in the volume

$$\frac{\partial}{\partial_t}(\iiint_{\mathscr{V}} (e + \frac{V^2}{2})\rho d\mathscr{V})$$

2.5 Energy Equation

Energy can be neither created or destroyed; It can only change in form.

$$B_1 + B_2 = B_3$$

$$\iiint_{\mathcal{V}} \dot{q}\rho d\mathcal{V} - \iint_S (p d\vec{S}) \cdot \vec{V} + \iiint_{\mathcal{V}} (\vec{f}\rho d\mathcal{V}) \cdot \vec{V}$$
$$= \frac{\partial}{\partial_t}\left(\iiint_{\mathcal{V}} (e + \frac{V^2}{2})\rho d\mathcal{V}\right) + \iint_S (e + \frac{V^2}{2})\rho d\vec{S} \cdot \vec{V} \quad (2.3)$$

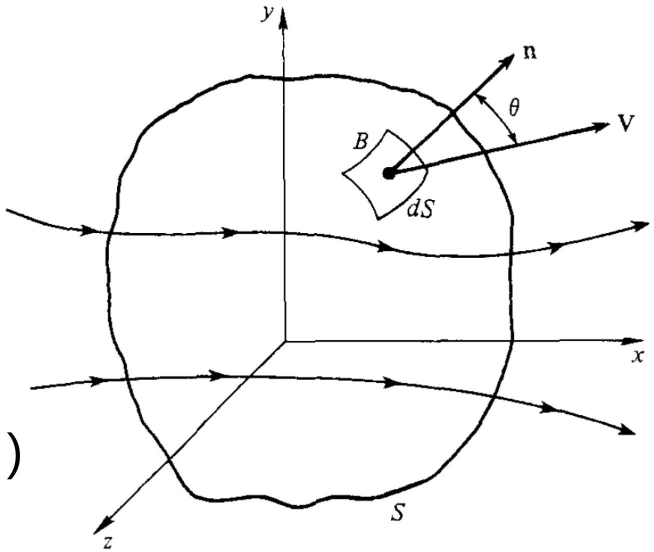


Figure 2.4 | Fixed control volume for derivation of the governing equations.

$$\dot{Q} + \dot{W}_{shaft} + \dot{W}_{viscous} - \iint_S (p d\vec{S}) \cdot \vec{V} + \iiint_{\mathcal{V}} (\vec{f}\rho d\mathcal{V}) \cdot \vec{V}$$

$$= \frac{\partial}{\partial_t}\left(\iiint_{\mathcal{V}} (e + \frac{V^2}{2})\rho d\mathcal{V}\right) + \iint_S (e + \frac{V^2}{2})\rho d\vec{S} \cdot \vec{V}$$

Summary (comp. & incomp.)

Continuity equation

$$\frac{\partial}{\partial t}\iiint_{\mathscr{V}} \rho d\mathscr{V} + \iint_S \rho \vec{V} \cdot d\vec{S} = 0$$

Momentum equation

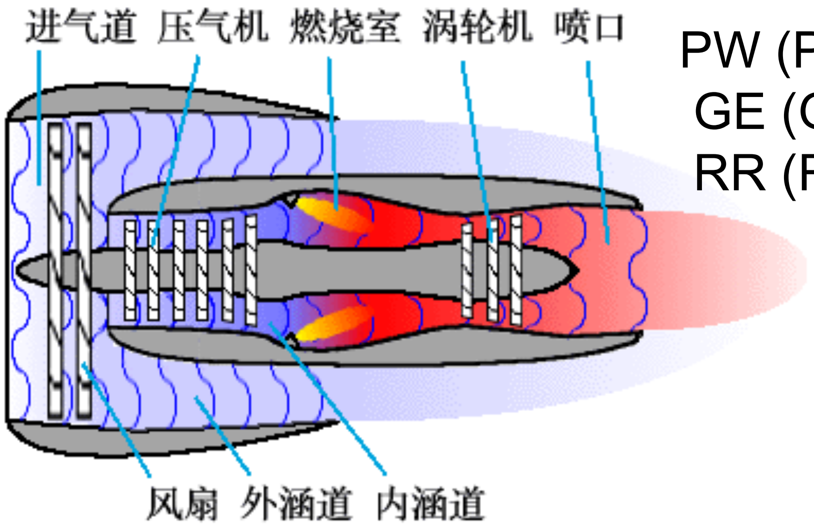
$$\iint_S (\rho \vec{V} \cdot d\vec{S})\vec{V} + \iiint_{\mathscr{V}} \frac{\partial}{\partial t}(\rho \vec{V}) d\mathscr{V} = \iiint_{\mathscr{V}} \rho \vec{f} d\mathscr{V} - \iint_S p d\vec{S}$$

Energy conservation

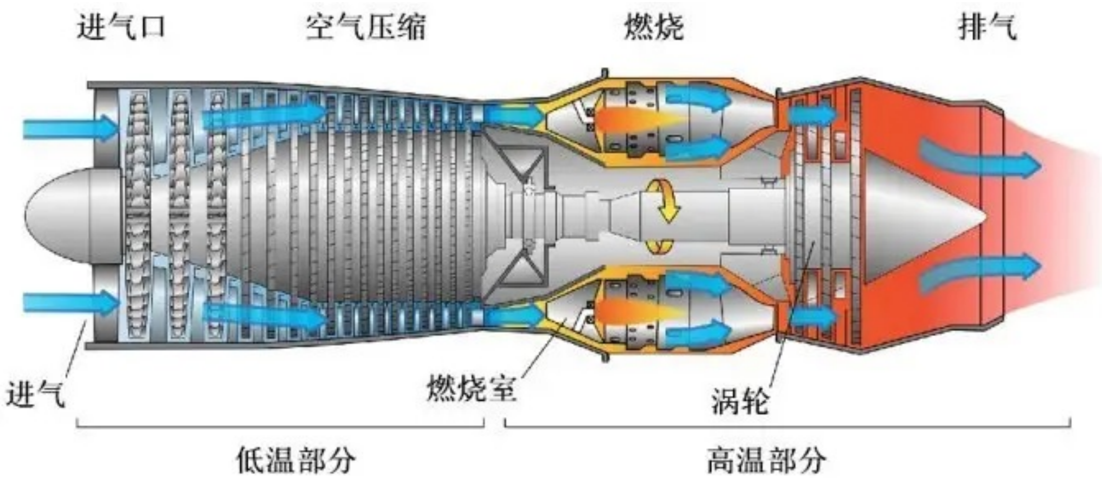
$$\iiint_{\mathscr{V}} \dot{q}\rho d\mathscr{V} - \iint_S (p d\vec{S}) \cdot \vec{V} + \iiint_{\mathscr{V}} (\vec{f}\rho d\mathscr{V}) \cdot \vec{V}$$
$$= \iiint_{\mathscr{V}} \frac{\partial}{\partial t}[(e + \frac{V^2}{2})\rho] d\mathscr{V} + \iint_S (e + \frac{V^2}{2})\rho d\vec{S} \cdot \vec{V}$$

EOS : $p=p(\rho,T)$; 6 equations, 6 variable (p, T, $\mathbf{V}$, ρ)
Given $\vec{f}$ and $\dot{q}$ the equations are closed. J/(kg*s)

2.6 Jet propulsion engine thrust(喷气式发动机推力)

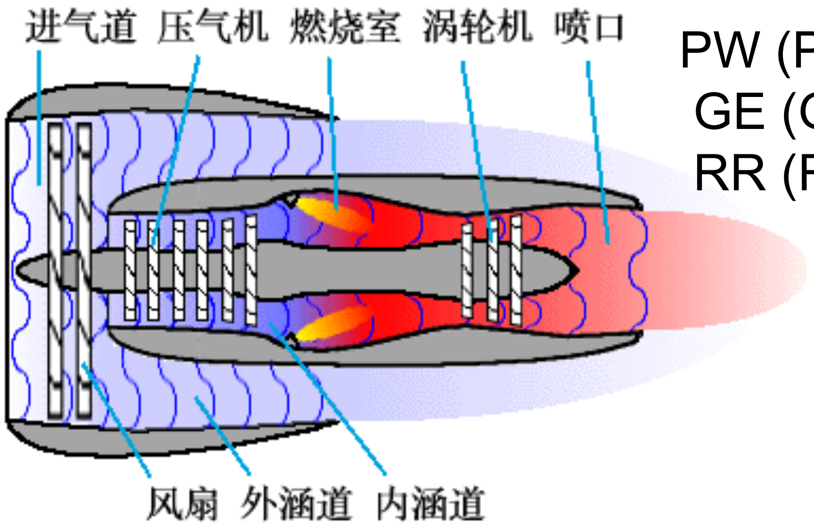


PW (Pratt and Whitney)
GE (General Electric)
RR (Rolls Royes)



How to calculate the thrust of a rocket engine?

2.6 Jet propulsion engine thrust(喷气式发动机推力)



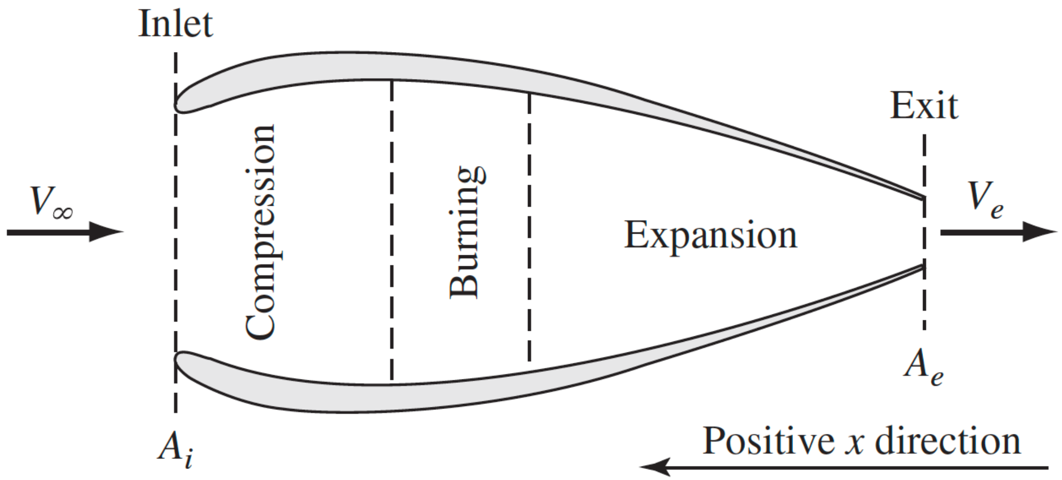
PW (Pratt and Whitney)
GE (General Electric)
RR (Rolls Royes)

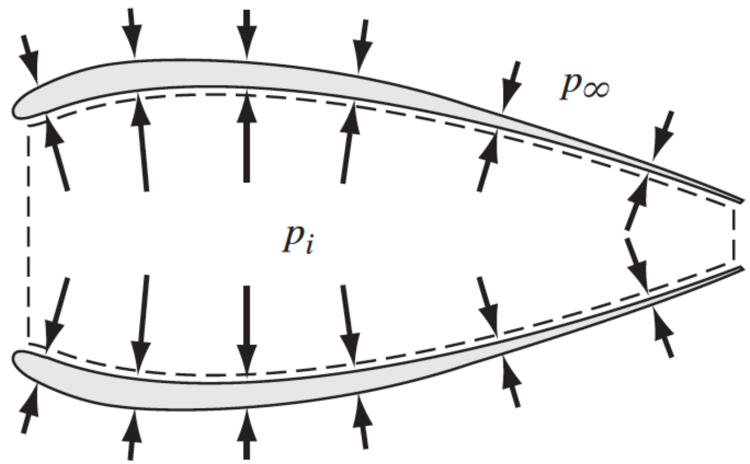
Momentum equation:

$$\iint_S (\rho\vec{V}\cdot d\vec{S})\vec{V} + \iiint_{\mathscr{V}} \frac{\partial}{\partial t}(\rho\vec{V})d\mathscr{V}$$

$$= \iiint_{\mathscr{V}} \rho\vec{f}d\mathscr{V} - \iint_S pd\vec{S}$$

2.6 Propulsion force of an air jet engine





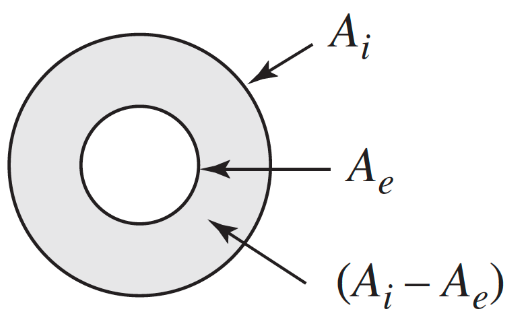
Total force: $\vec{F} = -\iint_S p d\vec{S}$

Thrust: $T = -\int (p_i dS)_x - \int (p_\infty dS)_x$

$$\int (p_\infty dS)_x = p_\infty \int (dS)_x$$

$$= p_\infty[-(A_i - A_e)] = p_\infty(A_e - A_i)$$

Force due to external pressure distribution



X方向的投影面积
Projected area along x direction

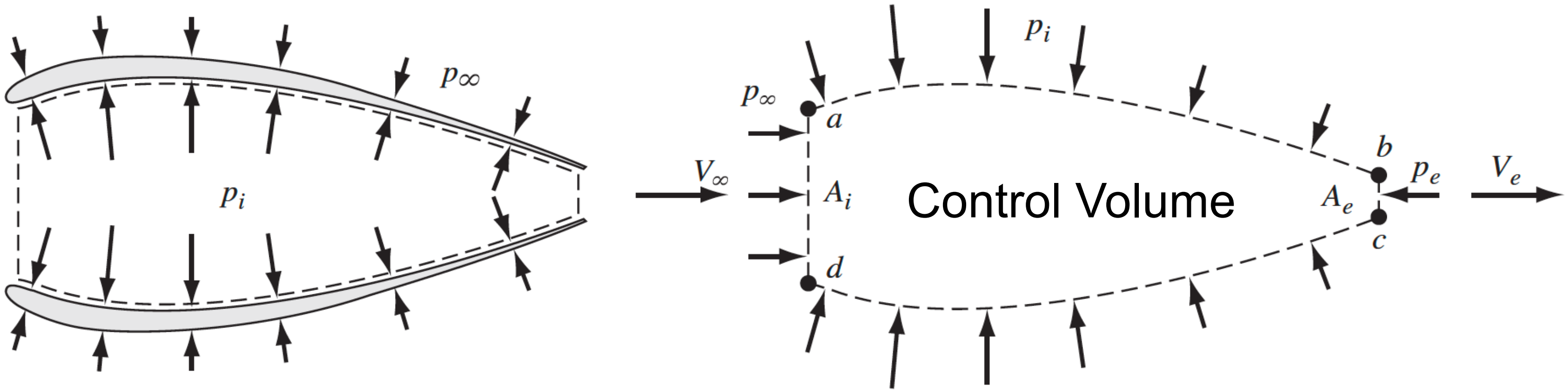
2.6 Propulsion force of an air jet engine

$$\text{Thrust: } T = -\int (p_i dS)_x - p_\infty (A_e - A_i)$$

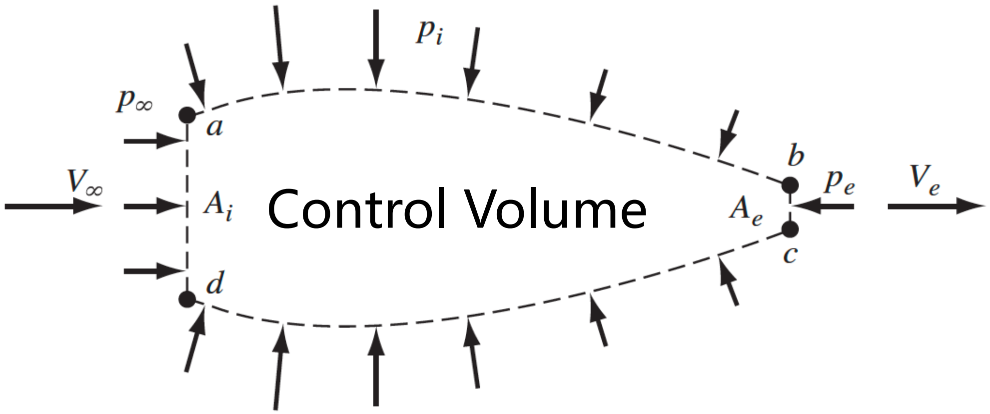
Force exerted to the internal surface by gas

Newton's 3rd law:
For every action there is an equal and opposite reaction.

1. Choose C.V.
2. Calculate forces exerted to fluid using momentum equation
3. Use Newton's third law to calculate force extorted to the surface by fluid



2.6 Propulsion force of an air jet engine



Momentum equation

$$\iint_S (\rho\vec{V}\cdot d\vec{S})\vec{V} + \iiint_{\mathscr{V}} \frac{\partial}{\partial t}(\rho\vec{V})d\mathscr{V} = \iiint_{\mathscr{V}} \rho\vec{f}d\mathscr{V} - \iint_S p d\vec{S}$$

Steady, no body force: $\iint_S (\rho\vec{V}\cdot d\vec{S})\vec{V} = -\iint_S p d\vec{S}$

X-component: $\iint (\rho\vec{V}\cdot d\vec{S})V_x = -\iint (p d\vec{S})_x$

$$-\rho_\infty V_\infty A_i(-V_\infty) + 0 + \rho_e V_e A_e(-V_e) + 0 \quad (*)$$
$$= -(p_\infty A_i + \int_a^b p_i dS_x - p_e A_e + \int_c^d p_i dS_x)$$

2.6 Propulsion force of an air jet engine

$$-\rho_\infty V_\infty A_i(-V_\infty) + 0 + \rho_e V_e A_e(-V_e) + 0 \quad (*)$$

$$= -(p_\infty A_i + \int_a^b p_i dS_x - p_e A_e + \int_c^d p_i dS_x)$$

$$\text{Let } \dot{m}_i = \rho_\infty V_\infty A_i \text{ and } \dot{m}_e = \rho_e V_e A_e$$

We have $\dot{m}_i V_\infty - \dot{m}_e V_e = -p_\infty A_i + p_e A_e - (\int_a^b p_i dS_x + \int_c^d p_i dS_x)$

壁面对气体的作用力

$$\text{Thrust: } T = -\int (p_i dS)_x - p_\infty (A_e - A_i)$$

$$T = -(\int_a^b p_i dS_x + \int_c^d p_i dS_x) - p_\infty (A_e - A_i)$$

气体对壁面的作用力

$$T = \dot{m}_e V_e - \dot{m}_i V_\infty + (p_e - p_\infty) A_e$$

thrust=rate of momentum change at inlet and exit+pressure at exit

2.6 Propulsion force of an air jet engine

Example 2.1

Consider a turbojet-powered airplane flying at a velocity of 300 m/s at an altitude of 10 km, where the free-stream pressure and density are 2.65×10^4 N/m^2 and 0.414 kg/m^3, respectively. The turbojet engine has inlet and exit areas of 2 m^2 and 1 m^2, respectively. The velocity and pressure of the exhaust gas are 500 m/s and 2.3×10^4 N/m^2 , respectively. The fuel-to-air mass ratio is 0.05. Calculate the thrust of the engine.

2.6 Propulsion force of an air jet engine

Example 2.2

Consider a liquid-fueled rocket engine burning liquid hydrogen as the fuel and liquid oxygen as the oxidizer. The hydrogen and oxygen are pumped into the combustion chamber at rates of 11 kg/s and 89 kg/s, respectively. The flow velocity and pressure at the exit of the engine are 4000 m/s and and 1.2×10^3 N/m^2, respectively. The exit area is 12 m^2. The engine is part of a rocket booster that is sending a payload into space. Calculate the thrust of the rocket engine as it passes through an altitude of 35 km, where the ambient pressure is 0.584×10^3 N/m^2.