§3.8 Fanno Flow
(One dimensional adiabatic flow with friction)

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This note derives the governing equations for one dimensional flow with friction from the wall, also known as the Fanno flow. The model system considered is one dimensional flow (constant area flow) with friction from the wall modeled as a wall shear stress τ_w. The length of the flow is L. Note the flow is still inviscid flow although this seems not correct in the first sense, but it helps understanding the physics behind.

§3.7.1 **The integral forms of the governing equations**

Consider adiabatic flow in a pipe. The wall friction is modeled as a stress exerted τ_w (in units of N/m^2) on the fluid from the wall. For high enough Reynolds number (true for the case of compressible flow), the boundary layers close to the wall occupies a very small fraction of the volume. We can thus ignore the boundary layers and assume that the flow is one dimensional constant area flow with area $A = \frac{\pi D^2}{4}$. Here D is the diameter of the pipe.

Figure 1 shows the Moody plot. It is seen that the friction factor $c_f = \frac{\tau_w}{\frac{1}{2}\rho u^2}$ changes with the Reynolds number Re. For the case of compressible flow, say $M > 0.3$ and $D = 10cm$, the corresponding Reynolds number is $Re = 6.37 \times 10^5$. $Re = 2.1 \times 10^6$ for $M = 1$. Please also note that roughness plays a key role.

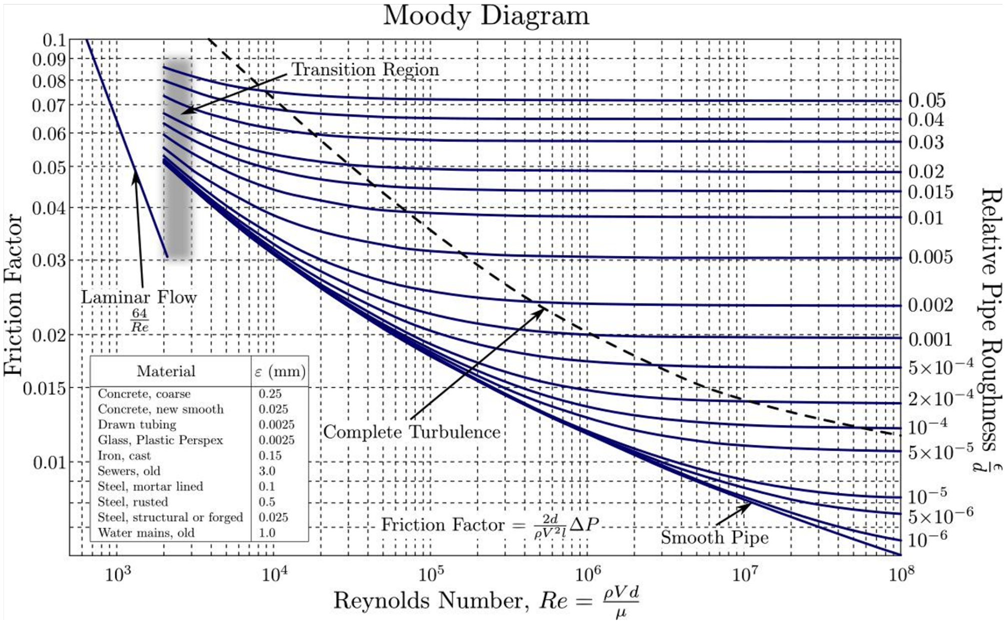

Figure 1. Moody graph showing the friction coefficient as a function of the Reynolds number Re and roughness height h.

1. The continuity equation:

$$\rho_1 u_1 = \rho_2 u_2 \tag{1}$$

2. The momentum equation for steady, adiabatic flow with friction from the wall:

$$\iint_s (\rho \vec{V} \cdot d\vec{S}) u = -\iint_S (pdS)x - \iint_S (\tau_w dS) \tag{2}$$

$$-\rho_1 u_1 A u_1 + \rho_2 u_2 A u_2 = p_1 A - p_2 A - \int_0^L \pi D \tau_w dx \tag{3}$$

Divide both sides of equation (3) with A, we have

$$p_2 - p_1 + \rho_2 u_2^2 - \rho_1 u_1^2 = -\frac{4}{D} \int_0^L \tau_w dx \tag{4}$$

3. The energy equation:

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \Rightarrow T_{01} = T_{02} \tag{5}$$

§3.7.2 **The differential forms of the governing equations**

To understand how friction changes the flow properties, let's derive differential equations that govern the flow property change.

1. The density change:

$$\frac{d\rho}{\rho} = -\frac{du}{u} \tag{6}$$

2. The temperature change:

   Recalling the definition of total temperature for adiabatic flow, we have $T_0 = T + \frac{u^2}{2c_p} = const.$. Differential this equation with respect to T, we have

$$dT + \frac{u}{c_p} du = 0 \tag{7}$$

$$\frac{dT}{T} = -(\gamma - 1) M^2 \frac{du}{u} \tag{8}$$

3. The pressure change:

   From EOS we have $p = \rho R T$, thus

$$dp = R\rho dT + RT d\rho \Rightarrow \frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T} \tag{9}$$

   Plugin the relation for $\frac{d\rho}{\rho}$ (equation 6) and $\frac{dT}{T}$ (equation 8), we have

$$\frac{dp}{p} = -[1 + (\gamma - 1)M^2]\frac{du}{u} \tag{10}$$

4. The entropy change:

$$ds = c_v\frac{dT}{T} - R\frac{d\rho}{\rho} \tag{11}$$

Substitute $\frac{dT}{T}$ (equation 8) abd $\frac{d\rho}{\rho}$ (equation 6) into the above equation, we have

$$ds = R(1 - M^2)\frac{du}{u} \tag{12}$$

5. The total pressure change:

Recalling that the entropy change $ds = c_p \ln \frac{dT_0}{T_0} - R\frac{dp_0}{p_0}$. For adiabatic flow $dT_0 = 0$, thus $ds = -R\frac{dp_0}{p_0}$. Then we have

$$\frac{dp_0}{p_0} = -\frac{ds}{R} = -(1 - M^2)\frac{du}{u} \tag{13}$$

6. The Mach number change: From the definition of Mach number, we know $M = \frac{u}{a} = \frac{u}{\sqrt{\gamma RT}}$. Differentiate this equation w.r.t. u and T, we have

$$dM = M\frac{du}{u} - \frac{1}{2}M\frac{dT}{T} \Rightarrow \frac{dM}{M} = \frac{du}{u} - \frac{1}{2}\frac{dT}{T} \tag{14}$$

$$\frac{dM}{M} = \frac{du}{u} - \frac{1}{2}[-(\gamma - 1)M^2\frac{du}{u}] = (1 + \frac{\gamma - 1}{2}M^2)\frac{du}{u} \tag{15}$$

7. The total temperature is conserved, thus $T_{01} = T_{02}$.

The flow property changes has been written in terms of $\frac{du}{u}$. But it is not straight forward to know how $\frac{du}{u}$ changes with friction. On the other hand, as the viscous stress from the wall is a dissipative process, the entropy of the system must increase when the flow passes pipes with friction, i.e., ds must be positive in this case. To Illustrate how the flow properties change with friction, let's now recast equations (6,8,10,12 and 13) in terms of ds. This can be done with ease. They are listed in the following.

(1) Velocity change:

$$\frac{du}{u} = \frac{1}{R(1-M^2)}ds \tag{16}$$

(2) Density change:

$$\frac{d\rho}{\rho} = -\frac{1}{R(1-M^2)}ds \tag{17}$$

(3) Temperature change:

$$\frac{dT}{T} = -(\gamma-1)M^2\frac{1}{R(1-M^2)}ds \tag{18}$$

(4) Pressure change:

$$\frac{dp}{p} = -[1+(\gamma-1)M^2]\frac{1}{R(1-M^2)}ds \tag{19}$$

(5) Total pressure change:

$$\frac{dp_0}{p_0} = -\frac{ds}{R} \tag{20}$$

(6) Mach number change:

$$\frac{dM}{M} = [1+\frac{(\gamma-1)}{2}M^2]\frac{1}{R(1-M^2)}ds \tag{21}$$

Now let's analyze how friction affect the flow properties. Note the influence for subsonic flow and supersonic flow are different.

	$ds \uparrow$
$M<1$	$\rho\downarrow$,$p\downarrow$,$s\uparrow$,$M\uparrow$, $p_0\downarrow$, $u\uparrow$, $T\downarrow$, $T_0 = const.$
$M>1$	$\rho\uparrow$,$p\uparrow$,$s\uparrow$, $M\downarrow$, $p_0\downarrow$, $u\downarrow$, $T\uparrow$ $T_0 = const.$

Table 1. Flow property change with friction action.

Now let's discuss the flow property change on the Fanno curve, i.e. $T - s$ curve. From equation (17), we can get

$$\frac{dT}{ds} = -(\gamma - 1)M^2 \frac{T}{R(1 - M^2)} = -\frac{M^2}{1 - M^2}\frac{T}{c_v} \tag{22}$$

For a given inlet condition in region 1, it corresponds to a point in the Fanno curve. Then the other points on the Fanno curve are collection of all possible states starting from region 1 with different length of the flow. For $M > 1$, the Fanno curve has a positive slope. For $M < 1$, the Fanno curve has a negative slope. In addition, $\frac{dT}{ds} \to \mp\infty$ when $M \to 1$.

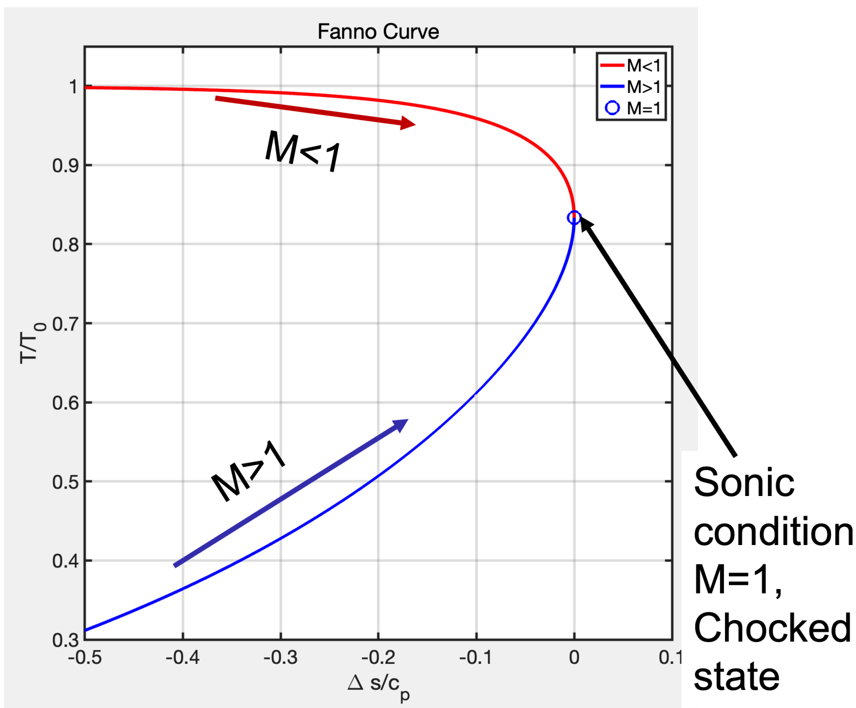

Figure 2. Fanno curve for one dimensional flow with friction.

Now let's discuss the flow properties on the Fanno curve, i.e, in the T-s curve. Several key points should be mentioned:

(1) There are two branches, one for subsonic flow, one for the super sonic flow.

(2) Friction increases the entropy of the flow. Thus it drives the flow towards the sonic condition, i.e., increase the Mach number for a subsonic flow and decrease the Mach number for a supersonic flow.

(3) It is not possible to achieve supersonic flow by friction.

(4) When the sonic condition is reached, the flow is said to be chocked, the corresponding pipe length is denoted as L^*. When $L > L*$, for subsonic flow, the wave propagates upstream to change the inlet conditions, reducing the mass flow rate and lower the inlet Mach number to accommodates the length. The new Mach number is determined when $L^* = L$. For supersonic flow, there will be a normal shock wave stand in the pipe. The location of the normal shock wave needs to be determined interactively.

Viscous chocking is very important in the design of pipe flows as it may reduce the total pressure and also the mass flow rate.

Example questions. See textbook Examples 3.15 (p.110), 3.16 (p.110).

§3.8.3 **Integral form of flow property change with friction**

1. **The Mach number relation**

   From equation (4), for a small dx, we have

$$d(p + \rho u^2) = -\frac{4}{D}\tau_w dx \tag{23}$$

   Introducing the friction factor $f = \tau_w/(1/2\rho u^2)$ into the above equation, one obtains

$$d(p + \rho u^2) = -\frac{1}{2}\rho u^2 \frac{4}{D} f dx \implies \frac{4}{D} f dx = -2d(p + \rho u^2)/(\rho u^2) \tag{24}$$

   Recalling $d(p + \rho u^2) = dp + \rho u du + u d(\rho u)$ and $d(\rho u) = 0$, equation (24) becomes

$$\frac{4}{D} f dx = -2(dp + \rho u du)/(\rho u^2) = -2(\frac{dp}{\rho u^2} + \frac{du}{u}) \tag{25}$$

   From EOS, we know $\rho = p/(RT)$. Substitute this into equation (25), one obtains

$$\frac{4}{D} f dx = -2(\frac{dp}{p}\frac{\gamma RT}{\gamma u^2} + \frac{du}{u}) = -2(\frac{1}{\gamma M^2}\frac{dp}{p} + \frac{du}{u}) \tag{26}$$

   Substitute equation (10) into the above equation, we have

$$\frac{4}{D} f dx = -2\{-\frac{1}{\gamma M^2}[1 + (\gamma - 1)M^2]\}\frac{du}{u} + \frac{du}{u}) = \frac{2(1 - M^2)}{\gamma M^2}\frac{du}{u} \tag{27}$$

Substitute equation (15) for $\frac{du}{u}$ into the above equation, we have

$$\frac{4}{D}fdx = \frac{(1-M^2)}{\gamma M^2}\frac{1}{(1+\frac{\gamma-1}{2}M^2)}\frac{dM^2}{M^2} \tag{28}$$

This equation related the Mach number change over a distance dx.

Integrating over L between $x_1(M_1)$ and $x_2(M_2)$, we have

$$\int_{x_1}^{x_2}\frac{4fdx}{D} = [-\frac{1}{\gamma M^2} - \frac{\gamma+1}{2\gamma}\ln(\frac{M^2}{1+\frac{\gamma-1}{2}M^2})]_{M_1}^{M_2} \tag{29}$$

2. **The total temperature relation**

$$T_{01} = T_{02} \tag{30}$$

3. **The static temperature relation**

$$\frac{T_2}{T_1} = \frac{T_2}{T_0}\frac{T_0}{T_1} = \frac{1+\frac{\gamma-1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_2^2} = \frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2} \tag{31}$$

4. **The pressure relation**

   From the continuity equation $\rho_1 u_1 = \rho_2 u_2$ and definition of speed of sound $a^2 = \frac{\gamma p}{\rho}$, we have

$$\frac{\gamma p_1 u_1}{a_1^2} = \frac{\gamma p_2 u_2}{a_2^2} \tag{32}$$

$$\frac{p_2}{p_1} = \frac{u_1}{u_2}\frac{a_2^2}{a_1^2} = \frac{M_1}{M_2}\frac{a_2}{a_1} = \frac{M_1}{M_2}(\frac{T_2}{T_1})^{1/2} \tag{33}$$

   Substitute equation (31) into the above equation, we have

$$\frac{p_2}{p_1} = \frac{M_1}{M_2}[\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}]^{1/2} \tag{34}$$

5. **The density relation**

$$\frac{\rho_2}{\rho_1} = \frac{p_2}{p_1}\frac{T_1}{T_2} = \frac{M_1}{M_2}[\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}]^{-1/2} \tag{35}$$

6. **The total pressure relation**

$$\frac{p_{02}}{p_{01}} = \frac{p_{02}}{p_2}\frac{p_2}{p_1}\frac{p_1}{p_{01}} = [\frac{2+(\gamma-1)M_2^2}{2+(\gamma-1)M_1^2}]^{\gamma/(\gamma-1)}\frac{M_1}{M_2}[\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}]^{1/2} = \frac{M_1}{M_2}[\frac{2+(\gamma-1)M_2^2}{2+(\gamma-1)M_1^2}]^{(\gamma+1)/[2(\gamma-1)]} \tag{36}$$

So how to solve a problem practically?

Given conditions in region 1, i.e., M_1, D and f, M_2 can be obtained using equation (29). Then using equations (31,34,35,36) other flow properties can be calculated. However it is alway difficult to calculate M_2 from equation (29).

A more practical way is to use a reference state as we did for Rayleigh flow. Let us choose a reference state with M=1 and solve the other state parameters with reference to this M=1 state. Let $M_2 = 1$ in equations 29,31, and 34-36, which corresponds to $L = L^*$, we have:

$$\frac{T}{T^*} = \frac{\gamma+1}{2+(\gamma-1)M^2} \tag{37}$$

$$\frac{p}{p^*} = \frac{1}{M}\frac{\gamma+1}{2+(\gamma-1)M^2} \tag{38}$$

$$\frac{\rho}{\rho^*} = \frac{1}{M}[\frac{\gamma+1}{2+(\gamma-1)M^2}]^{-1/2} \tag{39}$$

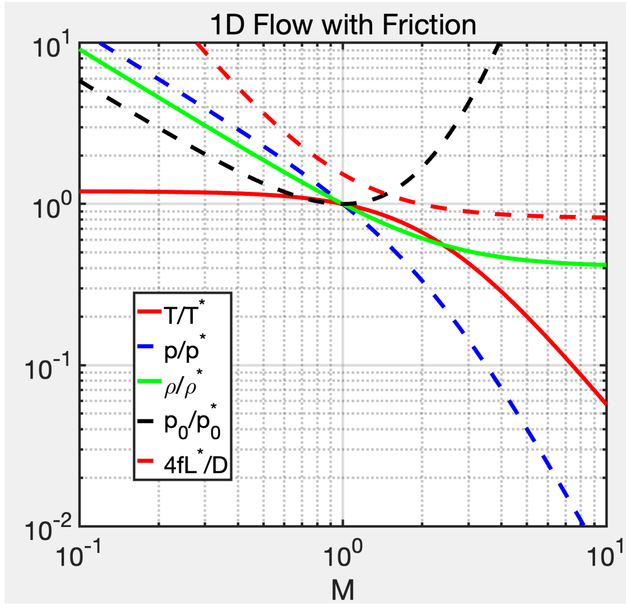

Figure 3. Change of flow properties w.r.t. the sonic reference state as a function of the Mach number.

$$\frac{p_0}{p_0^*} = \frac{1}{M}[\frac{2+(\gamma-1)M^2}{\gamma+1}]^{(\gamma+1)/[2(\gamma-1)]} \tag{40}$$

$$\int_0^{L^*} \frac{4fdx}{D} = [-\frac{1}{\gamma M^2} - \frac{\gamma+1}{2\gamma}\ln(\frac{M^2}{1+\frac{\gamma-1}{2}M^2})]_M^1 \tag{41}$$

$$\frac{4\bar{f}L^*}{D} = \frac{1-M^2}{\gamma M^2} + \frac{\gamma+1}{2\gamma}\ln(\frac{(\gamma+1)M^2}{2+(\gamma-1)M^2}) \tag{42}$$

Here $\bar{f} = \frac{1}{L*}\int_0^{L^*} f dx$ and $\frac{4\bar{f}L}{D} = \frac{4\bar{f}L_1^*}{D} - \frac{4\bar{f}L_2^*}{D}$

Equation 37-42 are plotted as a function of M in figure 3.

Example problems: see P115 example 3.17 and 3.18 from the textbook.