7.4 Linearized supersonic flow

Linearized perturbation velocity equation

$$\boxed{(1 - M_\infty^2)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0}$$

- Subsonic flow $\beta^2\phi_{xx} + \phi_{yy} = 0 \quad \beta = \sqrt{1 - M_\infty^2}$

  Elliptical partial differential equation (PDE)

- Supersonic flow $\lambda^2\phi_{xx} - \phi_{yy} = 0 \quad \lambda = \sqrt{M_\infty^2 - 1}$

  Hyperbolic PDE

7.4 Linearized supersonic flow

The model: supersonic flow over a surface with a small bump on two sides.

$$\lambda^2\phi_{xx} - \phi_{yy} = 0 \quad \lambda = \sqrt{M_\infty^2 - 1} \quad ①$$

The general solution of equation ① is the following:

$$\phi = f(x-\lambda y) + g(x+\lambda y)$$

Proof:

Proof:

$$\phi_x = \frac{\partial f}{\partial(x-\lambda y)}\frac{\partial(x-\lambda y)}{\partial x} + \frac{\partial g}{\partial(x+\lambda y)}\frac{\partial(x+\lambda y)}{\partial x}$$

$$= f' + g' \qquad f' = \frac{\partial f}{\partial(x-\lambda y)} \quad g' = \frac{\partial g}{\partial(x+\lambda y)}$$

$$\phi_{xx} = \frac{\partial f'}{\partial(x-\lambda y)}\cdot\frac{\partial(x-\lambda y)}{\partial x} + \frac{\partial g'}{\partial(x-\lambda y)}\cdot\frac{\partial(x+\lambda y)}{\partial x}$$

$$= f'' + g'' \qquad f'' = \frac{\partial^2 f}{\partial(x-\lambda y)^2} \quad g'' = \frac{\partial^2 g}{\partial(x+\lambda y)^2}.$$

$$\phi_y = \frac{\partial f}{\partial(x-\lambda y)}\frac{\partial(x-\lambda y)}{\partial y} + \frac{\partial g}{\partial(x+\lambda y)}\cdot\frac{\partial(x+\lambda y)}{\partial y}$$

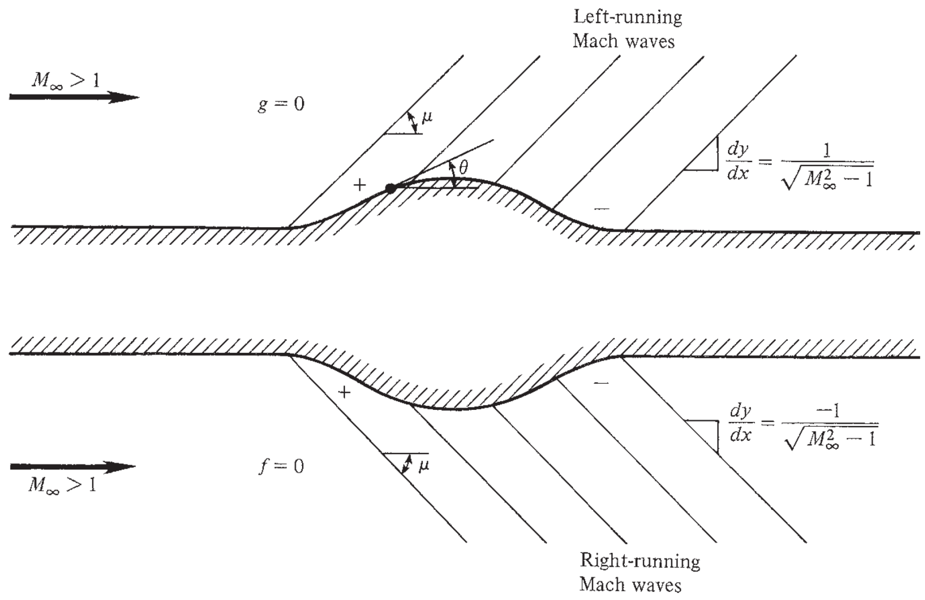
$$= f'(-\lambda) + g'\lambda$$

$$\phi_{yy} = \frac{\partial f'(-\lambda)}{\partial(x-\lambda y)}\frac{\partial(x-\lambda y)}{\partial y} + \frac{\partial g'\lambda}{\partial(x+\lambda y)}\frac{\partial(x+\lambda y)}{\partial y}$$

$$= \lambda^2 f'' + \lambda^2 g''$$

$$\lambda^2\phi_{xx} - \phi_{yy} = \lambda^2(f''+g'') - \lambda^2(f''+g'') = 0.$$

END.



7.4 *Linearized* supersonic flow

(1) General solution to equation ① $\phi = f(x - \lambda y) + g(x + \lambda y)$

- Let g=0, for constant ϕ, x-λy=const., thus $\dfrac{dy}{dx} = \dfrac{1}{\lambda} = \dfrac{1}{\sqrt{M_\infty^2 - 1}}$

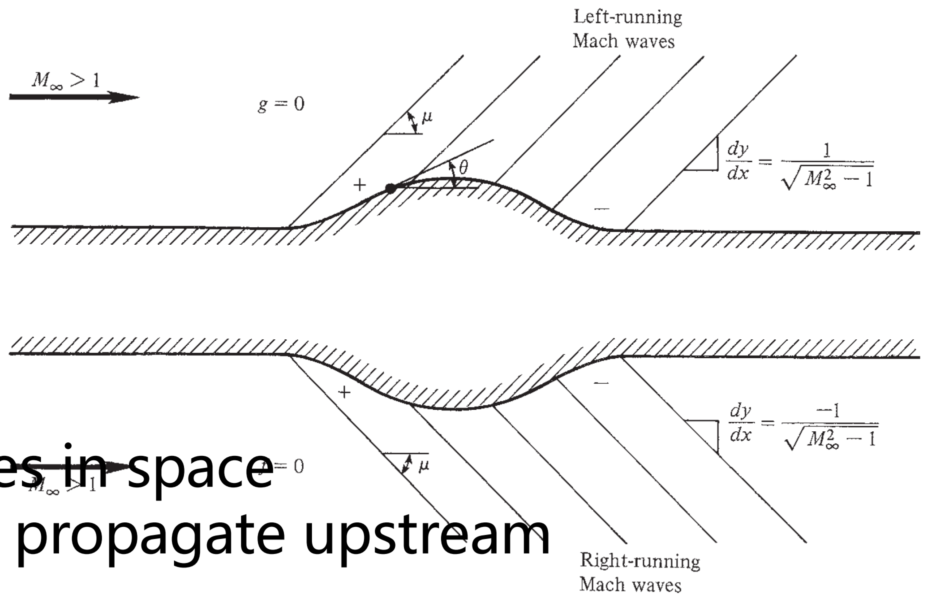
- Recalling Mach angle $\mu = \arcsin(1/M_\infty) = \arctan\left(1/\sqrt{M_\infty^2 - 1}\right)$

  Constant ϕ in this case means left running Mach waves.

- Let f=0, for constant ϕ, $x+\lambda y$=const., thus

$$\frac{dy}{dx} = \frac{-1}{\lambda} = \frac{-1}{\sqrt{M_\infty^2 - 1}}$$

Constant ϕ in this case means right running Mach waves.



Subsonic flow: perturbation propagates in space
Supersonic flow: perturbation can not propagate upstream

7.4 Linearized supersonic flow

(2) Linearized pressure coefficient $C_p = -\dfrac{2u'}{V_\infty}$

- Let g=0, corresponds to the upper Mach waves

$$\phi = f(x - \lambda y) \rightarrow \begin{cases} u' = \dfrac{\partial \phi}{\partial x} = f' \\ v' = \dfrac{\partial \phi}{\partial y} = -\lambda f' \end{cases} \rightarrow u' = -\dfrac{v'}{\lambda}$$

The BC at the surface $\tan\theta = \dfrac{dy}{dx} = \dfrac{v'}{V_\infty + u'}$

For small perturbations, $u' \ll V_\infty$ and $\tan\theta \approx \theta$.

$$\rightarrow v' = V_\infty \theta$$

$$u' = -\frac{V_\infty \theta}{\lambda}$$

$$C_p = -\frac{2u'}{V_\infty} = \frac{2\theta}{\lambda} \rightarrow \boxed{C_p = \frac{2\theta}{\sqrt{M_\infty^2 - 1}}}$$

Pressure coefficient for linearized supersonic flow

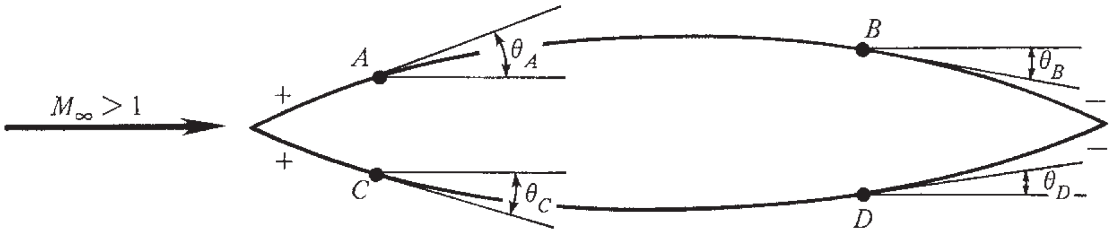
7.4 Linearized supersonic flow

(2) Linearized pressure coefficient

$$\boxed{C_p = \frac{2\theta}{\sqrt{M_\infty^2 - 1}}}$$

Pressure coefficient for linearized supersonic flow

- C_p is proportional to the local inclination angle with free stream direction
- Valid for any two-dimensional geometry
- For the right running Mach wave, we have $C_p = \dfrac{-2\theta}{\sqrt{M_\infty^2 - 1}}$
- Cp is positive on compression surface and negative on expansion surface
- Generation of wave drag for supersonic flow

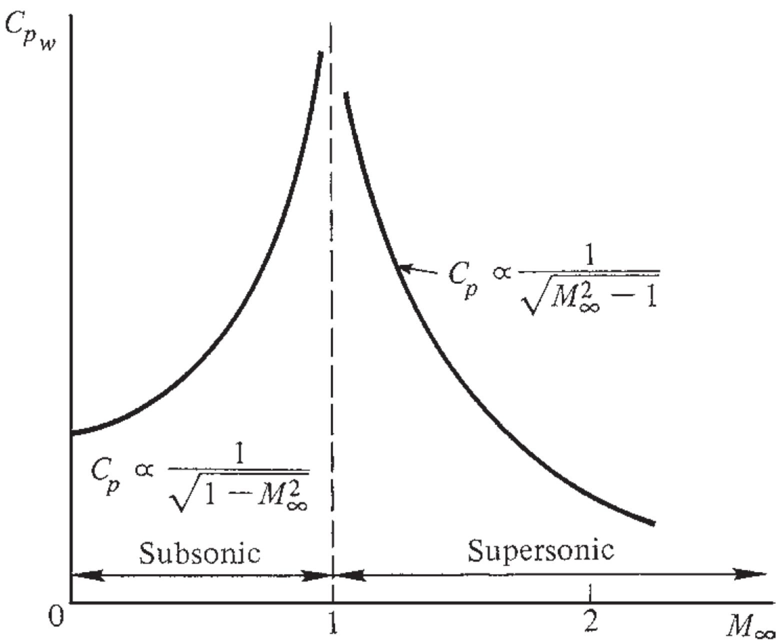


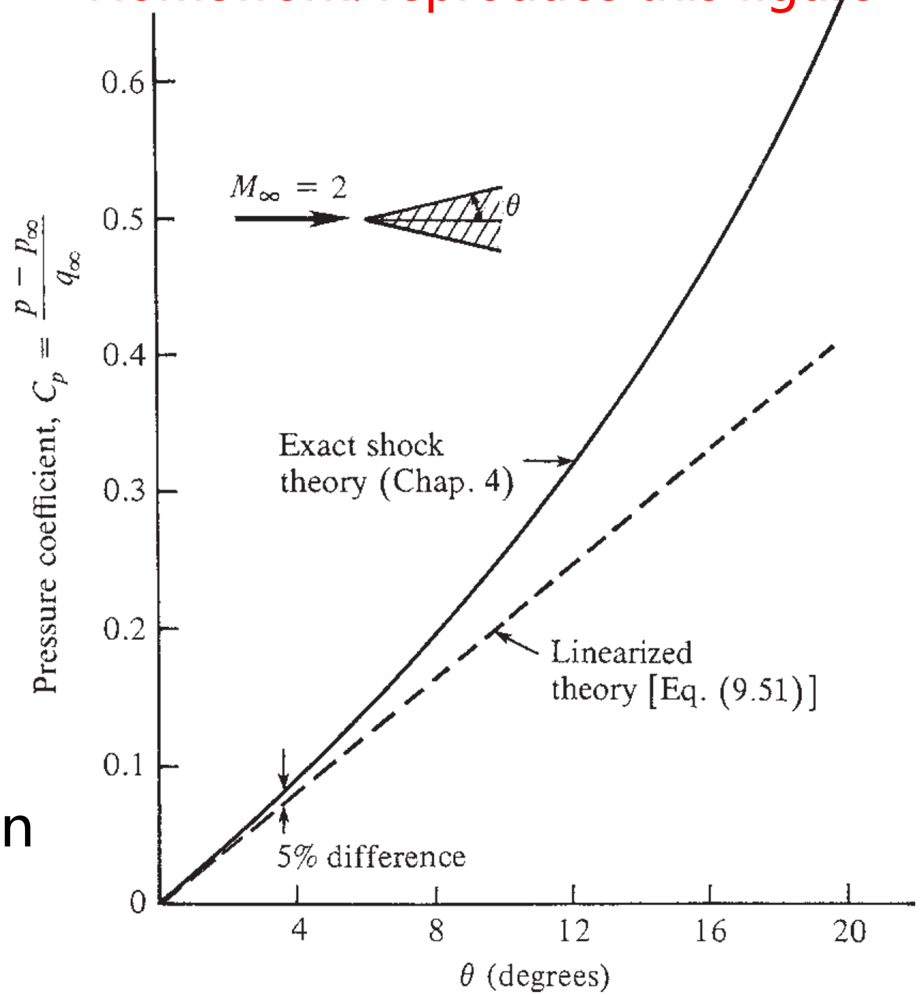
Biconvex airfoil

7.4 Linearized supersonic flow

- Effects of free stream Mach number

Homework: reproduce this figure





- Comparison between linearized theory and oblique shock solution

7.4 Linearized supersonic flow

Example: Supersonic flow over wavy surface

Consider a supersonic flow with an upstream Mach number of M_∞. This flow moves over a wavy wall with a contour given by $y_w = h\cos(2\pi x/l)$, where y_w is the ordinate of the wall. For small h, use linear theory to derive an equation for the velocity potential and surface pressure coefficient.

Solution

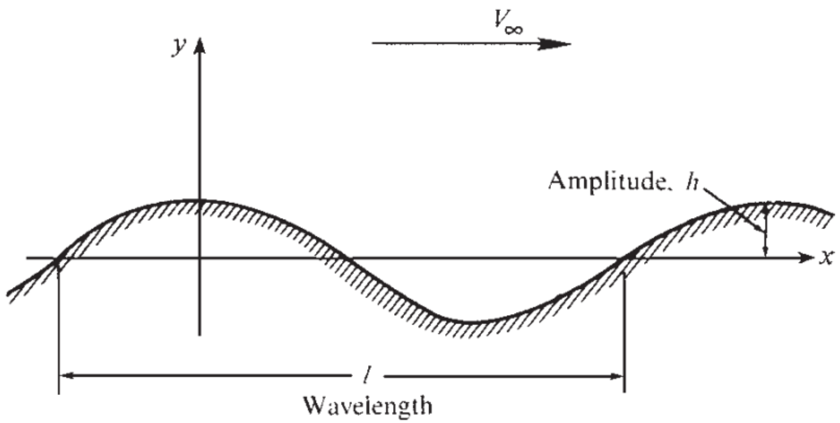
Governing equation $$\frac{\partial^2\phi}{\partial x} - \frac{1}{(M_\infty^2 - 1)}\frac{\partial^2\phi}{\partial y^2} = 0$$

General solution $$\phi(x,y) = f\left(x - \sqrt{M_\infty^2 - 1}y\right) + g\left(x + \sqrt{M_\infty^2 - 1}y\right)$$

Let g=0, $$\phi(x,y) = f\left(x - \sqrt{M_\infty^2 - 1}y\right)$$

$$\frac{\partial\phi}{\partial y} = \left[f'\left(x - \sqrt{M_\infty^2 - 1}y\right)\right]\left(-\sqrt{M_\infty^2 - 1}\right)$$

BC: $$\frac{dy_w}{dx} = \frac{v'_w}{V_\infty} = \frac{1}{V_\infty}\left(\frac{\partial\phi}{\partial y}\right)_w$$



7.4 Linearized supersonic flow

Thus, $\sqrt{M_\infty^2 - 1} f'(x) = V_\infty h \left(\frac{2\pi}{l}\right) \sin\left(\frac{2\pi x}{l}\right)$ at the wall

$$f'(x) = \frac{V_\infty h}{\sqrt{M_\infty^2 - 1}} \left(\frac{2\pi}{l}\right) \sin\left(\frac{2\pi x}{l}\right) \quad ②$$

Integration ② with respect to its arguments x

$$f(x) = -\frac{V_\infty h}{\sqrt{M_\infty^2 - 1}} \cos\left(\frac{2\pi x}{l}\right) + \text{const} \quad ③$$

Replacing x with $\left(x - \sqrt{M_\infty^2 - 1}\, y\right)$

Why g=0?!!

$$\boxed{\begin{aligned}\phi(x, y) &= f\left(x - \sqrt{M_\infty^2 - 1}\, y\right) \\ &= -\frac{V_\infty h}{\sqrt{M_\infty^2 - 1}} \cos\left[\frac{2\pi}{l}\left(x - \sqrt{M_\infty^2 - 1}\, y\right)\right] + \text{const}\end{aligned}}$$

7.4 Linearized supersonic flow

Supersonic flow $y_w = h\cos(2\pi x/l)$

$$C_p = -\frac{2u'}{V_\infty} = -\frac{2}{V_\infty}\frac{\partial\phi}{\partial x} = -\frac{4\pi}{\sqrt{M_\infty^2-1}}\left(\frac{h}{l}\right)\sin\left[\frac{2\pi}{l}\left(x-\sqrt{M_\infty^2-1}\,y\right)\right] \quad ④$$

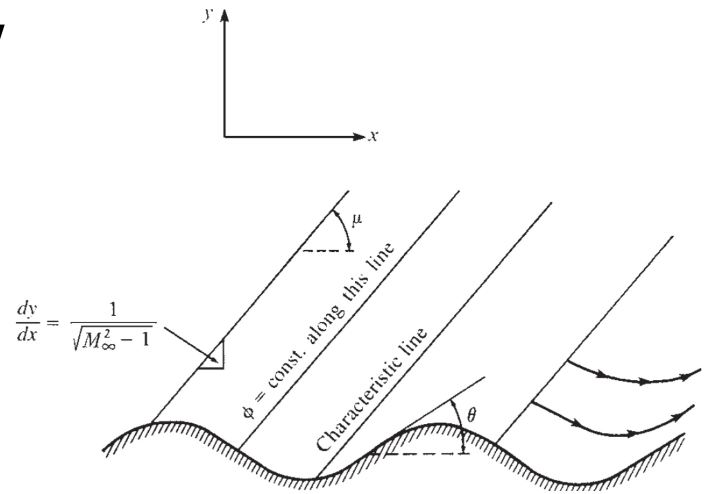
Subsonic flow

$$C_p = -\frac{2u'}{V_\infty} = -\frac{4\pi}{\sqrt{1-M_\infty^2}}\left(\frac{h}{l}\right)\boxed{\exp\left(\frac{-2\pi\sqrt{1-M_\infty^2}\,y}{l}\right)}\cos\left(\frac{2\pi x}{l}\right)$$

- No attenuation factor for supersonic flow, the perturbation propagates to infinity
- The magnitude of disturbance (ϕ or Cp) is a constant for given $\left(x-\sqrt{M_\infty^2-1}\,y\right)$

$$x-\sqrt{M_\infty^2-1}\,y = \text{const.} \rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{M_\infty^2-1}}$$

The lines are Mach lines.



7.4 Linearized supersonic flow

Supersonic flow $y_w = h\cos(2\pi x/l)$

$$C_{p_w} = -\frac{4\pi}{\sqrt{M_\infty^2 - 1}}\left(\frac{h}{l}\right)\sin\left(\frac{2\pi x}{l}\right) \quad ⑤$$

Subsonic flow

$$C_{p_w} = -\frac{4\pi}{\sqrt{1 - M_\infty^2}}\left(\frac{h}{l}\right)\cos\left(\frac{2\pi x}{l}\right)$$

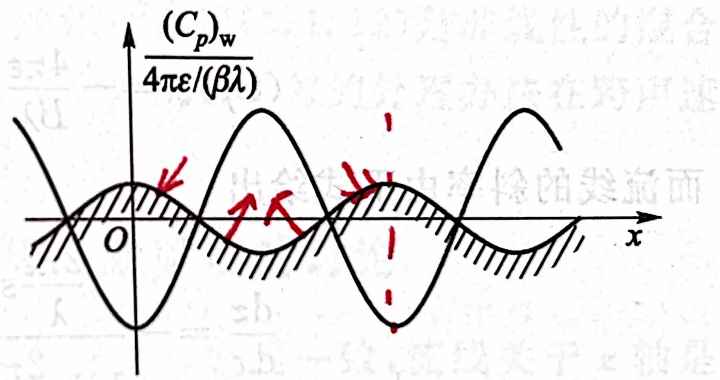


图 5.2.3 沿波形壁亚声速流的压力分布

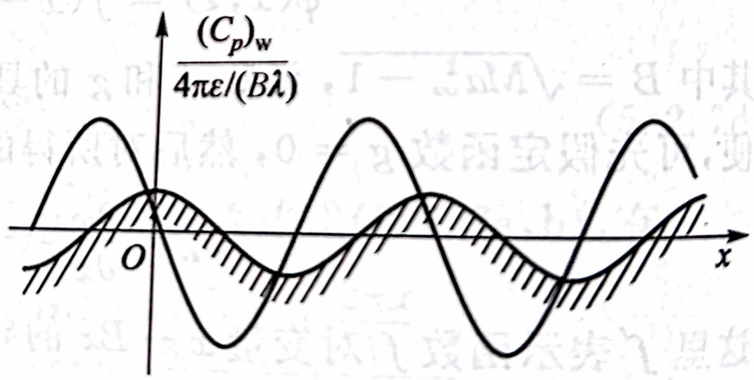


图 5.2.5 沿波形壁超声速流动的壁面压力分布

7.4 Linearized supersonic flow

(3) Recover the Prandtl-Glauert rule

$$y_w = h\cos\left(\frac{2\pi x}{l}\right)$$

$$\left.\begin{aligned}\frac{dy_w}{dx} &= -2\pi\left(\frac{h}{l}\right)\sin\left(\frac{2\pi x}{l}\right)\\ C_{p_w} &= -\frac{4\pi}{\sqrt{M_\infty^2-1}}\left(\frac{h}{l}\right)\sin\left(\frac{2\pi x}{l}\right)\end{aligned}\right\}\rightarrow C_p = \frac{2\left(\frac{dy_w}{dx}\right)}{\sqrt{M_\infty^2-1}}$$

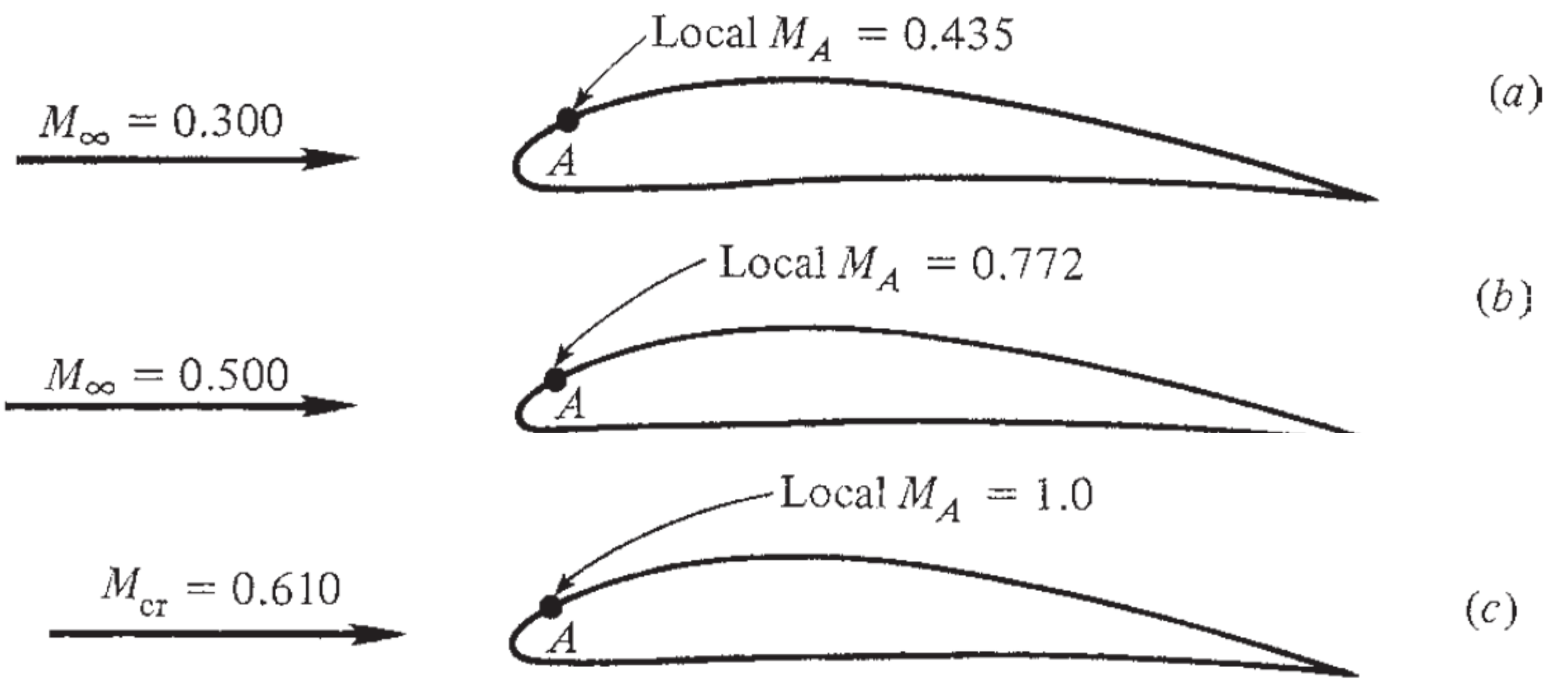
$$\left.\begin{aligned}C_p &= \frac{2\left(\frac{dy_w}{dx}\right)}{\sqrt{M_\infty^2-1}}\\ \tan\theta &= \frac{dy_w}{dx} \simeq \theta\end{aligned}\right\}\rightarrow \boxed{C_{p_w} = \frac{2\theta}{\sqrt{M_\infty^2-1}}}$$

Prandtl-Glauert rule

7.5 Critical Mach number

Linearized theory is not applicable when M∞≃1. Strong nonlinear effects.

(1) Definition of Critical Mach number



Critical Mach number: freestream Mach number at which sonic flow is first encountered on the airfoil.

7.5 Critical Mach number

(2) Calculation of Critical Mach number

Assume isentropic flow, we have $$\frac{p_A}{p_\infty} = \left(\frac{1 + \frac{\gamma - 1}{2} M_\infty^2}{1 + \frac{\gamma - 1}{2} M_A^2} \right)^{\gamma/(\gamma-1)}$$

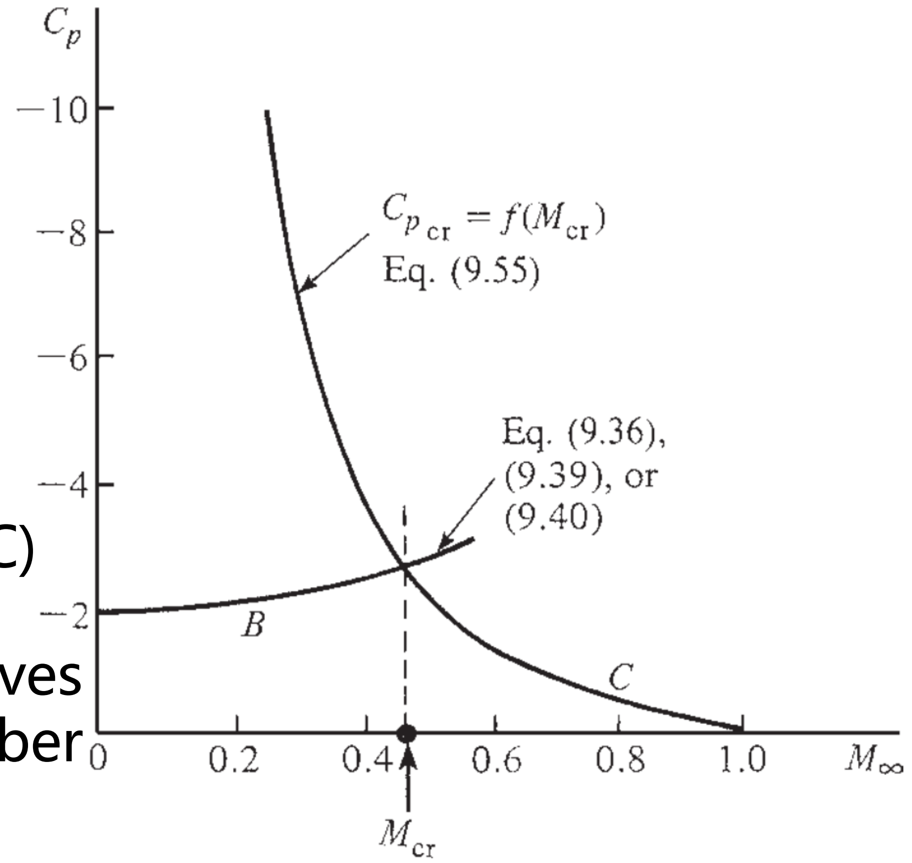
For pressure coefficient: $$C_{pA} = \frac{2}{\gamma M_\infty^2} \left[\left(\frac{1 + \frac{\gamma - 1}{2} M_\infty^2}{1 + \frac{\gamma - 1}{2} M_A^2} \right)^{\gamma/(\gamma-1)} - 1 \right]$$

When $M_A = 1$ $\quad M_\infty \equiv M_{cr}$

7.5 Critical Mach number

(2) Calculation of Critical Mach number

- Measure or calculate C_p for incompressible flow C_{p0}
- Use one of the compressibility correction plot C_p as a function of M_∞, (curve B)
- Plot C_{pcr} as a function of M_∞ (curve C)
- The intersection of the two curves determine the critical Mach number for a given airfoil

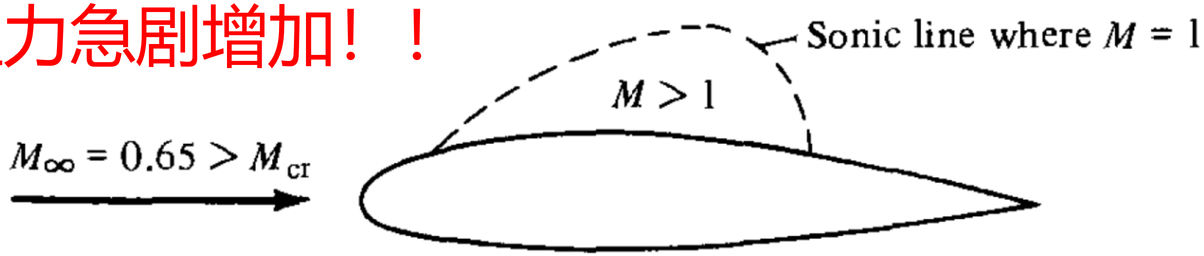


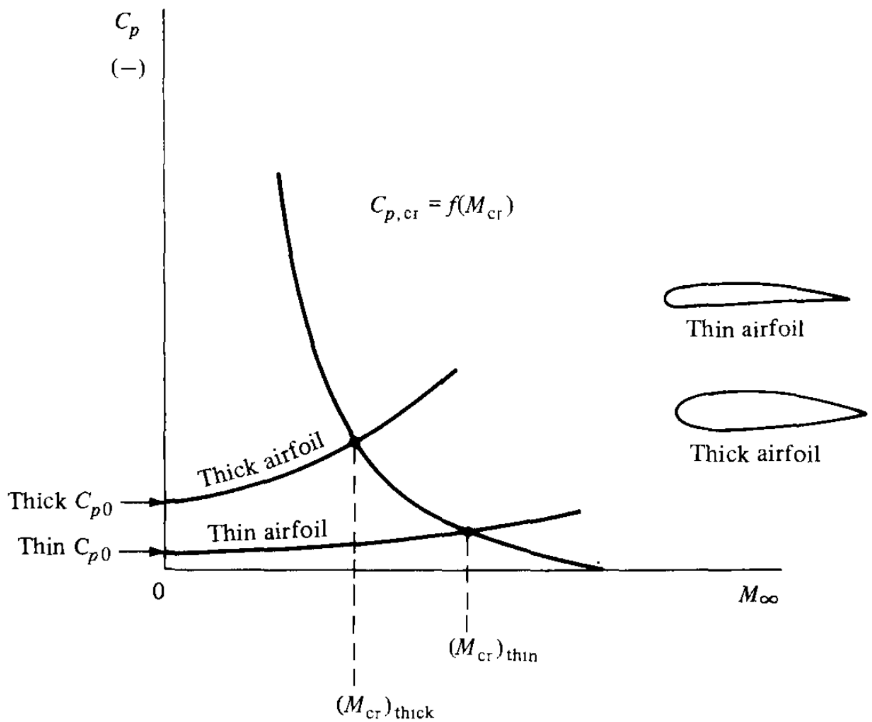
- Curve B depends on the geometry of the airfoil.
- Curve C is independent of the airfoil.

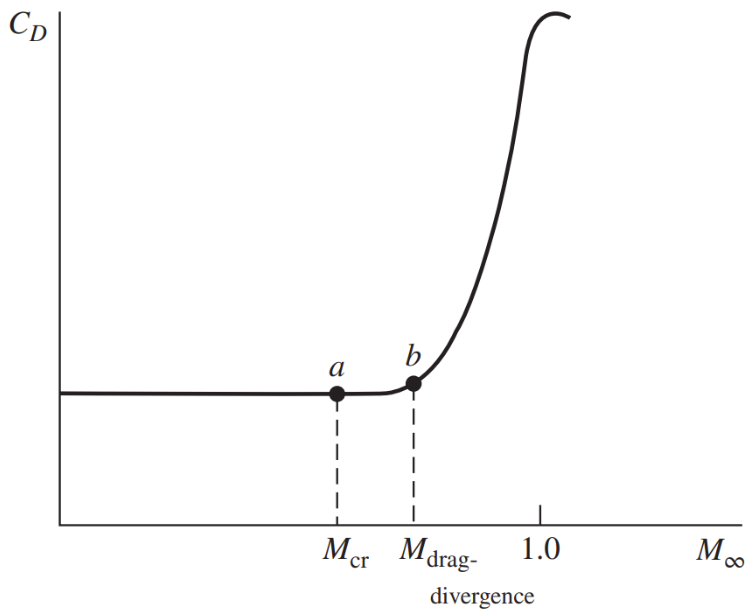
7.5 Critical Mach number

(3) Drag divergence and sound barrier

$Ma_\infty > Ma_{cr}$时翼型阻力急剧增加！！



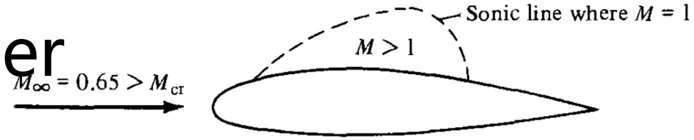




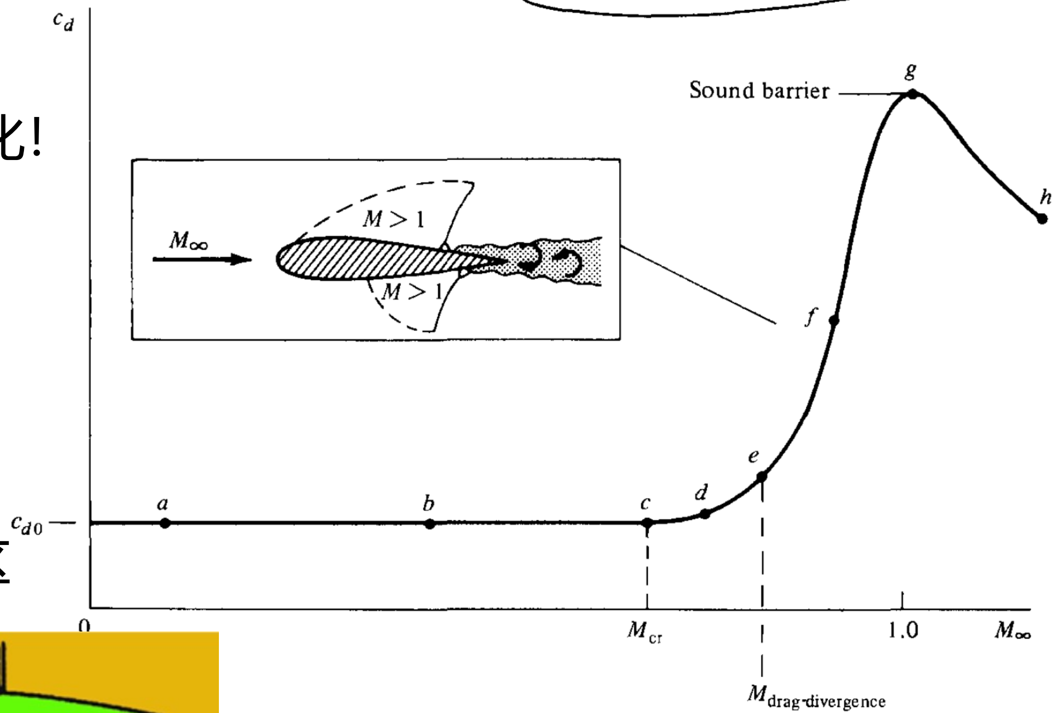
Drag divergence Mach no: the free stream Mach no when the drag start to rise.

7.5 Critical Mach number

(3) Drag divergence and sound barrier



1. $C_d \sim Ma$:



①$a - c$: $Ma_\infty < Ma_{cr}$, C_d不随Ma_∞变化!

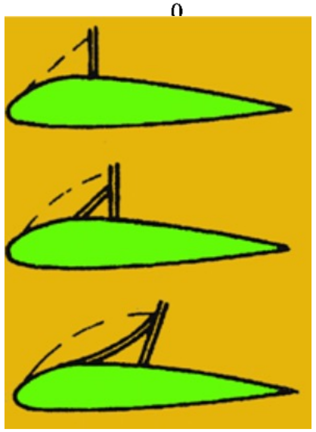
Ma_∞↑, Re↑，摩擦阻力↓,

$\frac{dC_p}{dx}$↑，分离区变大，压差阻力↑

→C_d变化小;

②$c - d$: $Ma_\infty > Ma_{cr}$, 上表面超声速区

→产生激波→阻力增加



③$c - d - e$: Ma_∞ ↑, 超声速区↑,

→激波↑ →阻力↑

7.5 Critical Mach number

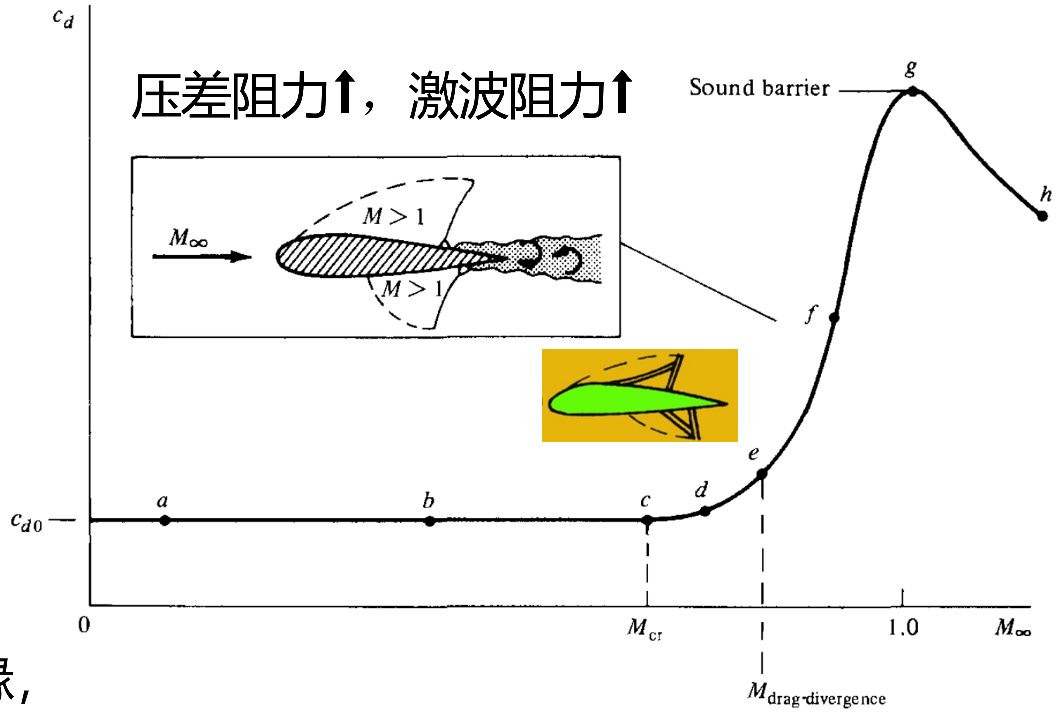
(3) Drag divergence and sound barrier

1. $C_d \sim Ma$:

④e: Ma_∞↑, 上下翼面超声速区扩大，
激波→逆压梯度↑,流动分离↑
→阻力开始急剧增加(阻力发散)。
阻力发散***Ma***:

$$dC_d/dMa_\infty = 0.1$$

g： C_d可增加到10 $C_{d,0}$。



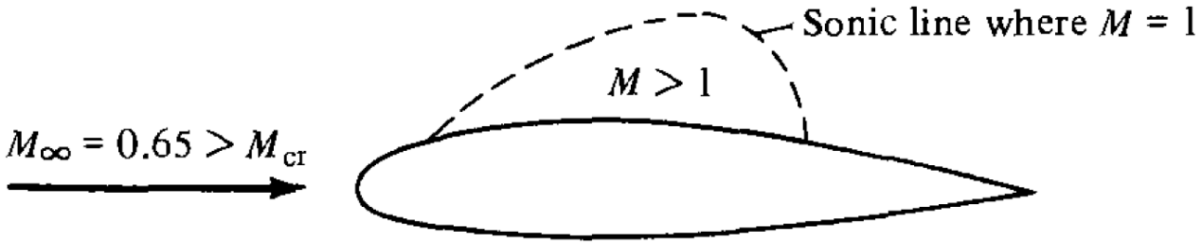
⑤g: Ma_∞→1, 上下翼面激波移至后缘，
→阻力达最大C_{dmax}。

⑥$g-f$: Ma_∞ >1, 进入超声速流，
Ma_∞↑C_d↓

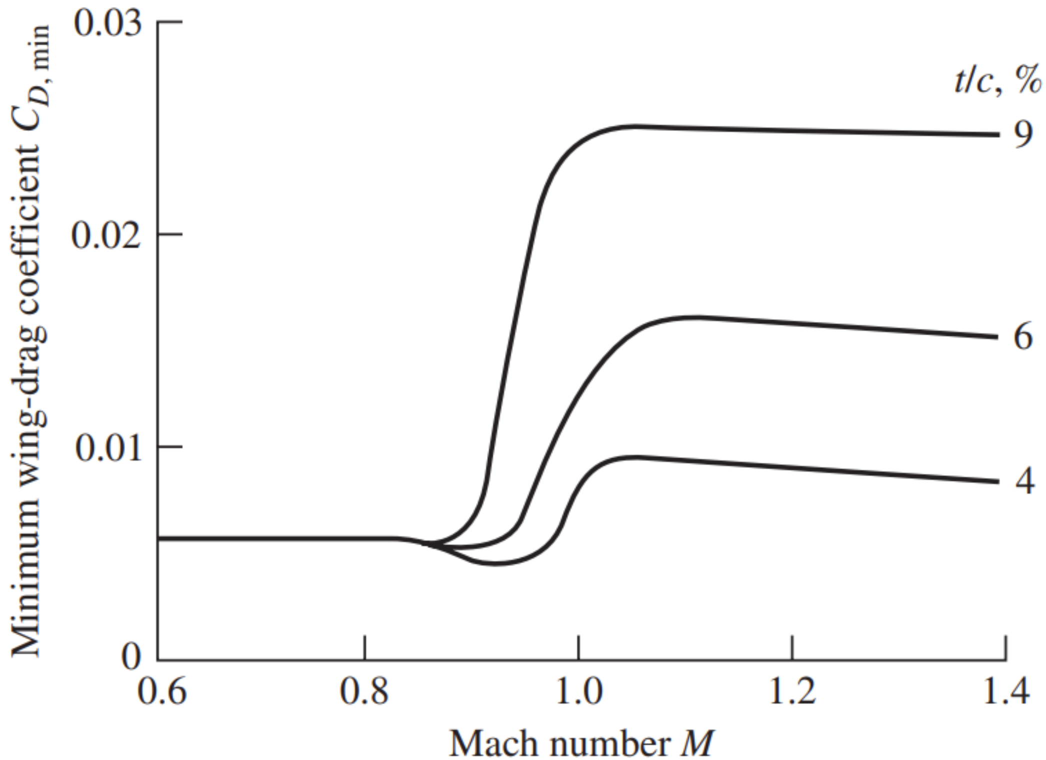
进入超声速流之前，必须跨越极大阻力！
音障（声障）

7.5 Critical Mach number

(3) Increasing the M_{cr}



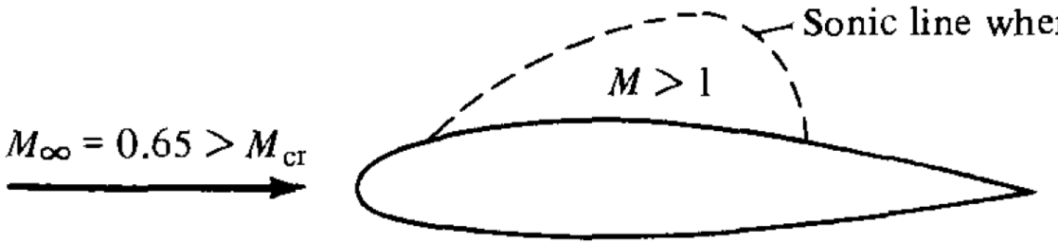
- 1.Make the wing thinner

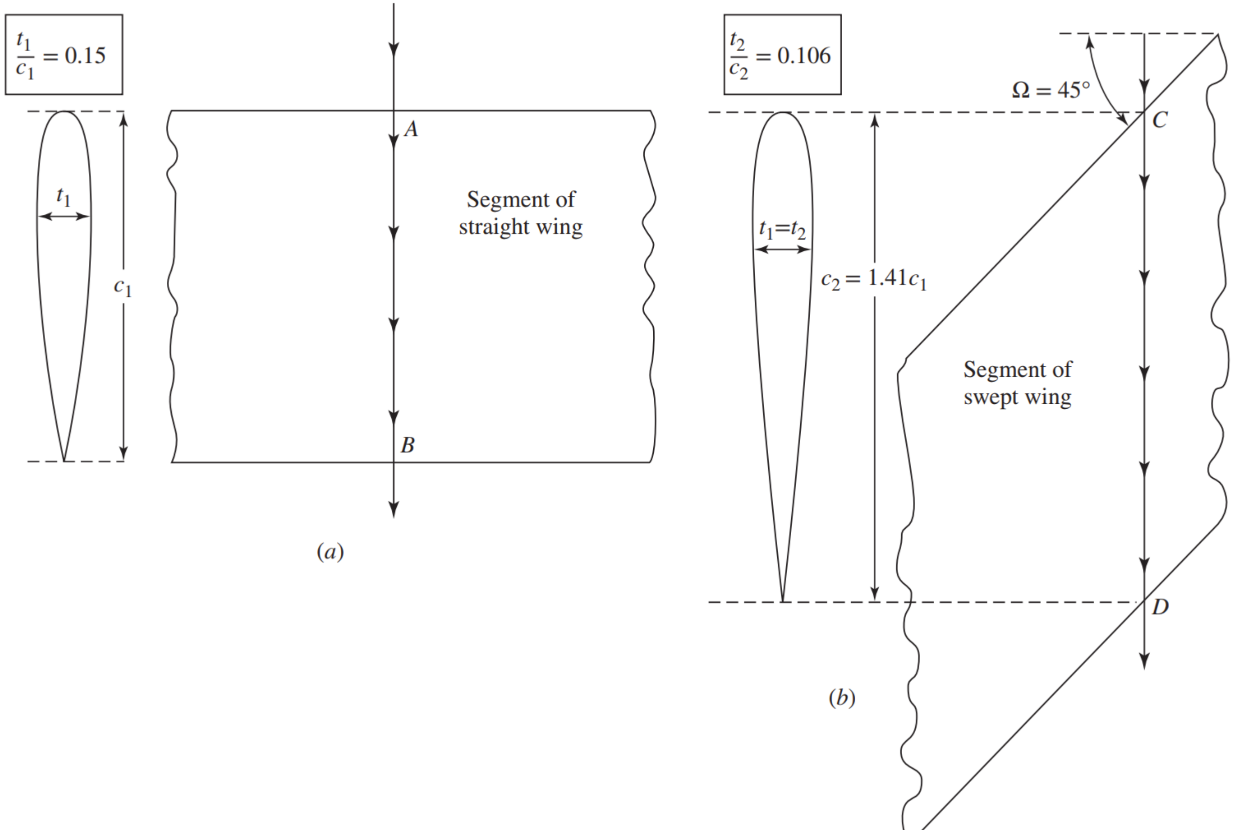


7.5 Critical Mach number

(3) Increasing the M_{cr}

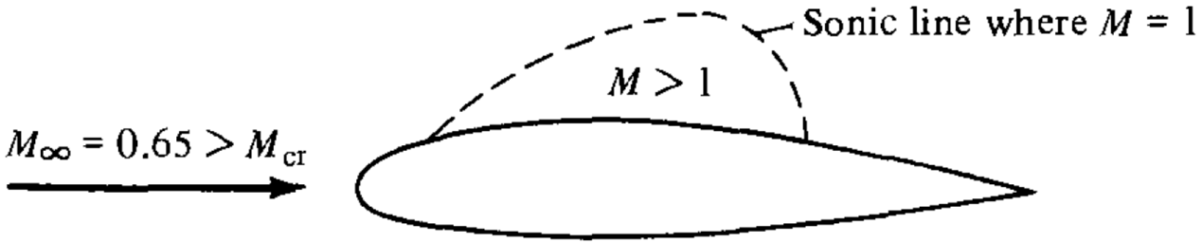
- 2.Sweep the wing





7.5 Critical Mach number

(3) Increasing the M_{cr}



- 2.Sweep the wing