西安交通大學
XI'AN JIAOTONG UNIVERSITY

气体动力学
Gas Dynamics

航天航空学院 工程力学系

谢毅超

yichao.xie@xjtu.edu.cn

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Ch3 One Dimensional Flow
一维流动

The Aeronautical engineer is pounding hard on the closed door leading into the field of supersonic motion.

Theodore von Karman, 1941

Review of Ch2

Continuity equation

$$\frac{\partial}{\partial t}\iiint_{\mathscr{V}} \rho d\mathscr{V} + \iint_S \rho \vec{V}\cdot d\vec{S} = 0$$

Momentum equation

$$\iint_S (\rho\vec{V}\cdot d\vec{S})\vec{V} + \frac{\partial}{\partial t}\iiint_{\mathscr{V}} (\rho\vec{V})d\mathscr{V} = \iiint_{\mathscr{V}} \rho\vec{f} d\mathscr{V} - \iint_S p d\vec{S}$$

Energy conservation

$$\iiint_{\mathscr{V}} \dot{q}\rho d\mathscr{V} - \iint_S (pd\vec{S})\cdot\vec{V} + \iiint_{\mathscr{V}} (\vec{f}\rho d\mathscr{V})\cdot\vec{V}$$
$$= \frac{\partial}{\partial t}\iiint_{\mathscr{V}} [(e+\frac{V^2}{2})\rho] d\mathscr{V} + \iint_S (e+\frac{V^2}{2})\rho d\vec{S}\cdot\vec{V}$$

EOS : $p=p(\rho,T)$; 6 equations, 6 variable (p, T, $\mathbf{V}$, ρ)
Given $\vec{f}$ and $\dot{q}$, the equations are closed.

Outline

- **CH3 One-dimensional flow**
  - Basic equations for 1D flow
    - C.E.
    - M.E.
    - E.E.
  - General considerations not limited to 1D flow
    - Speed of sound (Isentropic flow)
    - Alternative forms of the E.E.
  - Three typical 1D Flows
    - Normal shock waves
    - 1D flow heat addition
    - 1D flow with friction

Outline

3.1 Introduction

3.2 One dimensional flow equations (C.E. M.E. E.E.)

3.3 Speed of sound and Mach number (Isentropic flow)

3.4 Alternative forms of energy equation

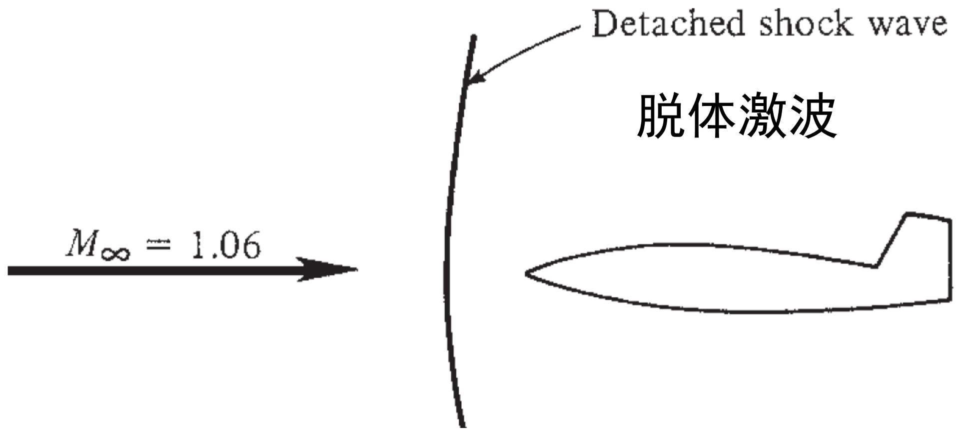
3.5 Normal shock relation (Nonisentropic flow)

3.6 Hugoniot Equations (雨纽贡关系)

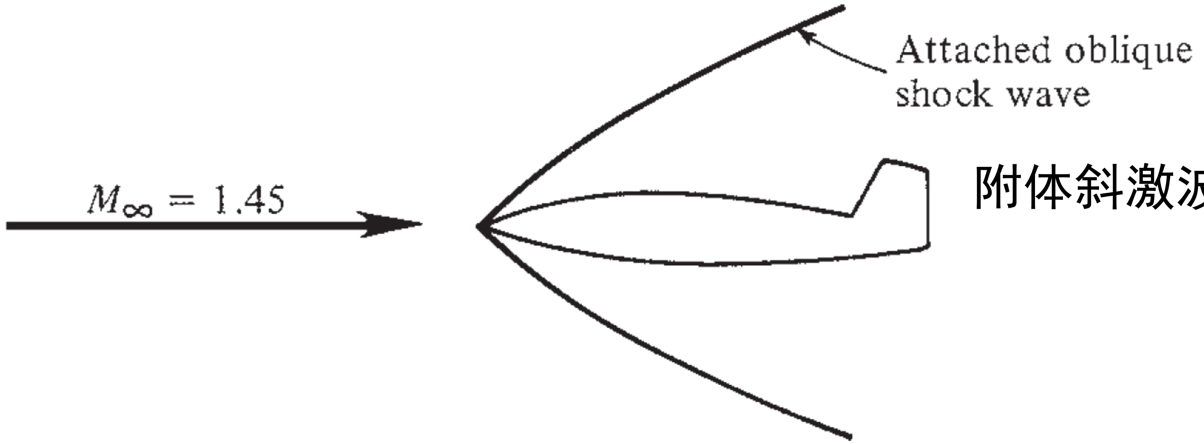
3.7 One dimensional flow with heat addition
(Rayleigh flow)

3.8 One dimensional flow with friction
(Fanno flow)

3.1 Introduction

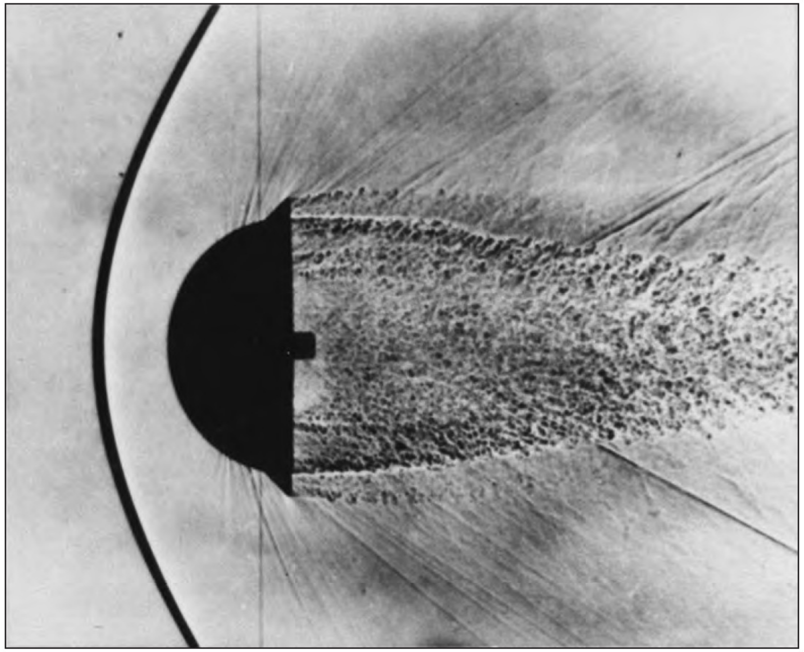


Bell XS-1 第一款有人驾驶的超音速飞机

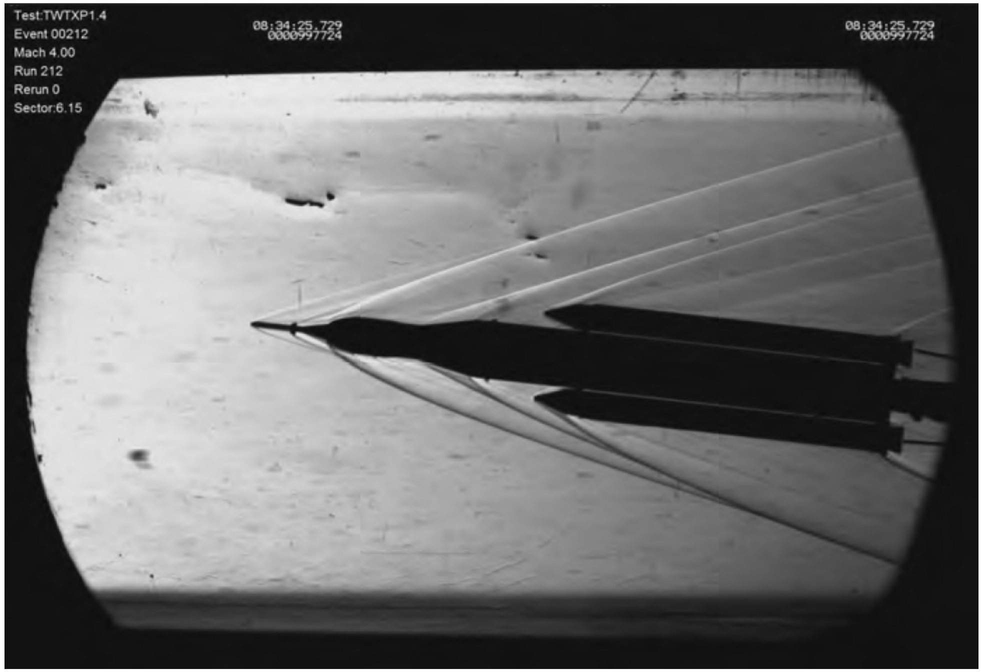


Shock wave (激波): a type of propagating disturbance that moves faster than the local speed of sound in the medium.

3.1 Introduction



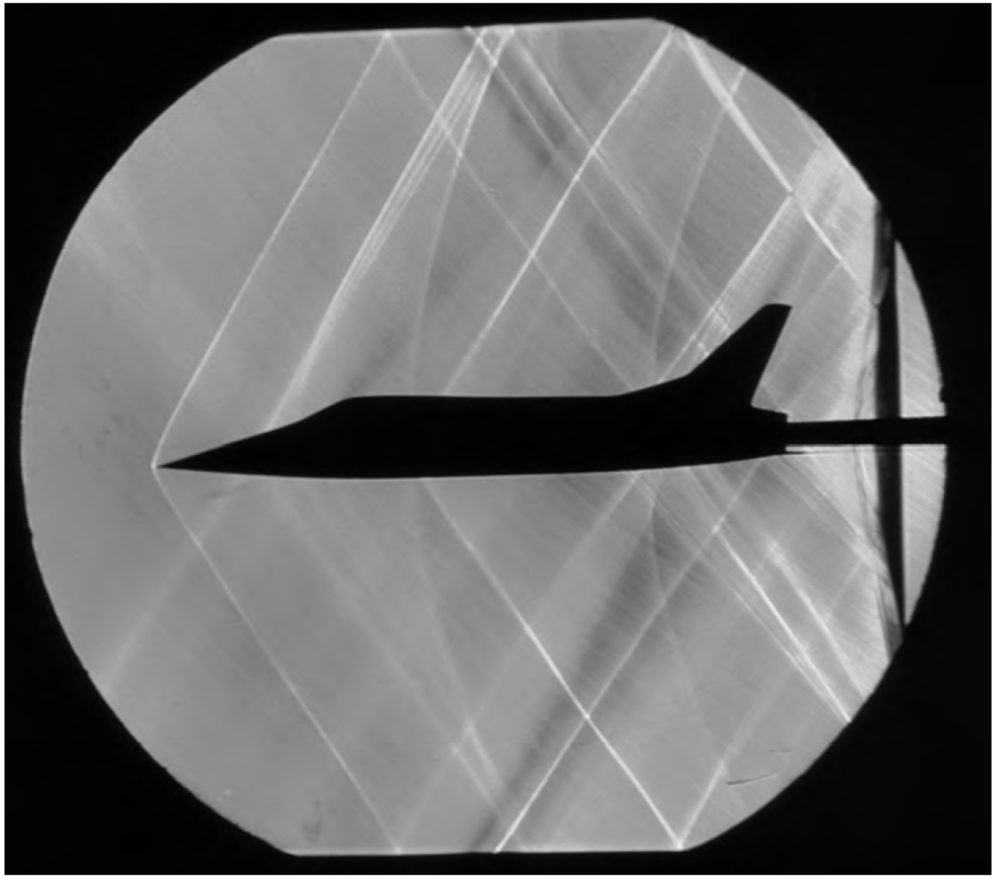
钝头体前的脱体激波



带攻角的火箭周围的斜激波

纹影法：利用光在介质中传播史折射率梯度依赖于流体密度，从而实现对流体密度分布的可视化。

3.1 Introduction

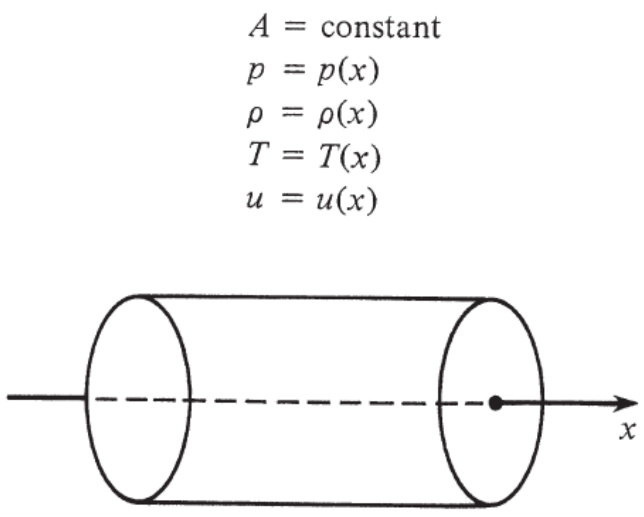


航天飞机周围的斜激波

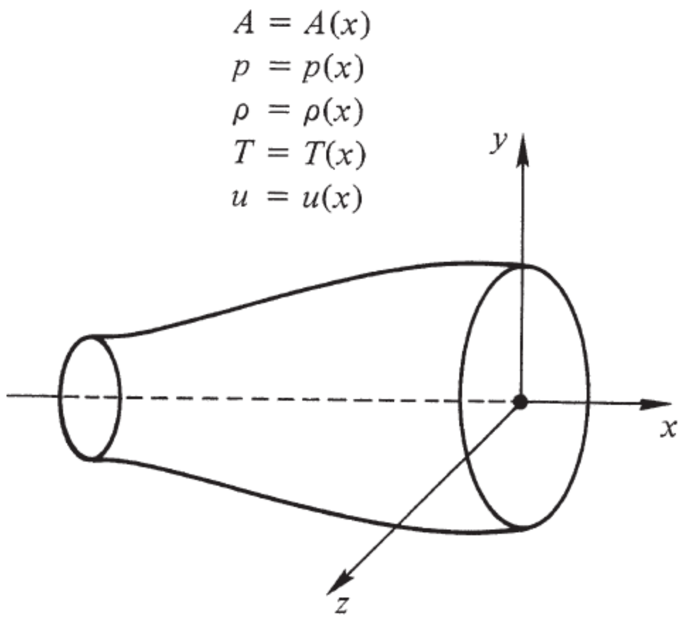
热湍流中的羽流结构

纹影法：利用光在介质中传播史折射率梯度依赖于流体密度，从而实现对流体密度分布的可视化。

3.2 One-dimensional flow equation



(a) One-dimensional flow

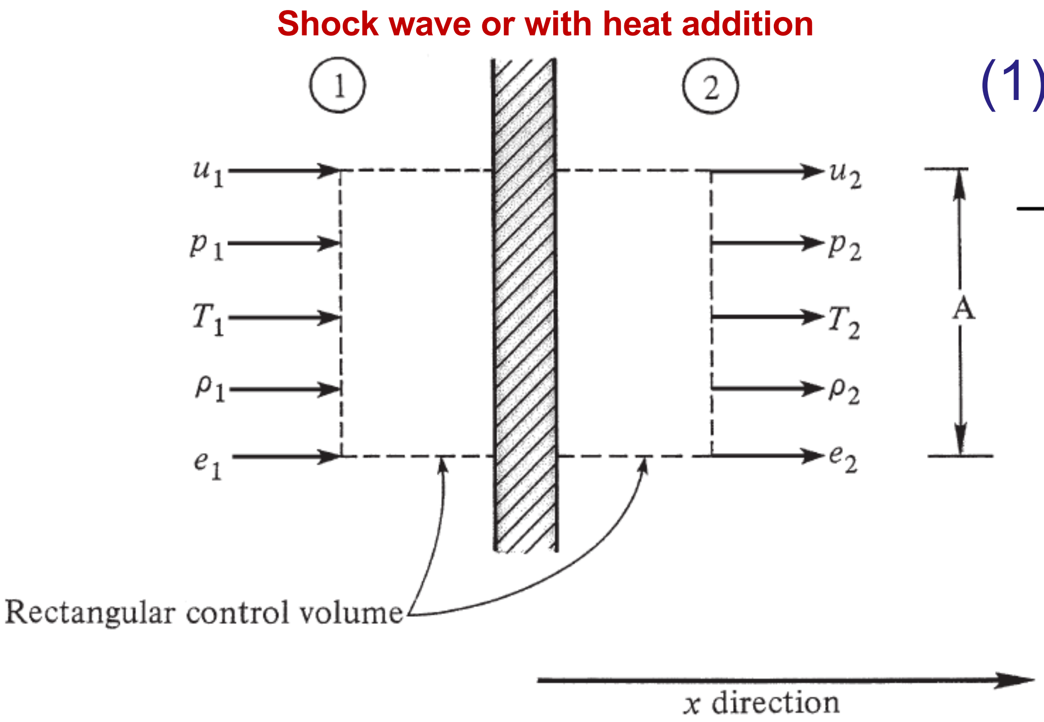


(b) Quasi-one-dimensional flow

1D flow: the physical properties of the flow is only a function of x (constant area flow)

Quasi-1D flow: the physical properties of the flow is only a function of x (the area changes gradually with x)

3.2 One-dimensional flow equation



(1) Continuity equation (Steady flow)

$$-\iint_S \rho\vec{V}\cdot d\vec{S} = \frac{\partial}{\partial t}\iiint_{\mathcal{V}} \rho d\mathcal{V}$$

$$\iint_S \rho\vec{V}\cdot d\vec{S} = 0$$

$$-\rho_1 u_1 A + \rho_2 u_2 A = 0$$

$$\rho_1 u_1 = \rho_2 u_2 \quad (3.1)$$

(2) Momentum equation (Steady flow, no body force)

$$\iint_S (\rho\vec{V}\cdot d\vec{S})\vec{V} + \iiint_{\mathcal{V}} \frac{\partial}{\partial t}(\rho\vec{V})d\mathcal{V} = \iiint_{\mathcal{V}} \rho\vec{f}d\mathcal{V} - \iint_S p d\vec{S}$$

$$\rho_1(-u_1 A)u_1 + \rho_2 u_2 A u_2 = p_1 A - p_2 A$$

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \quad (3.2)$$

3.2 One-dimensional flow equation

(3) Energy equation (Steady flow, no body force)

$$\iiint_{\mathscr{V}} \dot{q}\rho d\mathscr{V} - \iint_S (pd\vec{S}) \cdot \vec{V} + \iiint_{\mathscr{V}} (\vec{f}\rho d\mathscr{V}) \cdot \vec{V}$$

$$= \iiint_{\mathscr{V}} \frac{\partial}{\partial t}[(e + \frac{V^2}{2})\rho] d\mathscr{V} + \iint_S (e + \frac{V^2}{2})\rho d\vec{S} \cdot \vec{V}$$

$$\dot{Q} - (-p_1 A u_1 + p_2 A u_2) = (e_1 + \frac{u_1^2}{2})\rho_1(-Au_1) + (e_2 + \frac{u_2^2}{2})\rho_2 A u_2 \quad (*)$$

with $\dot{Q} = \iiint_{\mathscr{V}} \dot{q}\rho d\mathscr{V}$ being the total rate of heat added to the C.V.

divide both sides of (*) by equation (3.1) and A, we have:

$$\frac{\dot{Q}}{A\rho_1 u_1} + \frac{p_1}{\rho_1} + (e_1 + \frac{u_1^2}{2}) = \frac{p_2}{\rho_2} + e_2 + \frac{u_2^2}{2}$$

$\frac{\dot{Q}}{A\rho_1 u_1}$ being the heat added per unit mass q. Consider $h=e+pv$, we have:

$$\boxed{h_1 + \frac{u_1^2}{2} + q = h_2 + \frac{u_2^2}{2}} \quad (3.3)$$

3.2 One-dimensional flow equation

Governing equations for 1D steady flow with no body force:

Continuity equation:

$$\boxed{\rho_1 u_1 = \rho_2 u_2} \quad (3.1)$$

Momentum equation:

$$\boxed{p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2} \quad (3.2)$$

No body force and no viscous stress

Energy equation:

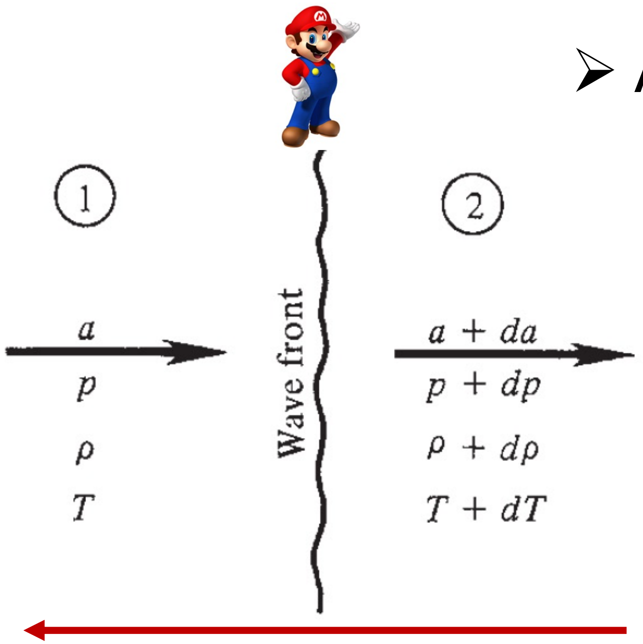
$$\boxed{h_1 + \frac{u_1^2}{2} + q = h_2 + \frac{u_2^2}{2}} \quad (3.3)$$

No shaft work, work done by viscous stress, heat conduction and diffusion and change in potential energy.

3.3 Speed of sound and Mach number

(1) Speed of sound

- Sound wave: weak pressure wave （isentropic process）
- Shock wave: strong pressure wave (velocity larger than sound)



direction of wave propagation

- Denote the sound velocity as a
- Apply the 1D flow equation to the sound wave:

$$\rho_1 u_1 = \rho_2 u_2$$

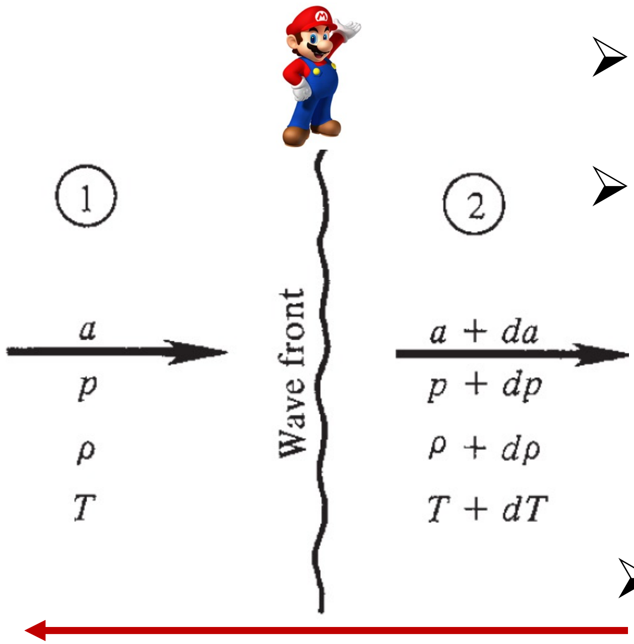
$$\rho a = (\rho + d\rho)(a + da)$$

$$\rho a = \rho a + a d\rho + \rho da + \cancel{d\rho da}$$

$$a = -\rho \frac{da}{d\rho}$$

3.3 Speed of sound and Mach number

(1) Speed of sound



direction of wave propagation

- Denote the sound velocity as a
- Apply the 1D flow equation to the sound wave:

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$$

$$p + \rho a^2 = p + dp + (\rho + d\rho)(a + da)^2$$

- Ignoring the second order terms

$$da = -\frac{dp + a^2 d\rho}{2a\rho}$$

From continuity equation we have $a = -\rho \frac{da}{d\rho}$

$$\Rightarrow \quad a^2 = \frac{dp}{d\rho}$$

The propagation of the sound wave is an isentropic process, no heat addition , no dissipation.

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

3.3 Speed of sound and Mach number

$$a^2 = (\frac{\partial p}{\partial \rho})_s$$

- From definition, we have $\rho = \frac{1}{v}$, thus $\partial \rho = -\frac{1}{v^2}\partial v$

- Hence, we have $a^2 = -v^2(\frac{\partial p}{\partial v})_s = \frac{v}{-\frac{1}{v}(\frac{\partial v}{\partial p})_s} = \frac{v}{\tau_s}$

$$a = \sqrt{\frac{v}{\tau_s}}$$

The speed of sound is a measure of the compressibility of the gas. For incompressible flow, a goes to infinity.

3.3 Speed of sound and Mach number

(2) Speed of sound for a calorically perfect gas

$$\left.\begin{array}{l}\frac{p}{\rho^\gamma}=Const.\\ a^2=\left(\frac{\partial p}{\partial \rho}\right)_s\end{array}\right\}\quad \left.\begin{array}{r} a^2=c\gamma\rho^{\gamma-1}=\gamma\frac{p}{\rho}\\ p=\rho RT\end{array}\right\}\quad \boxed{a=\sqrt{\gamma RT}}$$

- The speed of sound is a function of both γ and T.
- The speed of sound at standard sea level (T=15°C, p=1atm)

$$a_s=340.9m/s$$

- ☐ Newton assumes the propagation of sound is isothermal, leading to

$$a=\sqrt{RT}$$

At standard sea level, a=287.6m/s, which is 18% smaller than the actual value.

3.3 Speed of sound and Mach number

(3) The Mach number *M*

$$M=\frac{v}{a}=\frac{v}{\sqrt{\gamma RT}}\quad\begin{cases}M<1, \text{ subsonic flow}\\ M=1, \text{ sonic flow}\\ M>1, \text{ supersonic flow}\end{cases}$$

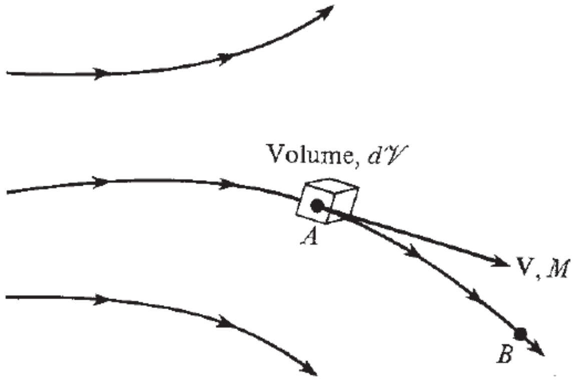
$$\frac{V^2/2}{e}=\frac{V^2/2}{c_v T}=\frac{V^2/2}{RT/(\gamma-1)}=\frac{\gamma V^2/2}{\gamma RT/(\gamma-1)}$$

$$=\frac{\gamma V^2/2}{a^2/(\gamma-1)}=\frac{\gamma(\gamma-1)}{2}M^2$$

- local property
- a function of temperature T
- a measure of the directed motion of the gas compared to the random thermal motion of the molecules
- a measure of the compressibility of the gas

3.3 Speed of sound and Mach number

(4) Characteristic Mach number M^* 特征马赫数



- Speed up (M<1) or slow down (M>1) A adiabatically
- Achieve an imaginary state with M=1
- Define the temperature at this state as T^*
- Define the speed of sound as a^*

$$a^* = \sqrt{\gamma R T^*}$$

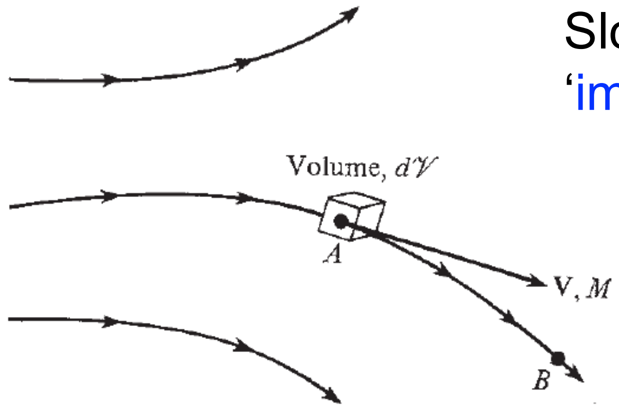
- Define characteristic Mach number M^*

$$M^* = \frac{V}{a^*}$$

3.3 Speed of sound and Mach number

(5) Total T_0, total p_0 and static T and static p

总压（滞止压）、总温（滞止温度）、 静温、 静压



Slow down the fluid element A isentropically to V=0
'imaginary'

The pressure and temperature which the fluid element achieves when V=0 are defined as total pressure p_0 and total temperature T_0

When A moving at velocity V with an actual pressure and temperature equal to p and T, we call p and T the static pressure and static temperature respectively.

Stagnation (total) speed $a_0 = \sqrt{\gamma R T_0}$

Stagnation (total) density $\rho_0 = p_0/(RT_0)$

3.4 Alternative forms of energy equation

(1) The energy equation

$$h_1 + \frac{u_1^2}{2} + q = h_2 + \frac{u_2^2}{2} \qquad (3.3)$$

- Assume no heat addition $h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$

- For calorically perfect gas: $h = c_p T$

$$c_p T_1 + \frac{u_1^2}{2} = c_p T_2 + \frac{u_2^2}{2}$$

- From thermal dynamics: $c_p = \frac{\gamma R}{\gamma - 1}$

$$\frac{\gamma R T_1}{\gamma - 1} + \frac{u_1^2}{2} = \frac{\gamma R T_2}{\gamma - 1} + \frac{u_2^2}{2} \qquad \frac{a_1^2}{\gamma - 1} + \frac{u_1^2}{2} = \frac{a_2^2}{\gamma - 1} + \frac{u_2^2}{2}$$

$$\frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} + \frac{u_1^2}{2} = \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2} + \frac{u_2^2}{2}$$

3.4 Alternative forms of energy equation

(2) Calculation of the total condition

- Slow down the fluid element A isentropically to $u_2=0$ 'imaginary'

$$c_pT+\frac{u^2}{2}=c_pT_0=h_0=\frac{\gamma}{\gamma-1}\frac{p_0}{\rho_0}=\frac{1}{\gamma-1}a_0^2$$

$$\frac{T_0}{T}=1+\frac{u^2}{2c_pT}=1+\frac{u^2}{2\gamma RT/(\gamma-1)}$$

$$=1+\frac{\gamma-1}{2}\frac{u^2}{a^2}=1+\frac{\gamma-1}{2}M^2$$

$$\boxed{\frac{T_0}{T}=1+\frac{\gamma-1}{2}M^2}$$

Total temperature is a measure of the total energy of the system.

- Isentropic relation

$$\frac{p_0}{p}=\left(\frac{\rho_0}{\rho}\right)^\gamma=\left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}}\Rightarrow$$

$$\boxed{\begin{aligned}\frac{p_0}{p}&=\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}\\ \frac{\rho_0}{\rho}&=\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{1}{\gamma-1}}\end{aligned}}$$

3.4 Alternative forms of energy equation

(3) Calculation of the characteristic condition

- Speed up (M<1) or slow down (M>1) A adiabatically to M=1 'Imaginary'

$$u = a = a^*$$

$$\frac{a^{*2}}{\gamma-1} + \frac{a^{*2}}{2} = \frac{a_0^2}{\gamma-1}$$

$$\left(\frac{a^*}{a_0}\right)^2 = \frac{2}{\gamma+1} = \frac{T^*}{T_0}$$

- Isentropic process, let M=1

$$\frac{p_0}{p} = \left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_0}{\rho} = \left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{1}{\gamma-1}}$$

$$\Rightarrow$$

$$\frac{p_0}{p^*} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_0}{\rho^*} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{1}{\gamma-1}}$$

For γ=1.4, we have $\frac{T^*}{T_o} = 0.833$ $\frac{p^*}{p_o} = 0.528$ $\frac{\rho^*}{\rho_o} = 0.634$

3.4 Alternative forms of energy equation

(3) Calculation of the characteristic condition

Divide $\frac{a^2}{\gamma-1}+\frac{u^2}{2}=\frac{a^{*2}}{\gamma-1}+\frac{a^{*2}}{2}$ by u^2, we have:

$$\frac{(a/u)^2}{\gamma-1}+\frac{1}{2}=\frac{(a^*/u)^2}{\gamma-1}+\frac{(a^*/u)^2}{2}=\frac{\gamma+1}{2(\gamma-1)}\left(\frac{a^*}{u}\right)^2$$

$$\frac{(1/M)^2}{\gamma-1}+\frac{1}{2}=\frac{\gamma+1}{2(\gamma-1)}\left(\frac{1}{M^*}\right)^2$$

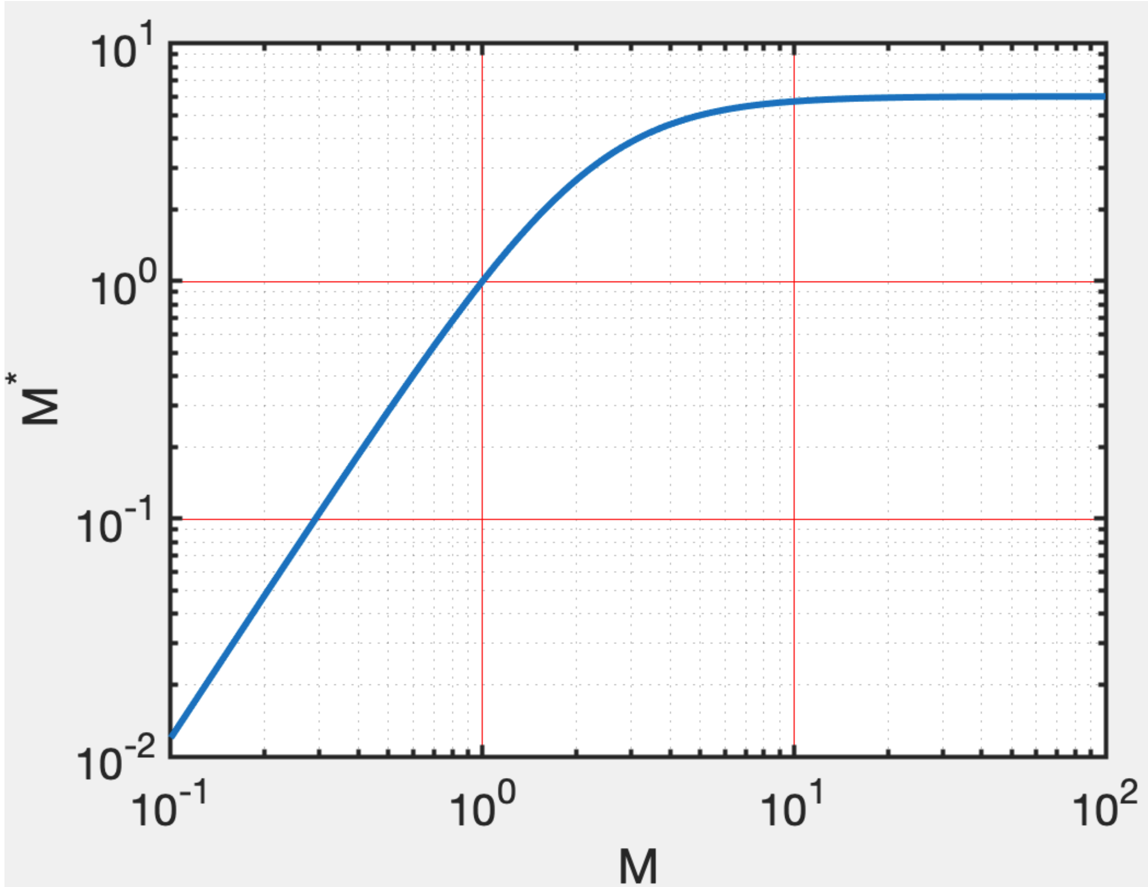
$$M^2=\frac{2}{(\gamma+1)/M^{*2}-(\gamma-1)}$$

$$M^{*2}=\frac{M^2(\gamma+1)}{[2+M^2(\gamma-1)]}$$

Relation between characteristic Mach number and Mach number

3.4 Alternative forms of energy equation

(3) Calculation of the characteristic condition



$M>1,\ M^*>1$

$M<1,\ M^*<1$

$M=1,\ M^*=1$

3.4 Alternative forms of energy equation

Exercise 3.1

空气流动中某点处的马赫数、静压和静温分别为3.5、0.3 atm 和180K, 计算该点的当地 p_0, T_0， a^* 和 M^*.

$$\frac{p_0}{p} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{1}{\gamma-1}}$$

$$\frac{T_0}{T} = 1 + \frac{\gamma-1}{2}M^2$$

等熵滞止，等熵关系表可知：

- M=3.5时， p_0/p=76.27, T_0/T=3.45

所以 p_0=76.27×p=22.881 atm, T_0=3.45×T=621K

$$M^{*2} = \frac{M^2(\gamma+1)}{[2+M^2(\gamma-1)]}$$

- 绝热减速到M=1时， T_0/T^*=1.2，所以T^*=T_0/1.2=517.5K

$a^* = \sqrt{\gamma R T^*}$ a^*=456m/s， a=269m/s， V=3.5a=941.5m/s

M^*=941.5/456=2.06

3.4 Alternative forms of energy equation

Exercise 3.2

储气罐中速度近乎为0，压力为海平面标准压力（p_0=1.013x10^{5}Pa, T_0=288K, c_p=1004J/(kg K)）的空气流过一个管道，分别用不可压缩气体假设和可压缩气体假设计算在出口流速加速到107m/s和300m/s时的出口压强。并基于计算结果讨论可压缩性的影响。

不可压缩流动假设下

$$p + \frac{1}{2}\rho v^2 = const.$$

$p_0=p+1/2\rho v^2$

所以$p=p_0-1/2\rho v^2$

$\rho=p/RT=1.23\text{kg/m}^3$

v=107m/s时，$p_i=0.9425\times10^5$Pa

v=300m/s时，$p_i=0.4595\times10^5$Pa

v=107m/s时，$p_c/p_i=1.007$

v=300m/s时，$p_c/p_i=1.22$

可压缩流动假设下

$$c_pT + \frac{u^2}{2} = c_pT_0$$ T=282.3K 和 243.2K

$$a = \sqrt{\gamma RT}$$ a=336.8m/s 和 312.6m/s

$M=v/a$, 分别为0.32 和0.96

查等熵流动表格：

M=0.32时p_0/p=1.074,

$p_c=p_0/1.074=0.9432\times10^5$ Pa

M=0.96时p_0/p=1.808,

$p_c=p_0/1.808=0.5602\times10^5$ Pa

M>0.3, 必须考虑可压缩效应！

Summary 1

- For 1D steady, no body force, pressure only flow

  Continuity equation: $\rho_1 u_1 = \rho_2 u_2$

  Momentum equation: $p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$

  Energy equation: $h_1 + \frac{u_1^2}{2} + q = h_2 + \frac{u_2^2}{2}$

Intensive quantity: *T, p, v*, specific quantities
强度量

Extensive quantity: *M, E, H, S*
广延量

- Speed of sound $a^2 = (\frac{\partial p}{\partial \rho})_s \quad a = \sqrt{\gamma RT}$

- Mach number $M = \frac{v}{a} = \frac{v}{\sqrt{\gamma RT}}$

$$\frac{T_0}{T} = 1 + \frac{\gamma-1}{2}M^2$$
$$\frac{p_0}{p} = (1 + \frac{\gamma-1}{2}M^2)^{\frac{\gamma}{\gamma-1}}$$
$$\frac{\rho_0}{\rho} = (1 + \frac{\gamma-1}{2}M^2)^{\frac{1}{\gamma-1}}$$

- Total (stagnant point) condition T_0, a_0 (adiabatic), p_0, ρ_0 (isentropic), h_0

- Characteristic condition T^*, a^*, M^*, p^*, ρ^*

$$\frac{T_*}{T_0} = \frac{2}{\gamma+1}$$
$$\frac{p_0}{p^*} = (1 + \frac{\gamma-1}{2})^{\frac{\gamma}{\gamma-1}}$$
$$\frac{\rho_0}{\rho^*} = (1 + \frac{\gamma-1}{2})^{\frac{1}{\gamma-1}}$$