Some progress on fixed subgroups and fixed points

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Rank of a group

Definition

For a finitely generated group G, the rank of G denoted $\mathrm{rk}(G)$ is the minimal number of generators of G.

Example

- For an abelian group G, if $H < G$, then $\mathrm{rk}(H) \leq \mathrm{rk}(G)$.
- Let F_n be a free group of rank $n > 1$. Then $F_n < F_2$ but $\mathrm{rk}(F_n) \geq \mathrm{rk}(F_2)$.

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Fixed subgroup: Scott conjecture

For a group G, denote the set of endomorphisms (resp. monomorphisms, automorphisms) of G by $\mathrm{End}(G)$ (resp. $\mathrm{Mon}(G)$, $\mathrm{Aut}(G)$).

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For an endomorphism $\phi \in \mathrm{End}(G)$, the fixed subgroup of ϕ is

$$\mathrm{Fix}(\phi) := \{g \in G \mid \phi(g) = g\}.$$

For a free group F_n of rank n:

Theorem (Dyer-Scott, 1975)

Let $\phi \in \mathrm{Aut}(F_n)$ be an automorphism with **finite order** *of F_n. Then*

$$\mathrm{rkFix}(\phi) \leq n.$$

Theorem (Bestvina-Handel, 1992)

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  - 3-manifold group (Z. '12&15, Lin-Wang '14, Jiang-Wang-Wang-Zheng '21)
  - Hyper. gp (Paulin '89, Neumann '92, Shor '99, Hsu-Wise '04)
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Fixed subgroup: surface & 3-manifold group

Theorem (Jiang-Wang-Z., 2011)

Suppose G is a **surface group**. *Then for any endomorphism $\phi \in \mathrm{End}(G)$,*

1. $\mathrm{rkFix}(\phi) \leq \mathrm{rk}(G)$, *with equality if and only if $\phi = \mathrm{id}$;*
2. $\mathrm{rkFix}(\phi) \leq \frac{1}{2}\mathrm{rk}(G)$ *if ϕ is not epimorphic.*

Nielsen considered the fixed subgroups of **automorphisms** of **orientable** surface group in 1920s.

Theorem (Lin-Wang, 2014)

Suppose ϕ is an automorphism of $G = \pi_1(M)$, where M is a hyperbolic 3-manifold. Then $\mathrm{rkFix}(\phi) < 2\mathrm{rk}(G)$.

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Fixed subgroup: inertia conjecture

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A subgroup A is inert in G if for every subgroup $H \leq G$,

$$\mathrm{rk}(A \cap H) \leq \mathrm{rk}(H).$$

$A \leq G$ is inert in $G \implies \mathrm{rk}(A) \leq \mathrm{rk}(G)$.

Theorem (Dicks-Ventura, 1996)

Let $\mathcal{F}$ be a family of **injective** *endomorphisms of F_n. Then*

$$\mathrm{Fix}\mathcal{F} := \{g \in F_n \mid \phi(g) = g, \forall \phi \in \mathcal{F}\} = \bigcap_{\phi \in \mathcal{F}} \mathrm{Fix}(\phi)$$

is inert in F_n, i.e., for every subgroup $H \leq G$

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In particular, $\mathrm{rk}(\mathrm{Fix}\mathcal{F}) \leq n$.

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The fixed subgroup of any family of endomorphisms of F_n is inert in F_n.

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The fixed subgroup of any family of **automorphisms** *of a surface group G is inert in G.*

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The Dicks-Ventura inertia conjecture holds not only in free groups but also in surface groups.

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Fixed subgroup: direct products

Theorem (Ventura-Wu-Z., 2015)

Let $G = \times_{i=1}^{n} G_i$ be a direct product of **surface** *groups and* **free** *groups. If neither of the factors is cyclic, then for $\phi \in \mathrm{Aut}(G)$,*

$$\mathrm{rkFix}(\phi) \leq \mathrm{rk}(G).$$

Otherwise, if G contains a non-cyclic factor and a factor $\mathbb{Z}$, then there exists $f \in \mathrm{Aut}(G)$ such that $\mathrm{Fix}(f)$ is not finitely generated.

Example

Let $f : F_2 \times \mathbb{Z} \to F_2 \times \mathbb{Z} = \langle a, b, t \mid [a, t], [b, t] \rangle$ be an automorphism such that

$$a \mapsto at, \quad b \mapsto b, \quad t \mapsto t.$$

Then $u \in \mathrm{Fix}(f)$ if and only if it has zero exponent sum in a. So $\mathrm{Fix}(f) \cong F_\infty \times \mathbb{Z}$ generated by the infinite set $\{t, \ a^i b a^{-i} | i \in \mathbb{Z}\}$.

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Definition

A group G is said to have the *finitely generated fixed subgroup property* of monomorphisms (resp. automorphisms, endomorphisms), abbreviated as $\mathrm{FGFP_m}$ (resp. $\mathrm{FGFP_a}$, $\mathrm{FGFP_e}$), if for any $f \in \mathrm{Mon}(G)$ (resp. $\mathrm{Aut}(G)$, $\mathrm{End}(G)$), the fixed subgroup $\mathrm{Fix}(f)$ is finitely generated.

Clearly, $\mathrm{FGFP_e} \Longrightarrow \mathrm{FGFP_m} \Longrightarrow \mathrm{FGFP_a}$.

Example

- Free groups and surface groups have $\mathrm{FGFP_e}$.
- F_2 and $\mathbb{Z}$ both have $\mathrm{FGFP_a}$ but their direct product $F_2 \times \mathbb{Z}$ don't.

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Fixed subgroup: free products

It is natural to ask

Question

1. If two groups G_1 and G_2 both have finitely generated fixed subgroup property ($\mathrm{FGFP_m}$, $\mathrm{FGFP_a}$ or $\mathrm{FGFP_e}$), then, what about their free product $G_1 * G_2$?
2. If the answer of (1) is affirmative, what is the quantitative relation among the explicit bounds of ranks of fixed subgroups of $G_1 * G_2$, G_1 and G_2?

Theorem (Lei-Z., 2023)

A free product $*_{i=1}^{n} G_i$ has $\mathrm{FGFP_m}$ (resp. $\mathrm{FGFP_a}$) if and only if the factor groups $G_1, G_2, \ldots, G_n$ all have $\mathrm{FGFP_m}$ (resp. $\mathrm{FGFP_a}$).

Fixed subgroup: free products

It is natural to ask

Question

1. If two groups G_1 and G_2 both have finitely generated fixed subgroup property ($\mathrm{FGFP_m}$, $\mathrm{FGFP_a}$ or $\mathrm{FGFP_e}$), then, what about their free product $G_1 * G_2$?
2. If the answer of (1) is affirmative, what is the quantitative relation among the explicit bounds of ranks of fixed subgroups of $G_1 * G_2$, G_1 and G_2?

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FP and UFP degree: definitions

To quantitatively analysis the ranks of fixed subgroups, we introduce

Definition

1. A group G is said to have k-FGFP, if for any $\phi \in \mathrm{Mon}(G)$,

$$\mathrm{rkFix}(\phi) \leq k \cdot \mathrm{rk}(G).$$

The FP degree for G is

$$\mathfrak{D}_f(G) := \sup\{\frac{\mathrm{rkFix}(\phi)}{\mathrm{rk}(G)} \mid \phi \in \mathrm{Mon}(G)\} \in [1, +\infty].$$

2. G is said to have k-UFGFP, ("U" for uniformly), if for every f.g. subgroup $H < G$ and $\phi \in \mathrm{Mon}(H)$, $\mathrm{rkFix}(\phi) \leq k \cdot \mathrm{rk}(H)$. The UFP degree for G is

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Proposition

Let G be a f.g. group and let $1 \neq H \leq G$ be a f.g. subgroup.

1. *If G has k-UFGFP, then its subgroup H also has k-UFGFP and hence has k-FGFP, i.e.,*

$$1 \leq \mathfrak{D}_f(H) \leq \mathfrak{D}_{uf}(H) \leq \mathfrak{D}_{uf}(G) \leq k.$$

2. $\mathfrak{D}_f(G) = \mathfrak{D}_{uf}(G) = 1$ *if G is one of the following: a free abelian group $\mathbb{Z}^n$, a free group F_n or a surfaces group.*

3. $\mathfrak{D}_f(F_2 \times \mathbb{Z}) = \infty$, *and hence* $\mathfrak{D}_{uf}(G) = \infty$ *if G contains a subgroup that is isomorphic to $F_2 \times \mathbb{Z}$.*

4. $\mathfrak{D}_f(*_{i=1}^n G_i) \leq n \cdot \max_{i=1}^n \mathfrak{D}_f(G_i)$, *where each G_i is a freely indecomposable group. In particular, if all the factors G_i have the same rank, then*

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FGFP in hyperbolic groups

[Paulin 1989]: the fixed subgroup of any automorphism of a Gromov hyperbolic groups is finitely generated.

[Shor 1999]: a torsion-free hyperbolic group contains, up to isomorphism, only finitely many fixed subgroups of automorphisms.

[Sela 1997]: every freely indecomposable torsion-free hyperbolic group is either co-Hopfian or infinite cyclic.

Theorem (Lei-Z., 2023)

For a torsion-free hyperbolic group G, we have $\mathfrak{D}_{\mathrm{f}}(G) < \infty$. In particular, for every monomorphism $\phi \in \mathrm{Mon}(G)$, the fixed subgroup $\mathrm{Fix}\phi$ is finitely generated.

A subgroup of a hyperbolic group may not be hyperbolic.

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For a hyperbolic group G, is $\mathfrak{D}_{\mathrm{uf}}(G)$ always finite?

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Fixed subgroup: hyperbolic group

A hyperbolic group G is called stably hyerbolic, if $\phi^m(G)$ is hyperbolic for arbitrary lager m and any $\phi \in \mathrm{End}(G)$.

Theorem (Lei-Z., 2023)

Let $G = *_{i=1}^{n} G_i$ be a torsion-free stably hyperbolic group, where each factor G_i has finite UFP *degree $\mathfrak{D}_{uf}(G_i)$. Then, for any* **endomorphism** $\phi \in \mathrm{End}(G)$,

$$\mathrm{rkFix}(\phi) \leq \frac{1}{4}\ell(\mathrm{rk}(G)+1)^2,$$

where the number $\ell = \max_{i=1}^{n} \mathfrak{D}_{uf}(G_i)$.

Conjecture (O'Neill-Turner, 2000): all hyperbolic groups are stably hyperbolic.

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Free products of free and surface groups

Theorem (Lei-Z., 2023)

Let $G = *_{i=1}^{t} G_i * F_s$ be a free product, where F_s is a free group of rank s, and each factor G_i is a **surface group**. *Then*

- *if $\phi \in \mathrm{End}(G)$, then*

$$\mathrm{rkFix}(\phi) \leq \frac{1}{4}(\mathrm{rk}(G) + 1)^2.$$

- *if $\phi \in \mathrm{Mon}(G)$, then $\mathrm{rkFix}(\phi) \leq (s + t)(\mathrm{rk}(G) - s - t + 1)$. Moreover, if $s = 0$ and all the surface groups G_i share the same rank, then*

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Graph groups

Theorem (Rodaro-Silva-Sykiotis, 2013)

*Let G be a **graph group** (RAAG). Then the following two conditions are equivalent*

- $\mathrm{Fix}(\phi)$ *is finitely generated for every endomorphism* $\phi \in \mathrm{End}(G)$;
- G *is a free product of finitely many free abelian groups of finite rank.*

Theorem (Lei-Z., 2023)

Let $G = *_{i=1}^{n} \mathbb{Z}^{t_i}$ be a free product of free abelian groups $\mathbb{Z}^{t_i}$ of rank t_i. Then for any endomorphism $\phi \in \mathrm{End}(G)$, we have

$$\mathrm{rkFix}(\phi) \leq n(\mathrm{rk}(G) - n + 1).$$

In particular, if $t_1 = t_2 = \cdots = t_n$, then $\mathrm{rkFix}(\phi) \leq \mathrm{rk}(G)$.

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3-manifold groups

Theorem (Lin-Wang, 2014)

Suppose ϕ is an automorphism of $G = \pi_1(M)$, where M is a hyperbolic 3-manifold. Then $\mathrm{rkFix}(\phi) < 2\mathrm{rk}(G)$.

Theorem (Lei-Z., 2023)

Let $M = \#_{i=1}^{n} M_i$ be a connected sum of finitely many hyperbolic 3-manifolds. Then the fundamental group $\pi_1(M)$ has FGFP_m (and hence FGFP_a). More precisely, for any monomorphism $f \in \mathrm{Mon}(\pi_1(M))$, we have

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What is Fixed Point Theory?

Let X be a space, and $f : X \to X$ a self-map.

- $x \in X$ is a fixed point of $f \iff f(x) = x$.
- $\mathrm{Fix} f := \{x \in X | f(x) = x\}$: the set of all fixed points of f.

Fixed Point Theory studies the nature of $\mathrm{Fix} f$ in relation to the space X and the map f, such as:

- Existence: is $\mathrm{Fix} f \neq \emptyset$?
- Number of fixed points $\#\mathrm{Fix} f$
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Fixed point class

Let X be a connected compact polyhedron, and $f : X \to X$ a self-map. The fixed point set splits into a disjoint union of **fixed point classes**

$$\mathrm{Fix} f := \{x \in X | f(x) = x\} = \bigsqcup_{\mathbf{F} \in \mathrm{Fpc}(f)} \mathbf{F}$$

Definition (path approach)

Two fixed points $x, x' \in \mathrm{Fix}(f)$ are in the same fixed point class $\iff$ there is a path c (called a Nielsen path) from x to x' such that $c \simeq f \circ c$ rel endpoints.

The index of a fixed point class $\mathbf{F}$ is the sum

$$\mathrm{ind}(f, \mathbf{F}) := \sum_{x \in \mathbf{F}} \mathrm{ind}(f, x) \in \mathbb{Z}.$$

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Index: examples

The index is defined by using homology.

- Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a diff. map, x a isolated fixed point. Then

$$\mathrm{ind}(f, x) = sgn\,\det(I - Df_x) = (-1)^k.$$

- If $f : \mathbb{C} \to \mathbb{C}$ has a complex analytic expression $z \mapsto f(z)$, then $\mathrm{ind}(f, z_0) =$ multiplicity of the zero z_0 of the function $z - f(z)$.

Theorem (Lefschetz-Hopf Fixed Point Theorem)

$$\sum_{\mathbf{F}} \mathrm{ind}(f, \mathbf{F}) = \sum_q (-1)^q \mathrm{Trace}(f_* : H_q(X;\mathbb{Q}) \to H_q(X;\mathbb{Q})) := L(f),$$

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$$L(f) \neq 0 \Longrightarrow \mathrm{Fix} f \neq \emptyset.$$

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Bounded Index Property

A compact polyhedron X is said to have the *Bounded Index Property (BIP)*(resp. *Bounded Index Property for Homeomorphisms (BIPH)*), if $\exists\ \mathcal{B} > 0$ s.t. for any map (resp. homeomorphism) $f : X \to X$,

$$|\mathrm{ind}(f, \mathbf{F})| \leq \mathcal{B}, \quad \forall \mathbf{F} \in \mathrm{Fpc}(f).$$

Question (Jiang, Math. Ann. 1998)

Suppose a compact polyhedron X is aspherical (i.e. $\pi_i(X) = 0$ for all $i > 1$). Does X have BIP or BIPH?

Many positive examples:

- [McCord, '92]: Infra-solvmanifolds have BIP;
- [Jiang-Wang, '92]: Geometric 3-manifolds have BIPH;
- [Jiang, '98][Kelly, '00]: Graphs & surfaces ($\chi < 0$) have BIP;
- [Ye-Z., 2019]: Products of negatively curved Riemannian manifolds have BIPH;
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A compact polyhedron X is said to have the *Bounded Index Property (BIP)*(resp. *Bounded Index Property for Homeomorphisms (BIPH)*), if $\exists\ \mathcal{B} > 0$ s.t. for any map (resp. homeomorphism) $f : X \to X$,

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Suppose a compact polyhedron X is aspherical (i.e. $\pi_i(X) = 0$ for all $i > 1$). Does X have BIP or BIPH?

Many positive examples:

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Fixed subgroups in $\pi_1(T^2 \# T^2) \times \mathbb{Z}^2$

Let $\Sigma_2 = T^2 \# T^2$ be the orientable surface of genus 2.

Proposition

For any integer $m > 0$, there is an automorphism ϕ of $\pi_1(\Sigma_2 \times T^2)$, such that

$$\mathrm{rkFix}(\phi) = 2m.$$

Proof.

Taking the presentation

$$\pi_1(\Sigma_2 \times T^2) = \langle a_1, b_1, a_2, b_2 \mid [a_1, b_1][a_2, b_2] \rangle \times \langle a, b \mid [a, b] \rangle.$$

Consider the automorphism $\phi : (u, v) \mapsto (u, r_\pi(u)\xi(v))$, where

$$r_\pi : \pi_1(\Sigma_2) \to \pi_1(T^2), \quad a_1 \mapsto a, \ b_1 \mapsto b, \ a_2, b_2 \mapsto 1,$$

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Lemma

Let $p : E = \Sigma_2 \times T^k \to \Sigma_2$ *be the projection to the first factor, where* T^k $(k \geq 1)$ *is a k-torus. Let* $f : E \to E$ *be a fiber-preserving map with induced self-map* $\bar{f}$ *on the base space.*

$$\begin{array}{ccc} E & \xrightarrow{f} & E \\ p\downarrow & & p\downarrow \\ \Sigma_2 & \xrightarrow{\bar{f}} & \Sigma_2 \end{array}$$

Then, for any essential fixed point class $\mathbf{F}$ *of* f*, the projection* $p(\mathbf{F})$ *is an essential fixed point class of* $\bar{f}$*, and*

$$|\mathrm{ind}(f, \mathbf{F})| = [\mathrm{Fix}\bar{f}_\pi : p_\pi(\mathrm{Fix} f_\pi)] \cdot |\mathrm{ind}(\bar{f}, p(\mathbf{F}))|,$$

where $f_\pi : \pi_1(E, e) \to \pi_1(E, e)$ *and* $\bar{f}_\pi : \pi_1(\Sigma_2, x) \to \pi_1(\Sigma_2, x)$ *for* $e = (x, y) \in \mathbf{F}$*, are the natural homomorphisms induced by* f *and* $\bar{f}$ *respectively.*

First examples DO NOT have BIP.

As a consequence, we give a negative answer to Question (Jiang, Math. Ann. 1998):

Theorem (Z.-Zhao, 2023)

- $(T^2 \# T^2) \times S^1$ *has BIPH, but does not have BIP;*
- $(T^2 \# T^2) \times T^2$ *does not have BIPH, and hence does not have BIP.*

A. Gogolev and J.-F. Lafont, *Aspherical products which do not support Anosov diffeomorphisms*, Ann. Henri Poincaré 17 (2016), 3005-3026.

J. Lei, P. Wang and Q. Zhang, *Classification of aut-fixed subgroups in free-abelian times surface groups*, 2023, 18pp. arXiv:2309.13540

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Thanks! 谢谢！