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Stretch diffusion and heat conduction in one-dimensional nonlinear lattices

Zhibin Gao,[1] Nianbei Li,[1,*] and Baowen Li[2,†]

[1]*Center for Phononics and Thermal Energy Science, School of Physics Science and Engineering, Tongji University, 200092 Shanghai, People's Republic of China*

[2]*Department of Mechanical Engineering, University of Colorado Boulder, Colorado 80309, USA*

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For heat conduction in one-dimensional (1D) nonlinear Hamiltonian lattices, it has been known that conserved quantities play an important role in determining the actual heat conduction behavior. In closed or microcanonical Hamiltonian systems, the total energy and stretch are always conserved. Depending on the existence of external on-site potential, the total momentum can be conserved or not. All the momentum-conserving lattices have anomalous heat conduction except the 1D coupled rotator lattice. It was recently claimed that "whenever stretch (momentum) is not conserved in a 1D model, the momentum (stretch) and energy fields exhibit normal diffusion." The stretch in a coupled rotator lattice was also argued to be nonconserved due to the requirement of a finite partition function, which enables the coupled rotator lattice to fulfill this claim. In this work, we will systematically investigate stretch diffusion and heat conduction in terms of energy diffusion for typical 1D nonlinear lattices. Contrary to what was claimed, no clear connection between conserved quantities and heat conduction can be established. The actual situation might be more complicated than what was proposed.

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I. INTRODUCTION

Anomalous heat conduction was first predicted for one-dimensional (1D) Fermi-Pasta-Ulam β (FPU-β) nonlinear lattices by Lepri *et al.* [1]. In that pioneering work, it was found numerically that the thermal conductivity κ diverges with the system size N as $\kappa \propto N^{\alpha}$ with $0 < \alpha < 1$, which violates Fourier's heat conduction law [1]. Numerical simulations also confirm this anomalous heat conduction in a diatomic Toda lattice [2], carbon nanotubes [3], and single polymer chains [4], to name a few. On the other hand, 1D nonlinear lattices with external on-site potential, such as Frenkel-Kontorova (FK) and ϕ^4 lattices, show normal heat conduction [5–7]. Much effort has been devoted to unraveling the physical mechanism behind normal and anomalous heat conduction in low-dimensional systems [8–42]. The consensus reached in this community is that momentum conservation and dimensionality play important roles in determining the actual heat conduction behavior [43–45]. Mode-coupling theory [43] predicts that $\kappa \propto N^{\alpha}$, $\kappa \propto \ln N$, and $\kappa \propto$ const for 1D, 2D, and 3D momentum-conserved systems, respectively. The numerics in 2D and 3D lattice systems were found to be consistent with these predictions [46–51]. In particular, the predictions of length-dependent anomalous heat conduction were also verified experimentally in 1D nanotubes [52], molecular chains [53], and 2D suspended graphene [54]. However, there is one exception of a 1D coupled rotator lattice that displays normal heat conduction behavior despite its momentum-conserving nature [55–58].

The traditional numerical methods used to investigate the heat conduction problem are the nonequilibrium molecular dynamics (NEMD) and equilibrium Green-Kubo (GK) methods [43,44]. A novel diffusion method in thermal equilibrium was proposed by Zhao [59], and it paved the way to explore the heat transport problem in nonlinear systems [58,60]. The mean-square displacement of energy diffusion generally follows a power-law time dependence as $\langle \Delta x^2(t) \rangle_E \propto t^{\beta}$ [59]. It has also been rigorously proven [61] that this energy diffusion method is equivalent to the Green-Kubo method, in which the connection relation of $\alpha = \beta - 1$ first proposed from particle diffusion analysis [21] can be derived.

There are also contining theoretical efforts ranging from early mode-coupling theory [8,9,26], renormalization-group analysis [20], hydrodynamical theory [29,42], and self-consistent mode-coupling theory [30] to recent nonlinear fluctuating hydrodynamical theory [62–69]. Although there is still debate about the actual classification and divergent exponents of the universal classes, there is no question that these theoretical works have greatly improved our understanding of the nature of anomalous heat transport in low-dimensional systems.

Most recently, it was claimed by Das and Dhar that "whenever stretch (momentum) is not conserved in a one-dimensional model, the momentum (stretch) and energy fields exhibit normal diffusion" [72]. The 1D coupled rotator lattice was taken as an example to support this claim. However, after carefully studying some typical 1D nonlinear lattices with normal heat conduction or energy diffusion behaviors, we found that no obvious connection between the conservation of stretch or momentum and the normal diffusion of energy and stretch can be established. Our numerical results indicate that the actual situation might be more complicated than what has been claimed.

This paper is organized as follows: In Sec. II, we present the detailed numerical results of stretch and energy diffusion for typical 1D nonlinear lattices such as ϕ^4, coupled rotator, FK, and combined (FK+ϕ^4) lattices. The conclusions are summarized in Sec. III.

*nbli@tongji.edu.cn

†Baowen.Li@Colorado.Edu

II. STRETCH DIFFUSION IN TYPICAL 1D NONLINEAR LATTICES

We consider the following Hamiltonian for general 1D lattices:

$$H = \sum_i H_i = \sum_i \left[\frac{p_i^2}{2} + V(q_{i+1} - q_i) + U(q_i)\right], \quad (1)$$

where q_i and p_i denote the displacement and momentum for the ith atom, respectively. The interaction potential $V(q_{i+1} - q_i)$ only depends on the displacement difference of $(q_{i+1} - q_i)$. The term $U(q_i)$ represents the external on-site potential, which breaks the conservation of total momentum. For simplicity, periodic boundary conditions $q_i = q_{N+i}$ are applied. The atom index i is assigned as $-(N-1)/2,\ldots,-1,0,1,\ldots,(N-1)/2$, where an odd number of lattice sizes N is chosen without loss of generality.

In the study of stretch diffusion and energy diffusion behaviors, one needs to define the corresponding correlation functions [59,61,62]. For energy diffusion, the excess energy distribution function $\rho_E(i,t)$ can be defined as [59]

$$\rho_E(i,t) = \frac{\langle \Delta H_i(t)\Delta H_0(0)\rangle}{\langle \Delta H_0(0)\Delta H_0(0)\rangle}, \quad (2)$$

where $\Delta H_i(t) = H_i(t) - \langle H_i\rangle$ and $\langle\cdot\rangle$ denotes the ensemble average or time average equivalently for ergodic systems. The stretch distribution function $\rho_D(i,t)$ can also be defined similarly as [62]

$$\rho_D(i,t) = \frac{\langle \Delta D_i(t)\Delta D_0(0)\rangle}{\langle \Delta D_0(0)\Delta D_0(0)\rangle}, \quad (3)$$

where the local stretch $D_i(t) \equiv q_{i+1}(t) - q_i(t)$ and $\Delta D_i(t) = D_i(t) - \langle D_i\rangle$.

For isolated or microcanonical systems with periodic boundary conditions, both the total energy $H = \sum_i H_i$ and the total stretch $D = \sum_i D_i = \sum_i (q_{i+1} - q_i)$ are conserved quantities. As a result, the excess energy distribution function $\rho_E(i,t)$ and the stretch distribution function $\rho_D(i,t)$ must satisfy the sum rules as $\sum_i \rho_E(i,t) = \sum_i \rho_D(i,t) = 0$ in microcanonical systems [39] by noticing that $\sum_i A_i(t) - \sum_i \langle A_i\rangle = 0$ with $A_i = H_i$ or D_i.

The spatiotemporal distribution functions $\rho_{E/D}(i,t)$ can be viewed as the fingerprint for its energy or stretch diffusion behaviors. The overall effect of diffusion can also be described by the mean-square displacement (MSD) $\langle\Delta x^2(t)\rangle_{E/D}$ of energy or stretch defined as

$$\langle\Delta x^2(t)\rangle_{E/D} = \sum_i i^2 \rho_{E/D}(i,t). \quad (4)$$

For example, if the distribution functions $\rho_{E/D}(i,t)$ follow the Gaussian distributions as $\rho_{E/D}(i,t) \sim \frac{1}{\sqrt{4\pi D_{E/D}t}} e^{-\frac{i^2}{4D_{E/D}t}}$ asymptotically, the MSD $\langle\Delta x^2(t)\rangle_{E/D} \sim 2D_{E/D}t$ depends linearly on time, indicating a normal diffusion behavior for energy or stretch.

One can also define a momentum distribution function $\rho_P(i,t)$ accordingly [58,59]. However, for lattices where total momentum is not conserved, the sum of $\rho_P(i,t)$ is not time-invariant as $\sum_i \rho_P(i,t) \neq 0$. In this situation, it is meaningless to discuss the momentum diffusion since the MSD $\langle\Delta x^2(t)\rangle_P$ of momentum is not well defined.

In numerical simulations, the fourth-order symplectic algorithm [70,71] will be used to integrate the equations of motions for 1D lattices. The time steps of $\Delta t = 0.1$ or 0.05 will be adopted. With this numerical setup, the sum of energy distribution $\sum_i \rho_E(i,t)$ and stretch distribution $\sum_i \rho_D(i,t)$ can be maintained within the range of the order of 10^{-5} and 10^{-14}, respectively. The energy density $E = H/N$ is the input parameter, and the temperature $T \equiv \langle p_i^2\rangle$ is a derived quantity as for isolated microcanonical systems.

A. ϕ^4 lattice

We first consider the 1D ϕ^4 lattice with the following Hamiltonian:

$$H = \sum_i \left[\frac{p_i^2}{2} + \frac{1}{2}(q_{i+1} - q_i) + \frac{1}{4}q_i^4\right]. \quad (5)$$

The 1D ϕ^4 lattice is a typical nonlinear lattice with on-site potential that exhibits normal heat conduction behavior [6,7]. The total momentum is not conserved due to the existence of external on-site potential. It has been verified that the energy diffusion is normal as well [59].

This normal diffusion for energy can be seen from Figs. 1(a) and 1(b). The excess energy distribution functions $\rho_E(i,t)$ collapse to an almost Gaussian distribution

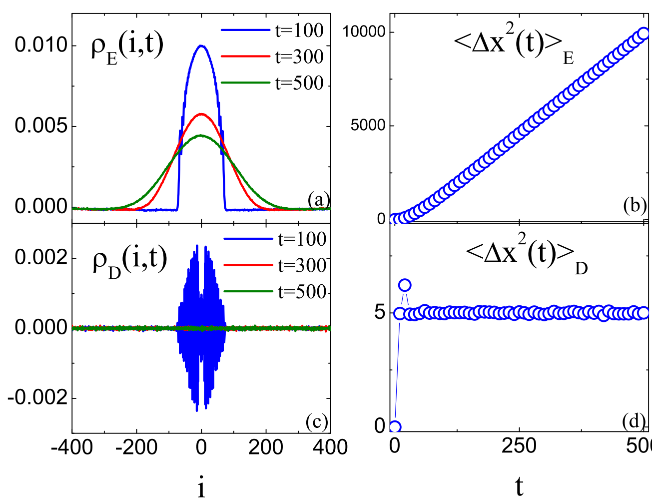

FIG. 1. The ϕ^4 lattice. (a) The excess energy distribution function $\rho_E(i,t)$ and (c) the stretch distribution function $\rho_D(i,t)$. (b) The MSD $\langle\Delta x^2(t)\rangle_E$ of energy and (d) $\langle\Delta x^2(t)\rangle_D$ of stretch. The excess energy distribution function $\rho_E(i,t)$ follows the Gaussian distribution when correlation time $t > 100$, while the stretch distribution function $\rho_D(i,t)$ fails to follow the Gaussian distribution. As a result, the MSD of energy depends on time linearly as $\langle\Delta x^2(t)\rangle_E \propto t$, displaying normal energy diffusion behavior. In contrast, the MSD $\langle\Delta x^2(t)\rangle_D$ of stretch saturates to a constant value after a short time scale, which is definitely not a normal diffusion behavior. In panel (a), the $\rho_E(i,t)$ is shifted upward with a constant value of $1/(N-1)$ to maintain the vanishing tails [39]. The energy density is set as $E = 0.4$ and the corresponding temperature is around $T \approx 0.44$. The lattice size is chosen as $N = 801$.

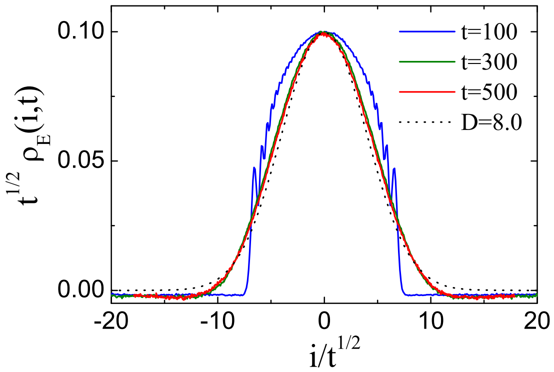

FIG. 2. The rescaled excess energy distribution function $t^{1/2}\rho_E(i,t)$ as the function $i/t^{1/2}$ for the ϕ^4 lattice. The data are taken from Fig. 1(a). The dotted black reference line corresponds to the normal Gaussian distribution with diffusion constant $D = 8.0$. It can be seen that when the correlation time is longer than some intrinsic relaxation time, the distribution of $\rho_E(i,t)$ approaches to the normal Gaussian distribution.

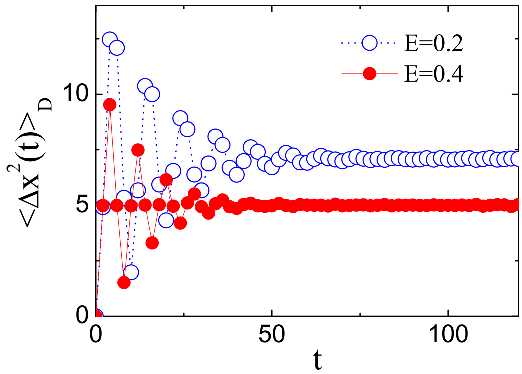

FIG. 3. Temperature dependence of the MSD of the stretch of $\langle\Delta x^2(t)\rangle_D$ for the ϕ^4 lattice. As the energy density (temperature) decreases, the transient time becomes longer before saturation, which might be the result of longer phonon relaxation times at lower temperatures. The saturation value also increases as the temperature decreases. The other parameters are the same as those used in Fig. 1. For energy density $E = 0.2$, the temperature is about $T \approx 0.22$.

$\rho_E(i,t) \sim \frac{1}{\sqrt{4\pi D_E t}} e^{-\frac{i^2}{4D_E t}}$ at long enough correlation times; see Fig. 2. As a result, the MSD $\langle\Delta x^2(t)\rangle_E$ of energy follows a linear time dependence as $\langle\Delta x^2(t)\rangle_E \sim 2D_E t$, asymptotically. Here D_E denotes the diffusion constant for energy.

However, the stretch distribution $\rho_D(i,t)$ fails to follow the Gaussian distribution, as can be seen in Fig. 1(c). The two humps existing at correlation time $t = 100$ spread rapidly over the lattice and disappear at larger correlation times. In Fig. 1(d), the MSD $\langle\Delta x^2(t)\rangle_D$ of stretch is plotted as a function of correlation time t. The $\langle\Delta x^2(t)\rangle_D$ saturates to a constant value after a short correlation time scale. It is definitely not the normal diffusion behavior that is predicted in Ref. [72]. The momentum is not conserved for a 1D ϕ^4 lattice, while its stretch diffusion is not normal.

From numerical simulations, we found that the transient time for $\langle\Delta x^2(t)\rangle_D$ is related to the phonon relaxation time. As temperature decreases, the phonon relaxation time increases giving rise to larger thermal conductivity [6,7]. This transient time follows the same trend as the phonon relaxation time as a function of temperature. It is found that the constant value saturated by $\langle\Delta x^2(t)\rangle_D$ also increases slightly as the temperature decreases (see Fig. 3).

B. Coupled rotator lattice

The 1D coupled rotator lattice has the following Hamiltonian:

$$H = \sum_i \left[\frac{p_i^2}{2} + [1 - \cos(q_{i+1} - q_i)]\right]. \tag{6}$$

Although it conserves the total momentum, it has normal heat conduction behavior [55,56] as well as normal energy diffusion behavior [58]. Furthermore, its momentum diffusion is also normal, which has never been expected [58].

The stretch conservation is a tricky issue for a coupled rotator lattice due to the 2π degeneracy of q_i. The dynamics of the system is invariant to the arbitrary shift of multiple 2π for every q_i as $q_i \to q_i + 2n\pi$, where n can be any integer number. Depending on how to limit the q_i or the local stretch D_i, the total stretch of a coupled rotator lattice can be adjusted as a conserved quantity or not. To illustrate this effect, we consider the following three limitations for q_i or D_i:

(i) No limitations. Nothing is done to the values of q_i and D_i. The variable q_i can take whatever it takes during the evolution of the system dynamics. In this situation, the total stretch is a conserved quantity as $D = \sum_i D_i = 0$, where periodic boundary conditions are applied. The local stretch $D_i = q_{i+1} - q_i$ can take a value from negative infinity to positive infinity, and the partition function is not well defined [67,72]. Although this effect will cause problems in theoretical analysis, the dynamics of the system will not be affected. The energy diffusion is normal, as can be seen from Fig. 4(a). In this situation, the stretch correlation functions $\rho_D(i,t)$ at

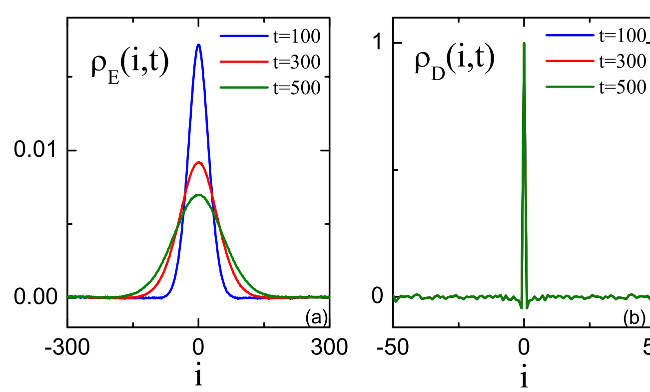

FIG. 4. The 1D coupled rotator lattice of case (i). (a) The excess energy distribution function $\rho_E(i,t)$ and (b) the stretch distribution function $\rho_D(i,t)$. The excess energy distribution functions $\rho_E(i,t)$ follow the Gaussian distributions at correlation times $t = 100$, 300, and 500, implying a normal energy diffusion behavior. However, the stretch distribution functions $\rho_D(i,t)$ at different correlation times all collapse to the same pattern curve as that at $t = 0$. The energy density is set as $E = 0.5$ and the corresponding temperature is around $T \approx 0.54$. The lattice size is chosen as $N = 601$.

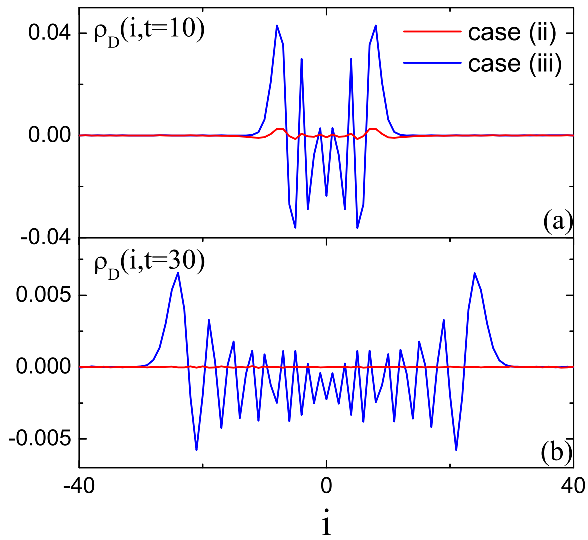

FIG. 5. The stretch distribution functions $\rho_D(i,t)$ for the 1D coupled rotator lattice in cases (ii) and (iii). At correlation time $t = 10$, both $\rho_D(i,t)$ show a similar pattern for cases (ii) and (iii), except that the amplitude in case (iii) is much larger than that in case (ii). At $t = 30$, $\rho_D(i,t)$ still maintains a clear pattern for case (iii) while the pattern disappears for case (ii). The parameters are the same as those used in Fig. 4.

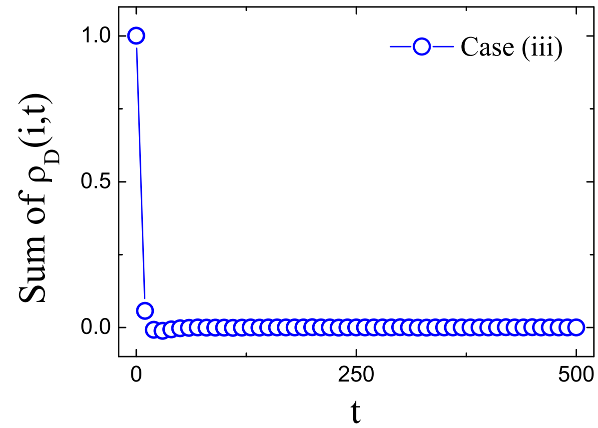

FIG. 6. The sum of $\rho_D(i,t)$ for the coupled rotator lattice in case (iii). It can be seen that $\sum_i \rho_D(i,t)$ decays from a finite value to 0 very quickly, which is consistent with the result in Ref. [72]. As a comparison, the sum $\sum_i \rho_D(i,t)$ for case (ii) can be maintained within the order of 10^{-14} in the whole correlation time range studied. These results indicate that the total stretch is conserved in case (ii) but not in case (iii). The parameters are the same as those used in Fig. 4.

different correlation times all collapse to the same pattern as that at $t = 0$, as can be seen in Fig. 4(b). This might be due to the unbounded nature of the values of q_i and D_i.

(ii) The q_i is limited within $-\pi < q_i \leqslant \pi$. After each time step in numerical simulations, the q_i is forced to reshifted into this region whenever it jumps out. As a result, the local stretch D_i lies within $-2\pi < D_i \leqslant 2\pi$. In this case, the partition function can be well defined. The total stretch is still a conserved quantity, which can be verified by the fact that the sum of $\sum_i \rho_D(i,t)$ can be maintained within the order of 10^{-14} for all times in numerical simulations. In Fig. 5(a), the $\rho_D(i,t)$ in case (ii) displays a similar spatial pattern at short correlation times to that in case (iii). The only difference is that the amplitude is much smaller for case (ii). At larger correlation times seen in Fig. 5(b), the $\rho_D(i,t)$ in case (ii) quickly loses its spatial pattern in comparison to that in case (iii). The MSD $\langle \Delta x^2(t)\rangle_D$ of stretch in this case saturates to a constant value after a short correlation time (not shown here), similar to that of the ϕ^4 lattice in Fig. 1(d).

(iii) The local stretch D_i is limited within $-\pi < D_i \leqslant \pi$. In this case, the q_i is not affected during the dynamical evolution. However, D_i is adjusted appropriately at every time step when it is recorded to generate the correlation function of stretch. The partition function is well defined. However, in this special situation, the total stretch is not a conserved quantity, as can be seen from Fig. 6, which is consistent with the result in Ref. [72]. As we have mentioned, it will be meaningless to discuss the diffusion behavior if the sum of $\rho_D(i,t)$ is not time-independent.

From the above results and discussions for three cases, it can be found that the conservation of total stretch is a very tricky issue in a coupled rotator lattice. Depending on the limitation of q_i or D_i, the stretch can be adjusted to be a conserved or nonconserved quantity. The stretch conservation or diffusion in a coupled rotator lattice will require further study, and it remains an open issue.

C. FK lattice

We then consider another 1D nonlinear lattice with on-site potential, namely the FK lattice with a Hamiltonian as

$$H = \sum_i \left[\frac{p_i^2}{2} + \frac{1}{2}(q_{i+1} - q_i)^2 + \frac{V}{2\pi}(1 - \cos 2\pi q_i) \right]. \quad (7)$$

The FK lattice also exhibits normal heat conduction [5] as well as normal energy diffusion behaviors. In Figs. 7(a) and 7(b), the excess energy distribution function $\rho_E(i,t)$ and the MSD $\langle \Delta x^2(t)\rangle_E$ of energy are plotted. The $\rho_E(i,t)$ follows the Gaussian distribution functions, and the $\langle \Delta x^2(t)\rangle_E$ is linearly proportional to the correlation time as $\langle \Delta x^2(t)\rangle_E \propto t$, indicating obvious normal diffusion behavior for energy.

In contrast to the ϕ^4 lattice, the stretch distribution function $\rho_D(i,t)$ of the FK lattice approaches the Gaussian distributions as $\rho_D(i,t) \sim \frac{1}{\sqrt{4\pi D_D t}} e^{-\frac{i^2}{4 D_D t}}$ at long enough correlation times, as can be seen in Fig. 7(c). The MSD $\langle \Delta x^2(t)\rangle_D$ of stretch follows the linear time dependence asymptotically as $\langle \Delta x^2(t)\rangle_D \sim 2D_D t$ in Fig. 7(d). Although both ϕ^4 and FK lattices have normal heat conduction and energy diffusion behaviors, they exhibit totally different stretch diffusion behavior. The stretch diffusion is normal for the FK lattice despite its stretch conservation nature. This effect is consistent with the prediction by Das and Dhar in Ref. [72]. Unlike the previous 1D coupled rotator lattice, there is no ambiguous space for the stretch conservation in the FK lattice (see Fig. 8). The value of q_i or D_i is not degenerated anymore due to the existence of the interaction potential term in Eq. (7).

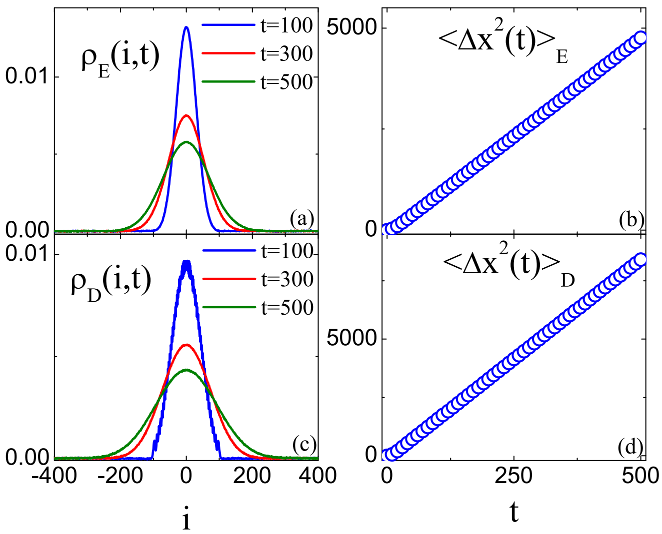

FIG. 7. The 1D FK lattice. (a) The energy distribution function $\rho_E(i,t)$ and (c) stretch distribution function $\rho_D(i,t)$. (b) The MSD $\langle\Delta x^2(t)\rangle_E$ of energy and (d) $\langle\Delta x^2(t)\rangle_D$ of stretch. Both the energy distribution function $\rho_E(i,t)$ and the stretch distribution function $\rho_D(i,t)$ follow the Gaussian distributions after long enough correlation times. As a result, both the energy and stretch diffusions are normal as $\langle\Delta x^2(t)\rangle_{E/D} \propto t$, asymptotically. These results are consistent with the claim in Ref. [72]. In panels (a) and (c), the $\rho_{E/D}(i,t)$ are shifted upward with a constant value of $1/(N-1)$ to maintain vanishing tails. The parameter for on-site coupling strength is set as $V = 1$. The energy density is set as $E = 1$ and the corresponding temperature is around $T \approx 0.86$. The lattice size is chosen as $N = 801$.

D. Combined (FK+ϕ^4) lattice

In the end, we consider the combined (FK+ϕ^4) lattice with Hamiltonian

$$H = \sum_i \left[\frac{p_i^2}{2} + \frac{1}{2}(q_{i+1} - q_i)^2 + \frac{V}{2\pi}(1 - \cos 2\pi q_i) + \frac{1}{4}q_i^4\right]. \tag{8}$$

The combined FK+ϕ^4 lattice should also have normal heat conduction behavior due to the existence of the on-site potential. This can be verified by examining the energy diffusion behavior in Figs. 9(a) and 9(b). $\rho_E(i,t)$ follows Gaussian distributions, and the MSD of energy depends linearly on time as $\langle\Delta x^2(t)\rangle_E \sim 2D_E t$, asymptotically.

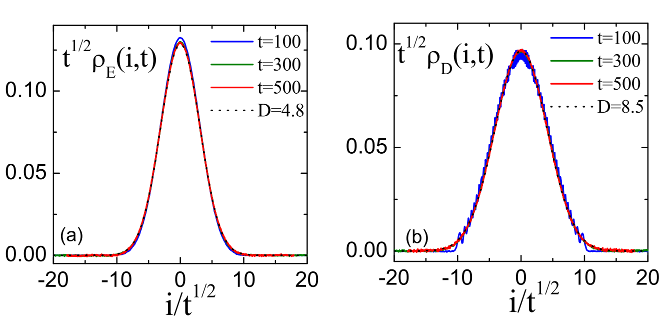

FIG. 8. The rescaled (a) excess energy distribution functions $t^{1/2}\rho_E(i,t)$ and (b) stretch distribution functions $t^{1/2}\rho_D(i,t)$ as a function of rescaled position $i/t^{1/2}$ for the FK lattice. The black dotted lines correspond to the normal Gaussian distributions with the diffusion constants $D_E = 4.8$ for energy and $D_D = 8.5$ for stretch. The data are taken from Figs. 7(a) and 7(c).

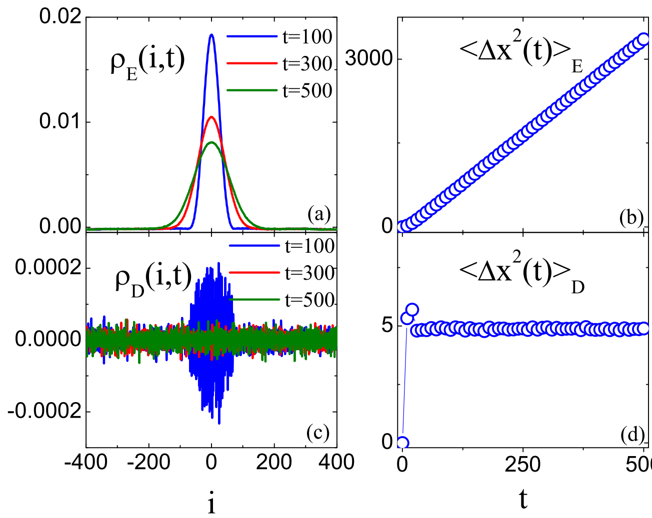

FIG. 9. The 1D combined FK+ϕ^4 lattice. (a) The energy distribution function $\rho_E(i,t)$ and (c) the stretch distribution function $\rho_D(i,t)$. (b) The MSD $\langle\Delta x^2(t)\rangle_E$ of energy and (d) $\langle\Delta x^2(t)\rangle_D$ of stretch. The overall behavior is similar to that of the ϕ^4 lattice. The excess energy distribution function $\rho_E(i,t)$ follows the Gaussian distribution, while the stretch distribution function $\rho_D(i,t)$ does not. As a result, only the energy diffusion is normal as $\langle\Delta x^2(t)\rangle_E \propto t$, asymptotically. In panel (a), the $\rho_E(i,t)$ is shifted upward with a constant value of $1/(N-1)$ to maintain vanishing tails. The parameter is set as $V = 0.5$. The energy density is set as $E = 0.5$ and the corresponding temperature is around $T \approx 0.47$. The lattice size is chosen as $N = 801$.

In Fig. 9(c), the stretch distribution functions $\rho_D(i,t)$ are plotted for correlation times $t = 100$, 300, and 500. No Gaussian-like distribution is observed, and the scenario is similar to that of the ϕ^4 lattice as in Fig. 1(c). As with the ϕ^4 lattice, the $\langle\Delta x^2(t)\rangle_D$ saturates to a constant value after a short time scale, as can be seen in Fig. 9(d). This is another counterexample to the claim by Das and Dhar since the momentum is not conserved here, while the stretch diffusion is also not normal.

III. CONCLUSIONS

In conclusion, we have systematically investigated the stretch diffusion as well as the energy diffusion for a few 1D nonlinear lattices with normal heat conduction behaviors. For isolated systems with periodic boundary conditions, both the total energy and the total stretch are conserved quantities. Depending on the existence of on-site potential, the total momentum can be conserved or nonconserved. For 1D ϕ^4 and combined (FK+ϕ^4) lattices, the total momentum is not conserved while the stretch diffusion is not normal, which are counterexamples to the claim in Ref. [72]. For a 1D coupled rotator lattice with normal momentum diffusion, the situation is tricky in the sense that its stretch conservation depends on the choices of limitation of q_i or D_i. Only for a 1D FK lattice is the total momentum not conserved and the stretch

TABLE I. The relation between conservation quantities and corresponding diffusion behaviors. The straight (waved) underline indicates that the numerical behavior is consistent (inconsistent) with theoretical predictions in Ref. [72]. If momentum or stretch is not conserved, there is no corresponding diffusion behavior for momentum or stretch, respectively.

Model	Conservation		Normal diffusion		
	Mom.	Stretch	Energy	Mom.	Stretch
ϕ^4	No	Yes	Yes		No
Rotator	Yes	Yes (I)	Yes	Yes	No
		Yes (II)	Yes	Yes	No
		No (III)	Yes	Yes	
FK	No	Yes	Yes		Yes
FK+ϕ^4	No	Yes	Yes		No

and energy diffusions are normal, which is consistent with the claim. The relation between conserved quantities and their corresponding diffusion behaviors is summarized in Table I. In conclusion, our numerical results do not support the definite claim that "whenever stretch (momentum) is not conserved in a one-dimensional model, the momentum (stretch) and energy fields exhibit normal diffusion" proposed in Ref. [72]. However, there is something of interest with regard to the lattices with cosine or bounded potentials. This remains an open issue, and we hope more efforts will be made in this direction in the near future.

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