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Some results on the lexicographic product of vertex-transitive graphs[☆]

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ABSTRACT

Many large graphs can be constructed from existing smaller graphs by using graph operations, for example, the Cartesian product and the lexicographic product. Many properties of such large graphs are closely related to those of the corresponding smaller ones. In this short note, we give some properties of the lexicographic products of vertex-transitive and of edge-transitive graphs. In particular, we show that the lexicographic product of *Cayley* graphs is a *Cayley* graph.

1. Introduction

Vertex-transitive and edge-transitive graphs are well suited for use as models for interconnection networks, as these graphs look the same viewed from any vertex [1,2]. Thus, in such networks the same routing algorithm may be used by each processor. In recent years, the problem of how to use new finite groups techniques to study vertex-transitive graphs has received a lot of attention (see e.g. [3–6]).

The Cayley graph is also an important connection pattern of interconnection networks, which has been studied extensively, with more results obtained [7,8]. As a result, how to obtain large Cayley graphs has become an interesting topic not only in its own right but also practically. Nedela and Škoviera [9] studied the Cayley graph of the generalized Petersen graphs. Xu [10] proved that the Cartesian product of two Cayley graphs is a Cayley graph. For further results and references, the reader is referred to the recent paper [10].

Some large graphs can be constructed from existing smaller graphs by using, for example, the Cartesian product and the lexicographic product [1,11]. Many properties of such large graphs are associated strongly with those of the corresponding smaller ones [12].

In this note, we consider the lexicographic product of graphs. Our main objective is to study the properties of lexicographic products of vertex-transitive and of edge-transitive graphs, and of the Cayley graphs. We show that the lexicographic product of vertex-transitive (edge-transitive) graphs is a vertex-transitive (edge-transitive) graph and, in particular, the lexicographic product of *Cayley* graphs is a *Cayley* graph.

2. The main results

We start by fixing some notation.

Let $G = (V, E)$ be a simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and the edge set $E = \{e_1, e_2, \ldots, e_m\}$. Let $\mathrm{Aut}(G)$ denote the automorphism group of G. A graph G is vertex-transitive (resp. edge-transitive) if $\mathrm{Aut}(G)$ acts transitively on $V(G)$

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(resp. on $E(G)$). Let Γ be a finite group and S a subset of Γ that is closed under taking the inverse and does not contain the identity. *CayleyC*$_\Gamma(S)$ is a graph with vertex set Γ and edge set $E(C_\Gamma(S)) = \{gh : hg^{-1} \in S\}$.

Let G_1 and G_2 be two graphs. The *lexicographic product*, denoted by $G_1 \odot G_2$, is a graph with vertex set $V(G_1) \times V(G_2)$, and there is an edge from (u_1, u_2) to (v_1, v_2) if either there is an edge from u_1 to v_1 in G_1, or $u_1 = v_1$ and there is an edge from u_2 to v_2 in G_2. For other terminology and notation not defined here, see [13].

Theorem 2.1. *Let G_i $(i = 1, 2, \dots, n)$ be vertex-transitive graphs. Then the lexicographic product graph $G_1 \odot G_2 \odot \cdots \odot G_n$ is a vertex-transitive graph.*

Proof. Suppose that $G = G_1 \odot G_2 \odot \cdots \odot G_n$, and let $x = x_1x_2, \dots, x_n$ and $y = y_1y_2, \dots, y_n$ be any two vertices of the graph G, where $x_i, y_i \in V(G_i)$ $(i = 1, 2, \dots, n)$. Since G_i is vertex-transitive, there exists $\sigma_i \in \mathrm{Aut}(G_i)$ such that $\sigma_i(x_i) = y_i$ $(i = 1, 2, \dots, n)$. Now define the mapping ϕ as follows:

$$\phi(x_1x_2, \dots, x_n) = \sigma_1(x_1)\sigma_2(x_2), \dots, \sigma_n(x_n).$$

It is easy to verify that ϕ is an element of $\mathrm{Aut}(G)$, and $\phi(x) = y$. Thus, G is vertex-transitive. □

Similarly, we have the following:

Theorem 2.2. *Let G_i $(i = 1, 2, \dots, n)$ be edge-transitive graphs. Then the lexicographic product $G_1 \odot G_2 \odot \cdots \odot G_n$ is an edge-transitive graph.*

The proof of the following result is simple, but it is very important in the theory of graph embedding.

Theorem 2.3. *Suppose that $G = G_1 \odot G_2 \odot \cdots \odot G_n$ and $G' = G'_1 \odot G'_2 \odot \cdots \odot G'_n$. If G_i is a subgraph of G'_i for $i = 1, 2, \dots, n$, then G is a subgraph of G'.*

From the above theorems, we know that the lexicographic product of the vertex-transitive (edge-transitive) graphs is a vertex-transitive (edge-transitive) graph. It is well known that the Cayley graph is vertex-transitive [3], but the reverse need not to be true (e.g., for the Petersen graph). It is then natural to ask whether the lexicographic product of *Cayley* graphs is a *Cayley* graph. The following theorem gives an affirmative answer to this question.

Theorem 2.4. *The lexicographic product of Cayley graphs is a Cayley graph.*

Proof. Let $G_i = C_{\Gamma_i}(S_i)$ be the *Cayley* graph for a finite group $\Gamma_i = (X_i, \circ_i)$ on the set S_i; then $G = G_1 \odot G_2 \odot \cdots \odot G_n$ is the Cayley graph $C_\Gamma(S)$ for the group $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$ on the set

$$S = \bigcup_{i=1}^{n} \{e_1 \cdots e_{i-1}\} \times Q_i \times Q_{i+1} \times \cdots \times Q_n,$$

where

$$Q_i = \begin{cases} \{e_i\} & x_i = y_i \\ S_i & (x_i, y_i) \in E(G_i) \\ \Phi_i & (x_i, y_i) \notin E(G_i) \end{cases}$$

and e_i is the unit element of the Γ_i, $i = 1, 2, \dots, n$.

We only need to consider the case $n = 2$. Then, $G = G_1 \odot G_2$, $\Gamma = \Gamma_1 \times \Gamma_2$, and $S = (\{e_1\} \times S_2) \cup (S_1 \times S_2) \cup (S_1 \times \Phi_2) \cup (S_1 \times \{e_2\})$, for any two elements x_1x_2 and y_1y_2, where $x_i, y_i \in X_i$, $i = 1, 2$. We only need to prove that

$$(x_1x_2, y_1y_2) \in E(G) \Longleftrightarrow (x_1x_2)^{-1} \circ (y_1y_2) \in S.$$

By the definition of the lexicographic product,

$$(x_1x_2, y_1y_2) \in E(G) \Leftrightarrow \begin{cases} (x_1, y_1) \in E(G_1), & \text{otherwise} \\ x_1 = y_1, & (x_2, y_2) \in E(G_2) \end{cases}$$

since $G_i = C_{\Gamma_i}(S_i)$, $i = 1, 2$. Thus, we have

$$(x_i, y_i) \in E(G_i) \Leftrightarrow x_i^{-1} \circ_i y_i \in S_i.$$

Next, we distinguish the following cases:

Case (1):

$$\begin{aligned} &x_1 = y_1,\ (x_2, y_2) \in E(G_2) \Leftrightarrow \\ &(x_1x_2)^{-1} \circ (y_1y_2) = (x_1^{-1}x_2^{-1}) \circ (y_1y_2) \\ &\qquad = (x_1^{-1} \circ_1 y_1)(x_2^{-1} \circ_2 y_2) \\ &\qquad = (x_1^{-1} \circ_1 x_1)(x_2^{-1} \circ_2 y_2) \\ &\qquad = e_1(x_2^{-1} \circ_2 y_2) \in \{e_1\} \times S_2 \subseteq S. \end{aligned}$$

Case (2):

$$\begin{aligned} & x_2 = y_2,\ (x_1, y_1) \in E(G_1) \Leftrightarrow \\ (x_1x_2)^{-1} \circ (y_1y_2) &= (x_1^{-1}x_2^{-1}) \circ (y_1y_2) \\ &= (x_1^{-1} \circ_1 y_1)(x_2^{-1} \circ_2 y_2) \\ &= (x_1^{-1} \circ_1 y_1)(x_2^{-1} \circ_2 x_2) \\ &= (x_1^{-1} \circ_1 y_1)e_2 \in S_1 \times \{e_2\} \subseteq S. \end{aligned}$$

Case (3):

$$\begin{aligned} & (x_1, y_1) \in E(G_1),\ (x_2, y_2) \in E(G_2) \Leftrightarrow \\ (x_1x_2)^{-1} \circ (y_1y_2) &= (x_1^{-1}x_2^{-1}) \circ (y_1y_2) \\ &= (x_1^{-1} \circ_1 y_1)(x_2^{-1} \circ_2 y_2) \in S_1 \times S_2 \subseteq S. \end{aligned}$$

Case (4):

$$\begin{aligned} & (x_1, y_1) \in E(G_1),\ (x_2, y_2) \notin E(G_2) \Leftrightarrow \\ (x_1x_2)^{-1} \circ (y_1y_2)U &= (x_1^{-1}x_2^{-1}) \circ (y_1y_2) \\ &= (x_1^{-1} \circ_1 y_1)(x_2^{-1} \circ_2 y_2) \in S_1 \times \Phi_2 \subseteq S. \end{aligned}$$

These show that $G = G_1 \odot G_2$ is the Cayley graph $C_\Gamma(S)$ for the group $\Gamma = \Gamma_1 \times \Gamma_2$ on the subset $S = (\{e_1\} \times S_2) \cup (S_1 \times S_2) \cup (S_1 \times \Phi_2) \cup (S_1 \times \{e_2\})$. This completes the proof of the theorem. □

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