ADVANCED
QUANTUM
TECHNOLOGIES

Supporting Information

for *Adv. Quantum Technol.*, DOI 10.1002/qute.202300136

Optical Router and 4 × 1 Multiplexer of Coexisting Crystal Field and Non-Hermitian Autler-Townes Splitting Controlled by Photon–Phonon Dressing in Eu^{3+}: $BiPO_4$

Anas Mujahid, Huanrong Fan, Yafen Wang, Faizan Raza, Irfan Ahmed, Yaxin Guo, Faisal Munir, Muhammad Usman, Bo Li and Yanpeng Zhang**

Supporting Information: Optical router and 4x1 multiplexer of coexisting crystal field and Autler-Townes splitting controlled by photon-phonon dressing in Eu^{3+}: $BiPO_4$

Anas Mujahid†, Huanrong Fan†, Yafen Wang, Faizan Raza, Irfan Ahmed, Yaxin Guo, Faisal Munir, Muhammad Usman, Bo Li* and Yanpeng Zhang*

Key Laboratory for Physical Electronics and Devices of the Ministry of Education & Shaanxi Key Lab of Information Photonic Technique, School of Electronic Science and Engineering, Faculty of Electronics and Information, Xi'an Jiaotong University, Xi'an 710049, China

Emails: *ypzhang@mail.xjtu.edu.cn, and libophy@xjtu.edu.cn
†These authors contribute equally

In this Supplementary Material, we provide more details formulas of Parity Time (PT) symmetry and display the FL and SFWM signals Eigenvalues based on the real and imaginary parts with respect to linewidth for different dressings.

SI. Analytical Calculations for the Non-Hermitian Quantization or Exceptional points control:

The second-order Fluorescence (FL) system is given as;

$$\rho_{FL}^{(2)} = \frac{|G_1|^2}{(\Gamma_{02} + i\Delta_1 + |G_1|^2/(\Gamma_{00} + 2i\Delta_1) + |G_{p1}|^2/(\Gamma_{01} + i\Delta_1 + i\Delta_{p1}))\Gamma_{22}} \quad \text{(S1)}$$

Firstly, from $\Gamma_{02} + i\Delta_1 + |G_1|^2/(\Gamma_{00} + 2i\Delta_1)$ the fraction Δ_1 can have two values. Secondly, by splitting the above level, from $|G_{p1}|^2/(\Gamma_{01} + i\Delta_1 + i\Delta_{p1})$ the fraction can have two values, so Δ_1 have three values in total, $\Gamma_{20} + i\Delta_1 + \frac{|G_1|^2}{(\Gamma_{00} + 2i\Delta_1)} = 0$, $(\Gamma_{20} + i\Delta_1)(\Gamma_{00} + 2i\Delta_1) + |G_1|^2 = 0$, and $\Gamma_{20}\Gamma_{00} - i\Delta_1\Gamma_{00} + 2i\Delta_1\Gamma_{20} - 2\Delta_1^2 + |G_1|^2 = 0.$

SI.1 Real part quantization:

Substitute $\Delta_1 = a$ into the formula in the above photon dressing formulas, we get: $\Gamma_{20}\Gamma_{00} - ia\Gamma_{00} + 2ia\Gamma_{20} - 2a^2 + 4iab + |G_1|^2 = 0$. Find the root of the real part of the denominator by ignoring b^2 (reason: b^2 is very small), and we get: It can be solved for $a = \pm\sqrt{\frac{\Gamma_{20}\Gamma_{00} + |G_1|^2}{2}}$, which corresponds to linewidth. Now substitute $\Delta_1 = b$ into formula (2), we get: $-i4ab + ia(\Gamma_{20} + 2\Gamma_{00}) = 0$ $b = \frac{\Gamma_{00} - 2\Gamma_{20}}{4}$, where $b = \frac{\Gamma_{00} - 2\Gamma_{20}}{4}$ corresponding to the lifetime (Γ). Secondly Solved for: $(\Delta_1 - x_1)(\Delta_1 - x_2) = 0$, and $\Delta_1\Gamma_{01} + i\Delta_1^2 - \Delta_1 i x_1 + i\Delta_{p1}\Delta_1 - x_1\Gamma_{01} + x_1 i\Delta_1 + ix_1^2 + x_1 i\Delta_{p1} + |G_{p1}|^2 = 0$. Putting Δ_1 =c and $\Delta_1^2 = c^2 + i2cd$ we get $c\Gamma_{01} + ic^2 - 2cd + i\Delta_{p1}c - x_1\Gamma_{01} + 2x_1 ic + x_1 i\Delta_{p1} + ix_1^2 + |G_{p1}|^2 = 0$. Again, Putting x1= a+ib with b=0 $c\Gamma_{01} + ic^2 - 2cd + i\Delta_{p1}c - a\Gamma_{01} + 2iac + ia\Delta_{p1} + i(a+ib)^2 + |G_{p1}|^2 = 0$, and $c^2 + (\Delta_{p1} + a)c = 0$. A further solution is $c = \frac{-(\Delta_{p1} + 2a) \pm \sqrt{(\Delta_{p1} + 2a)^2 - 4(1)(a\Delta_{p1} - b^2)}}{2}$, $c\Gamma_{01} - 2cd - a\Gamma_{01} - 2ab + |G_{p1}|^2 = 0$ $2cd = c\Gamma_{01} - 2cd - a\Gamma_{01} - 2ab + |G_{p1}|^2$, $d = \frac{c\Gamma_{01} - 2cd - a\Gamma_{01} - 2ab + |G_{p1}|^2}{2c}$.

SI.2 Imaginary part quantization:

Substitute $\Delta_1 = ib$ in the above photon dressing formulas, we get:

$c\Gamma_{01} + ic^2 - 2cd + i\Delta_{p1}c - (ib)\Gamma_{01} + 2i(ib)c + i(ib)\Delta_{p1} + i(a+ib)^2 + |G_{p1}|^2 = 0$. Find the root of the imaginary part of the denominator a^2, ignoring (reason: a^2 very small), we get: $c^2 + \Delta_{p1}c - b(\Gamma_{01} + b) = 0$, $c = \frac{-\Delta_{p1} \pm \sqrt{(\Delta_{p1})^2 - 4(b(\Gamma_{01} + b))}}{2}$, $c\Gamma_{01} - 2cd - (2c + \Delta_{p1} + 2a)b + |G_{p1}|^2 = 0$ $d = \frac{c\Gamma_{01} - (2c + \Delta_{p1} + 2a)b + |G_{p1}|^2}{2c}$.

Secondly Solved $(\Delta_1 - x_1)(\Delta_1 - x_2) = 0$ and $\Delta_1\Gamma_{01} + i\Delta_1^2 - \Delta_1 ix_1 + i\Delta_{p1}\Delta_1 - x_1\Gamma_{01} + x_1 i\Delta_1 + ix_1^2 + x_1 i\Delta_{p1} + |G_{p1}|^2 = 0$. Put Δ_1 =c and $\Delta_1^2 = c^2 + i2cd$, $c\Gamma_{01} + ic^2 - 2cd + i\Delta_{p1}c - x_1\Gamma_{01} + 2x_1 ic + x_1 i\Delta_{p1} + ix_1^2 + |G_{p1}|^2 = 0$. Putting x1=a+ib with a=0, $c\Gamma_{01} + ic^2 - 2cd + i\Delta_{p1}c - b\Gamma_{01}i\text{-}2bc\text{-}b\Delta_{p1} + i(a+ib)^2 + |G_{p1}|^2 = 0$ and $c^2 + \Delta_{p1}c + b\Gamma_{01} - b^2 = 0$. The further solution we get $c = \frac{-(\Delta_{p1}) \pm \sqrt{(\Delta_{p1})^2 - 4(b\Gamma_{01} - b^2)}}{2}$, $c\Gamma_{01} - 2cd - 2bc - b\Delta_{p1} - 2ab + |G_{p1}|^2 = 0$, $2cd = c\Gamma_{01} - 2ab - b\Delta_{p1} - 2ab + |G_{p1}|^2$, and $d = \frac{c\Gamma_{01} - 2ab - b\Delta_{p1} - 2ab + |G_{p1}|^2}{2c}$.

Numerical solution: directly solve the quadratic equation in one variable with the root-finding formula, and obtain the expression in the simulation program. The EP point can be obtained by setting the value under the root sign to 0. When the square root part is zero, the real and imaginary parts will be equal that correspond to the Exception point (EP point).

SI.3 Real Non-Hermitian quantization or EP Control

$\mathrm{Re}(EP1) = \left(\sqrt{\frac{\Gamma_{20}\Gamma_{00} + |G_1|^2}{2}} - \sqrt{\frac{\Gamma_{20}\Gamma_{00} + |G_1|^2}{2}}\right)/2 = 0$, $\mathrm{Re}(EP2) = \frac{-(\Delta_{p1} + 2a)}{2}$ Where $a = \frac{\pm\sqrt{8\Gamma_{20}\Gamma_{00}}}{2}$ refer to the real part.

SI.4 Imaginary Non-Hermitian quantization or EP control

$\mathrm{Im}(EP1) = \mathrm{Im}(EP2) = \frac{-\Delta_{p1}}{2}$. Moreover, we show the summary of the Eigenvalues, Linewidth, and Exceptional points of the four kinds of dressing i.e. single, parallel, cascade, and nested in table form below.

Table S1. Shows the FL real and imaginary parts of Eigenvalues and linewidth for different dressing

Dressing	**Eigen Value**		**Linewidth**	
	Real part	Imaginary part	Real part	Imaginary part
Single	$A_1 = (-\Delta_{p1} \pm \sqrt{U_1})/2$	$B_1 = (\Gamma_{10} + \Gamma_{20} \pm \sqrt{W_1})/2$	$\Gamma_{e1} = (\Gamma_{10} + \Gamma_{20})/2$	$\Delta_{e1} = (d\Delta_{p1} - \Gamma_{20}\Delta_{p1})/2d$
$\Gamma^* = \delta_1 - \delta_2$	$\Gamma_r^* = \sqrt{U_1}$	$\Gamma_i^* = \sqrt{W_1}$		
Parallel	$A_{21} = \pm\sqrt{\Gamma_{20}^2 + \|G_1\|^2}/2$	$B_{21} = \pm\sqrt{\Gamma_{10}^2 + \|G_{p1}\|^2}/2$	$\Gamma_{e21} = (\Gamma_{22}\Gamma_{20} + \|G_1\|^2)/\Gamma_{22}$	$\Delta_{e21} = \Gamma_{22}\Gamma_{20} + \|G_1\|^2/\Gamma_{22}$

$\Gamma_P^* = \delta_1 - \delta_3$	$\Gamma_r^* = \sqrt{\Gamma_{20}^2 + \lvert G_1 \rvert^2}$	$\Gamma_i^* = \sqrt{\Gamma_{10}^2 + \lvert G_{p1} \rvert^2}$		
	$A_{22} = (-\Delta_{p1} \pm \sqrt{U_{22}})/2$	$B_{22} = (\Gamma_{10} + \Gamma_{20} \pm \sqrt{W_{22}})/2$	$\Gamma_{e22} = (\Gamma_{10} + \Gamma_{20})/2$	$\Delta_{e22} = (d\Delta_{p1} - \Gamma_{20}\Delta_{p1})/2d$
	$\Gamma_r^* = \sqrt{U_{22}}$	$\Gamma_i^* = \sqrt{W_{22}}$		
Cascade	$A_{31} = \pm\sqrt{8\Gamma_{20}\Gamma_{00}}/2$	$B_{31} = -\Delta_{p1} \pm \sqrt{W_{31}}/2$	$\Gamma_{e31} = (\Gamma_{00} - 2\Gamma_{20})/4$	$\Delta_{31} = (c\Gamma_{01} + X_{31})/2c$
$\Gamma_C^* = \delta_1 - \delta_3$	$\Gamma_r^* = \sqrt{8\Gamma_{20}\Gamma_{00}}$	$\Gamma_i^* = \sqrt{W_{31}}$		
	$A_{32} = -(\Delta_{p1} + 2a) \pm \sqrt{U_{32}}/2$	$B_{32} = -(\Delta_{p1}) \pm \sqrt{W_{32}}/2$	$\Gamma_{e32} = c\Gamma_{01} - V_{32}/2c$	$\Delta_{e32} = c\Gamma_{01} + X_{32}/2c$
	$\Gamma_r^* = \sqrt{U_{32}}$	$\Gamma_i^* = \sqrt{W_{32}}$		
Nested	$A_{41} = \pm\sqrt{\Gamma_{20}\Gamma_{00} + \lvert G_1 \rvert^2}/2$	$B_{41} = (\Delta_{p1} \pm \sqrt{W_{41}})/4$	$\Gamma_{e41} = (-2\Gamma_{20} - \Gamma_{00})/4$	$\Delta_{e41} = c\Gamma_{10} + X_{41}/4c$
$\Gamma_N^* = \delta_1 - \delta_3$	$\Gamma_r^* = \sqrt{\Gamma_{20}\Gamma_{00} + \lvert G_1 \rvert^2}$	$\Gamma_i^* = \sqrt{W_{41}}/2$		
	$A_{42} = (\Delta_{p1} + 4a) \pm \sqrt{U_{42}})/4$	$B_{42} = 4b - \Delta_{p1} \pm \sqrt{W_{42}}/4$	$\Gamma_{e42} = c\Gamma_{10} - 4ab + \Gamma_{10}a + \lvert G_{p1} \rvert^2/4c$	$\Delta_{e42} = d\Gamma_{10} - X_{42}/4d$
	$\Gamma_r^* = \sqrt{U_{42}}/2$	$\Gamma_i^* = \sqrt{W_{42}}/2$		

$U_1 = \lvert \Delta_{p1} \rvert^2 - 4\Gamma_{10}\Gamma_{20} \lvert G_{p1} \rvert^2$, $W_1 = (\Gamma_{10} + \Gamma_{20})^2 - 4\Gamma_{10}\Gamma_{20} \lvert G_{p1} \rvert^2$, $U_{22} = \lvert \Delta_{p1} \rvert^2 - 4\Gamma_{10}\Gamma_{20} \lvert G_{p1} \rvert^2$, $W_{22} = (\Gamma_{10} + \Gamma_{20})^2 - 4\Gamma_{10}\Gamma_{20} \lvert G_{p1} \rvert^2$,
$W_{31} = \Delta_{p1}{}^2 - 4b(\Gamma_{01} + b)$, $U_{32} = (\Delta_{p1} + 2a)^2 - 4(1)(a\Delta_{p1} - b^2)$, $W_{32} = (\Delta_{p1})^2 - 4(b\Gamma_{01} - b^2)$, $W_{41} = \Delta_{p1}{}^2 - 42ib^2 + i4b\Gamma_{10}$,
$U_{42} = (-\Delta_{p1} - 4a)^2 + 4a(2a + \Delta_{p1})$, $W_{42} = (\Delta_{p1} - 4b)^2 - 8(2b^2 + \Delta_{p1}b)$, $X_{31} = (-2c + \Delta_{p1} + 2a)b + \lvert G_{p1} \rvert^2$,
$X_{32} = -2ab - b\Delta_{p1} - 2ab + \lvert G_{p1} \rvert^2$, $V_{32} = 2cd - a\Gamma_{01} - 2ab + \lvert G_{p1} \rvert^2$, $X_{41} = -4ab + 4bc + b\Delta_{p1} + \lvert G_{p1} \rvert^2$, $X_{42} = 4ab + \Gamma_{10}b + \lvert G_{p1} \rvert^2$

FL Table

Exponential Point table

Dressing	Real part		Imaginary part	
	Resonant position	linewidth	Resonant position	linewidth
Single (Δ_{FL})	$-\Delta_{p1}$	$(\Gamma_{10} + \Gamma_{20})/2$	$(\Gamma_{10} + \Gamma_{20})/2$	$(d\Delta_{p1} - \Gamma_{20}\Delta_{p1})/2$

Exponential Point table

Dressing	Real part	Imaginary part

	Resonant position	linewidth	Resonant position	linewidth
Parallel(Δ_{FL})	$-(\Gamma_{22}\Gamma_{20}+\|G_1\|^2)/i\Gamma_{22}$	0	$(\Gamma_{22}\Gamma_{20}+\|G_1\|^2)/\Gamma_{22}$	0
	$c=-\Delta_{p1}/2$	$(\Gamma_{10}+\Gamma_{20})/2$	$(\Gamma_{10}+\Gamma_{20})/2$	$(d\Delta_{p1}-\Gamma_{20}\Delta_{p1})/2d$

Exponential Point table

Dressing	Real part		Imaginary part	
	Resonant position	linewidth	Resonant position	linewidth
Cascade(Δ_{FL})	0	$(\Gamma_{00}-2\Gamma_{20})/4$	$(\Delta_{p1}-2a)/2$	$c\Gamma_{01}-2cd-a\Gamma_{01}-2ab+\|G_{p1}\|^2/2c$
	$-\Delta_{p1}/2$	$(c\Gamma_{01}-(2c+\Delta_{p1}+2a)b+\|G_{p1}\|^2)/2c$	$-\Delta_{p1}/2$	$c\Gamma_{01}-2ab-b\Delta_{p1}-2ab+\|G_{p1}\|^2/2c$

Exponential Point table

Dressing	Real part		Imaginary part	
	Resonant position	linewidth	Resonant position	linewidth
Nested (Δ_{FL})	0	$-(2\Gamma_{20}+\Gamma_{00})/4$	$(\Delta_{p1}+4a)/4$	$(c\Gamma_{10}-4ab+\Gamma_{10}a+\|G_{p1}\|^2)/4c$
	$\Delta_{p1}/4$	$(c\Gamma_{10}--4ab+4bc+b\Delta_{p1}+\|G_{p1}\|^2)/4c$	$(4b-\Delta_{p1})/4$	$(d\Gamma_{10}-4ab+\Gamma_{10}b+\|G_{p1}\|^2)/4d$

Table S2. Shows the SFWM real and imaginary parts of Eigenvalues and linewidth for different dressing

Dressing	Eigen Value		Linewidth	
	Real part	Imaginary part	Real part	Imaginary part
Single	$a_1=((\Delta_{p1}-2\Delta_1^{`})\pm\sqrt{u_1})/2$	$b_1=((\Gamma_{20}+\Gamma_{10})\pm\sqrt{w_1})/2$	$\Gamma_{e1}=(\Gamma_{20}+\Gamma_{10})/2+v_1$	$\Delta_{e1}=(\Delta_{p1}-2\Delta_1)/2+x_1$

$\Gamma_S^* = \delta_1 - \delta_2$	$\Gamma_r^* = \sqrt{u_1}$	$\Gamma_i^* = \sqrt{w_1}$		
Parallel $\Gamma_p^* = \delta_1 - \delta_3$	$a_{21} = \pm\sqrt{(\Gamma_{20}\Gamma_{00} + \lvert G_1 \rvert^2)/2}$	$b_{21} = \left((\Gamma_{20} + 2\Gamma_{00}) \pm \sqrt{w_{21}}\right)/4$	$\Gamma_{e21} = (\Gamma_{20} + 2\Gamma_{00})/4$	$\Delta_{e21} = 0$
	$\Gamma_r^* = \sqrt{\Gamma_{20}\Gamma_{00} + \lvert G_1 \rvert^2}$	$\Gamma_i^* = \sqrt{w_{21}}$		
	$a_{22} = \left((\Delta_{p1} - 2\Delta_1^{`}) \pm \sqrt{u_{22}}\right)/2$	$b_{22} = ((\Gamma_{20} + \Gamma_{10}) \pm \sqrt{w_{22}})/2$	$\Gamma_{e22} = (\Gamma_{20} + \Gamma_{10})/2 + v_{22}$	$\Delta_{e22} = (\Delta_{p1} - 2\Delta_1^{`})/2 + x_{22}$
	$\Gamma_r^* = \sqrt{u_{22}}$	$\Gamma_i^* = \sqrt{w_{22}}$		
Cascade $\Gamma_C^* = \delta_1 - \delta_3$	$a_{31} = \left((\Delta_1' + 2\Delta_1) \pm \sqrt{u_{31}}\right)/2$	$b_{31} = -((\Gamma_{00} + \Gamma_{20}) \pm \sqrt{w_{31}})/2$	$\Gamma_{e31} = -(\Gamma_{00} + \Gamma_{20})/2 + v_{31}$	$\Delta_{e31} = \left(\Delta_1' + 2\Delta_1\right)/2 + x_{31}$
	$\Gamma_r^* = \sqrt{u_{31}}$	$\Gamma_i^* = \sqrt{w_{31}}$		
	$a_{32} = \left((2a + \Delta_1 + \Delta_{p1}) \pm \sqrt{u_{32}}\right)/2$	$b_{32} = \left(-(\Gamma_{10} - 2b + \Delta_1 + \Delta_{p1}) \pm \sqrt{w_{32}}\right)/2$	$\Gamma_{e32} = \left((a-c)\Gamma_{10} - 2ab + 2bc - v_{32}\right)/2c$	$\Delta_{e32} = \left(a\Gamma_{10} - 2ab + 2ad + x_{32}\right)/2d$
	$\Gamma_r^* = \sqrt{u_{32}}$	$\Gamma_i^* = \sqrt{w_{32}}$		
	$-\Delta_1'$	Γ_{20}	0	0
Nested $\Gamma_N^* = \delta_1 - \delta_3$	$a_{41} = (2\Delta_1 \pm \sqrt{u_{41}})/2$	$b_{41} = ((-\Gamma_{00} - \Gamma_{20}) \pm \sqrt{w_{41}})/2$	$\Gamma_{e41} = (\Gamma_{00} + \Gamma_{20})/2 - v$	$\Delta_{e41} = \Delta_1(\Gamma_{00} + \Gamma_{20})/2b + \Delta_1$
	$\Gamma_r^* = \sqrt{u_{41}}$	$\Gamma_i^* = \sqrt{w_{41}}$		
	$a_{42} = (2a + \Delta_2 - \Delta_{p1} \pm \sqrt{u_{42}})/2$	$b_{42} = (-\Gamma_{10} + 2b - \Delta_2 + \Delta_{p1} \pm \sqrt{w_{42}})/2$	$\Gamma_{e42} = ((a-c)\Gamma_{10} + v_{42})/2c$	$\Delta_{e42} = (a\Gamma_{10} - 2ab + x_{42})/2d$
	$\Gamma_r^* = \sqrt{u_{42}}$	$\Gamma_i^* = \sqrt{w_{42}}$		
	Δ_1	Γ_{20}	0	0

Where r and i represents the real and imaginary parts.

Exponential Point table

Dressing	Real part		Imaginary part	
	Resonant position	linewidth	Resonant position	width
Single (δ_{AS})	$\Delta_{p1} - 2\Delta_1^{`}$	$(\Gamma_{20} + \Gamma_{10})/2 + v_1$	$(\Gamma_{20} + \Gamma_{10})/2$	$(\Delta_{p1} - 2\Delta_1)/2 + x_1$

Exponential Point table

Dressing	Real part		Imaginary part	
	Resonant position	linewidth	Resonant position	linewidth
Parallel(δ_{AS})	0	$(\Gamma_{20} + 2\Gamma_{00})/4$	$(\Gamma_{20} + 2\Gamma_{00})/4$	0

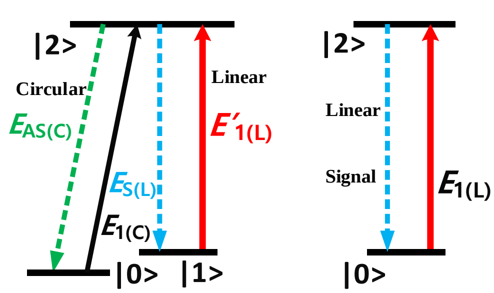
	$(\Delta_{p1} - 2\Delta_1^{`})/2$	$(\Gamma_{20} + \Gamma_{10})/2 + v_{22}$	$(\Gamma_{20} + \Gamma_{10})/2$	$(\Delta_{p1} - 2\Delta_1^{`})/2 + x_{22}$

Exponential Point table

Dressing	Real part		Imaginary part	
	Resonant position	linewidth	Resonant position	linewidth
Cascade(δ_{AS})	$(\Delta_1' + 2\Delta_1)/2$	$-(\Gamma_{00}+\Gamma_{20})/2 + v_{31}$	$-(\Gamma_{00} + \Gamma_{20})/2$	$(\Delta_1' + 2\Delta_1)/2 + x_{31}$
	$(2a + \Delta_1 + \Delta_{p1})/2$	$((a-c)\Gamma_{10} - 2ab + 2bc - v_{32})/2c$	$(\Gamma_{10} + 2b - \Delta_1 - \Delta_{p1})/2$	$(a\Gamma_{10} - 2ab + 2ad + x_{32})/2d$
	$-\Delta_1^{'}$	0	Γ_{20}	0

Exponential Point table

Dressing	Real part		Imaginary part	
	Resonant position	linewidth	Resonant position	linewidth
Nested (δ_{AS})	Δ_1	$(\Gamma_{00} + \Gamma_{20})/2 - v$	$(-\Gamma_{00} - \Gamma_{20})/2$	$\Delta_1(\Gamma_{00} + \Gamma_{20})/2b + \Delta_1$
	$(2a + \Delta_2 - \Delta_{p1})/2$	$((a-c)\Gamma_{10} + v_{42})/2c$	$(-\Gamma_{10} + 2b - \Delta_2 + \Delta_{p1})/2$	$(a\Gamma_{10} - 2ab + x_{42})/2d$
	Δ_1	0	Γ_{20}	0



The perturbation chains $\rho_{11}^{(0)} \xrightarrow{E_1'(L)} \rho_{21}^{(1)} \xrightarrow{E_{AS}(C)} \rho_{20}^{(2)} \xrightarrow{E_1(C)} \rho_{22(S)}^{(3)}$ and $\rho_{00}^{(0)} \xrightarrow{E_1(C)} \rho_{20}^{(1)} \xrightarrow{E_S(L)} \rho_{21}^{(2)} \xrightarrow{E_1'(L)} \rho_{22(AS)}^{(3)}$

$$\rho_{AS(C)}^{(3)} = \frac{-iG_S c_1 \cos(\theta_1) G_1 G_1'}{(\Gamma_{20} + i\Delta_1 + |G_{p1}|^2/(\Gamma_{10} + i\Delta_1 - i\Delta_{p1}))(\Gamma_{21} + i\Delta_1 + c_1^2 \cos^2(\theta_1)|G_1|^2/(\Gamma_{20} + 2i\Delta_1))(\Gamma_{20} + i\Delta_1 + i\Delta_1'))}, \text{(S1)}$$

$$\rho_{S(C)}^{(3)} = \frac{-iG_{AS} c_{AS} c_1 \cos^2(\theta_1) G_1 G_1'}{(\Gamma_{20} + i\Delta_1' + |G_{p1}|^2/(\Gamma_{10} + i\Delta_1' - i\Delta_{p1}))(\Gamma_{21} + i\Delta_1' + c_1^2 \cos(\theta_1)|G_1|^2/(\Gamma_{20} + i\Delta_1' + i\Delta_1))(\Gamma_{20} + i\Delta_1 + i\Delta_1'))}. \text{(S2)}$$

where c is the anisotropic factor in different directions. θ is the rotated angle.

One linear polarization is equal to one left circular polarization + one right circular polarization. So by the perturbation chains $\rho_{00}^{(0)} \xrightarrow{E_1} \rho_{20}^{(1)} \xrightarrow{E_1'} \rho_{22}^{(2)}$, we get

$$\begin{aligned} \rho_{FL(C)}^{(2)} &= \rho_{FL(C(left))}^{(2)} + \rho_{FL(C(right))}^{(2)} \\ &= \frac{-|G_1|^2 c_x^2 \cos^2(\varphi)}{(\Gamma_{20} + i\Delta_1 + |G_{p1}|^2/(\Gamma_{10} + i\Delta_1 - i\Delta_{p1}))(\Gamma_{22} + c_x^2 \cos^2(\varphi)|G_1|^2/(\Gamma_{20} + i\Delta_1))} \\ &+ \frac{-|G_1|^2 c_y^2 \cos^2(\varphi)}{(\Gamma_{20} + i\Delta_1 + |G_{p1}|^2/(\Gamma_{10} + i\Delta_1 - i\Delta_{p1}))(\Gamma_{22} + c_y^2 \cos^2(\varphi)|G_1|^2/(\Gamma_{20} + i\Delta_1))} \end{aligned} \quad \text{(S3)}$$

Table S3. Eigenvalues of parallel two dressing for SFWM

Energy level	Real part	imaginary part
$\|-1>$	$a_1 = +\sqrt{(\Gamma_{20}\Gamma_{00} + \|G_1\|^2)/2}$ $a_2 = -\sqrt{(\Gamma_{20}\Gamma_{00} + \|G_1\|^2)/2}$	$b_1 = ((\Gamma_{20} + 2\Gamma_{00}) + \sqrt{w_{21}})/4$ $b_2 = ((\Gamma_{20} + 2\Gamma_{00}) - \sqrt{w_{21}})/4$
$\|1>$	$a_3 = ((2a_1 + \Delta_1 + \Delta_{p1}) + \sqrt{u_{32}})/2$ $a_4 = ((2a_1 + \Delta_1 + \Delta_{p1}) - \sqrt{u_{32}})/2$	$b_3 = (-(\Gamma_{10} - 2b_1 + \Delta_1 + \Delta_{p1}) + \sqrt{w_{32}})/2$ $b_4 = (-(\Gamma_{10} - 2b_1 + \Delta_1 + \Delta_{p1}) - \sqrt{w_{32}})/2$