Genuine Tripartite Non-Gaussian Entanglement

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Triple-photon states generated by three-mode spontaneous parametric down-conversion are the paradigm of unconditional non-Gaussian states, essential assets for quantum advantage. How to fully characterize their non-Gaussian entanglement remains however elusive. We propose here a set of sufficient and necessary conditions for separability of the broad family of spontaneously generated three-mode non-Gaussian states. We further derive state-of-the-art conditions for genuine tripartite non-Gaussian entanglement, the strongest class of entanglement. We apply our criteria to triple-photon states revealing that they are fully inseparable and genuinely entangled in moments of order $3n$. Our results establish a systematic framework for characterizing the entanglement of triple-photon states and thus fostering their application in quantum information protocols.

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Entanglement is a physical property describing the inseparability of quantum systems composed by multiple elements. This core concept of contemporary physics dates back to 1935, when Einstein, Podolsky, and Rosen proposed a *gedanken* experiment criticizing the nonlocality of quantum mechanics and pointing out at a possible incompleteness of the theory [1]. Nowadays entanglement is prepared regularly in a number of physical systems [2]. For instance, entangled quadratures of electromagnetic fields are generated by single- and two-mode squeezing in parametric amplifiers and oscillators [3,4]. These squeezed states are the cornerstone of multipartite entangled quantum networks and have greatly promoted the development of quantum optics [5–8] and quantum information science [9–11]. They exhibit Gaussian statistics and their entanglement properties are completely specified by their covariance matrix [12]. However, it has been shown that non-Gaussian entangled states—inseparable states with non-Gaussian statistics–are an essential resources for universal quantum computing [13–17], demonstrating superior performance in many continuous variable protocols, such as quantum key distribution [18], quantum teleportation [19–21], and quantum metrology [22]. Remarkably, the lack of passive separability of non-Gaussian entanglement has been recently pointed out as resource for quantum computational advantage [23]. Non-Gaussian entangled states have been probabilistically created by photon addition and subtraction on Gaussian states along the last decades [24–26], but a source of deterministic non-Gaussian entangled states was still missing.

Triple-photon states (TPS) are quantum states obtained through third-order spontaneous parametric down-conversion where a pump photon is converted into three photons with different energies. They constitute true three-mode unconditional non-Gaussian entangled states. They have been demonstrated in the microwave regime [27], and there is a great effort developing new platforms that produce them at optical wavelengths [28–31]. These new TPS are expected to extend the development of quantum optics and break the probabilistic nature induced by non-Gaussian operations in existing quantum information technologies.

Non-Gaussian entangled states are not only challenging to obtain in laboratories but are also hard to characterize. Along the last two decades, efficient tools to detect multipartite Gaussian entanglement have been developed and experimentally tested countless times [32,33]. However, these criteria fail to detect fully non-Gaussian entanglement [34–36]. Several sufficient criteria have been proposed to detect the entanglement of known non-Gaussian states [37–39]. However, non-Gaussian states encompass a huge state space and necessary conditions of entanglement only arise when specific state features are taken into account [40]. For TPS, sufficient conditions have been recently derived [35,41]. These conditions are nevertheless not necessary—they do not fully or even unambiguously reveal the non-Gaussian entanglement of states–, only work in a limited parameter range, and are experimentally very demanding. Thus, a systematic framework that fully characterizes TPS non-Gaussian entanglement and hence its operational usefulness [23], sensitive at any parameter range and experimentally accessible is necessary.

In this Letter, we propose a set of full separability criteria for tripartite continuous variable states. These criteria are a series of inequalities fulfilled by any three-mode biseparable state, based on linear combinations of experimentally accessible high-order operators. Violation of these inequalities is a sufficient condition for three-mode full inseparability. By analogy with the covariance matrix, we construct the high-order covariance matrices based on the high-order operators. Using these high-order covariance matrices we find that the inequalities provide sufficient and necessary conditions for the full separability of spontaneously-generated three-mode non-Gaussian states, such as TPS. This establishes a systematic framework for the characterization of TPS non-Gaussian entanglement. Furthermore, based on our full separability conditions we derive a series of stringent conditions which rule out mixtures of biseparable states resulting in a criterion of genuine tripartite non-Gaussian entanglement. Finally, we demonstrate the entanglement structure of TPS in a experimentally relevant parameter space by means of numerical simulations.

We start our analysis by considering the interaction Hamiltonian describing the nondegenerate three-mode spontaneous parametric down-conversion

$$\hat{H}_I = i\hbar\kappa(\hat{a}_1^\dagger\hat{a}_2^\dagger\hat{a}_3^\dagger\hat{a}_4 - \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4^\dagger), \tag{1}$$

where κ is the 3rd-order coupling constant. The annihilation operators $\hat{a}_1$, $\hat{a}_2$, $\hat{a}_3$, and $\hat{a}_4$ describe, respectively, the three down-converted modes and the pump mode. Under the evolution of vacuum or thermal states driven by the Hamiltonian (1), it has been demonstrated theoretically [35,36] and experimentally [27] that third order -coskewness- is the lowest-order correlation of TPS. Because of the shape of Hamiltonian (1), TPS presents also 6th- and 9th-order quantum correlations, and even higher [42]. Here, we refer to $\langle\hat{\varsigma}_1^n\hat{\varsigma}_2^n\hat{\varsigma}_3^n\rangle$ as a $3n$th-order correlation, where $\hat{\varsigma}_i \equiv \{\hat{q}_i^1, \hat{p}_i^1\}$ is a canonical quadrature operator corresponding to mode $i = 1, 2, 3$ and $n \in \mathbb{Z}^+$. This means that the well-developed entanglement criteria involving second-order correlations are no longer applicable to TPS [34]. From now on we will refer to non-Gaussian entanglement as the entanglement that is not detected by Gaussian entanglement witnesses based on second order moments—the covariance matrix—such as van Loock-Furusawa or Reid criteria [33,43].

To fully probe the high-order-moment nature of TPS entanglement, we introduce the high-order quadrature operators $\hat{q}_k^n = (\hat{a}_k^{\dagger n} + \hat{a}_k^n)/2$ and $\hat{p}_k^n = i(\hat{a}_k^{\dagger n} - \hat{a}_k^n)/2$ for the mode k, satisfying the commutation relation $[\hat{q}_k^n, \hat{p}_k^n] = i\hat{f}_k^n$, where the full expressions of $\hat{f}_k^n$ are given in the Supplemental Material [42]. Likewise, the two-mode operators $\hat{q}_{lm}^n = (\hat{a}_l^{\dagger n}\hat{a}_m^{\dagger n} + \hat{a}_l^n\hat{a}_m^n)/2$ and $\hat{p}_{lm}^n = i(\hat{a}_l^{\dagger n}\hat{a}_m^{\dagger n} - \hat{a}_l^n\hat{a}_m^n)/2$ are defined for the modes l and m, which follow the commutation relation $[\hat{q}_{lm}^n, \hat{p}_{lm}^n] = i\hat{f}_{lm}^n$ [42]. We also define the following linear combinations

$$\hat{u}_{k,lm}^n = g_{k,n}\hat{q}_k^n - \frac{\hat{q}_{lm}^n}{g_{k,n}}, \qquad \hat{v}_{k,lm}^n = g_{k,n}\hat{p}_k^n + \frac{\hat{p}_{lm}^n}{g_{k,n}}, \tag{2}$$

for a given permutation $\{k, l, m\}$ of $\{1, 2, 3\}$, where $g_{k,n}$ is an arbitrary real number.

The standard approach to witness tripartite entanglement is to examine the separability of the three possible bipartitions of the system. Thus, let us consider the tripartite density operator $\rho = \sum_i \eta_i \rho_k^i \otimes \rho_{lm}^i$ with $\Sigma_i \eta_i = 1$. For brevity, we denote it as $\rho_{k,lm}$. We derive in the Supplemental Material that for the biseparable state $\rho_{k,lm}$, the total variance of a pair of operators $\hat{u}_{k,lm}^n$ and $\hat{v}_{k,lm}^n$ satisfies the inequality [42]

$$\langle\Delta(\hat{u}_{k,lm}^n)^2\rangle + \langle\Delta(\hat{v}_{k,lm}^n)^2\rangle \geq \mathfrak{B}_{k,lm}^n, \tag{3}$$

where $\mathfrak{B}_{k,lm}^n = g_{k,n}^2 f_k^n + f_{lm}^n/g_{k,n}^2$ and $f_{k/lm}^n \equiv \langle\hat{f}_{k/lm}^n\rangle$. Thus we have the following theorem concerning full tripartite inseparability:

Theorem 1.—Violation of all three inequalities

$$F_1^n \equiv \langle\Delta(\hat{u}_{1,23}^n)^2\rangle + \langle\Delta(\hat{v}_{1,23}^n)^2\rangle - \mathfrak{B}_{1,23}^n \geq 0, \tag{4a}$$

$$F_2^n \equiv \langle\Delta(\hat{u}_{2,13}^n)^2\rangle + \langle\Delta(\hat{v}_{2,13}^n)^2\rangle - \mathfrak{B}_{2,13}^n \geq 0, \tag{4b}$$

$$F_3^n \equiv \langle\Delta(\hat{u}_{3,12}^n)^2\rangle + \langle\Delta(\hat{v}_{3,12}^n)^2\rangle - \mathfrak{B}_{3,12}^n \geq 0, \tag{4c}$$

for any n is sufficient to confirm fully inseparable tripartite entanglement.

Inequality (4a) is a necessary condition for the separability of the bipartition 1–23. Once inequality (4a) is violated with any n, we can conclude that the state cannot be described by $\rho_{1,23}$. Similarly, inequalities (4b) and (4c) are implied by biseparable states $\rho_{2,13}$ and $\rho_{3,12}$, respectively. Therefore, violating the three inequalities for any n negates all possible bipartitions, thus proving the full inseparability of the state.

The three sets of bounds given in Eq. (4) must be fulfilled for the three biseparable states. This naturally raises two questions: (i) Do the states that violate these bounds at any order n necessarily possess non-Gaussian entanglement? (ii) Are these bounds strong enough to ensure that the state satisfying the three sets of inequalities is fully separable?

The answer to the first question is negative. Although the violation of inequalities (4) depends on correlations like $\langle \hat{q}_k^n \hat{q}_{lm}^n \rangle > 0$ and $\langle \hat{p}_k^n \hat{p}_{lm}^n \rangle < 0$, these high-order moments do not identify the statistical properties of states. In fact, we can easily find a three-mode Gaussian state that simultaneously violates the three inequalities in Eq. (4), such as $\rho = 1/2(|\alpha\rangle\langle\alpha| \otimes |\psi(r)\rangle\langle\psi(r)| + |\psi(r)\rangle\langle\psi(r)| \otimes |\alpha\rangle\langle\alpha|)$, where $|\alpha\rangle$ is a coherent state with classical complex amplitude α and $|\psi(r)\rangle$ is a two-mode squeezed vacuum state with squeezing parameter r [42]. This suggests that the above criterion involving high-order moments have downward compatibility [44,45], i.e., they can diagnose some Gaussian entanglement. Note that, however, zero-mean-field Gaussian states would never violate inequalities (4a)–(4c).

For the second question we have however a positive answer. We find that the inequalities (4a)–(4c) are indeed sufficient and necessary conditions for the full separability of spontaneously generated states from the Hamiltonian (1), such as TPS. The proof steps are as follows.

We first collect the high-order quadrature and built-up operators in the vector $\hat{R}^n = (\hat{q}_k^n, \hat{p}_k^n, \hat{q}_{lm}^n, \hat{p}_{lm}^n)^T$ and write the commutation relations as

$$[\hat{R}_i^n, \hat{R}_j^n] = i\Omega_{ij}^n, \qquad i, j = 1, \ldots, 4, \tag{5}$$

where $\Omega^n = i\hat{f}_k^n s_y \oplus i\hat{f}_{lm}^n s_y$ and s_y represents the y Pauli matrix. Analogous to the Gaussian states case, the high-order covariance matrices V^n is defined as $V_{ij}^n = \langle \Delta\hat{R}_i^n \Delta\hat{R}_j^n + \Delta\hat{R}_j^n \Delta\hat{R}_i^n \rangle/2$, where $\Delta\hat{R}^n = \hat{R}^n - \langle \hat{R}^n \rangle$ and the high-order local moment $\langle \hat{R}^n \rangle = \mathrm{tr}[\hat{R}^n \rho]$, with ρ being the density operator of the system. Then we have $V_{ij}^n + i\langle \Omega_{ij}^n \rangle/2 = \langle \hat{R}_i^n \hat{R}_j^n \rangle$, where the commutation relation (5) and the property of spontaneously generated states $\langle \hat{R}_i^n \rangle = 0$ are used [42]. Hence we have the following statement of the uncertainty principle

$$V^n + \frac{i}{2}\langle \Omega^n \rangle \geq 0. \tag{6}$$

Every physical state that satisfies $\langle \hat{R}_i^n \rangle = 0$, i.e., with zero local moments, must conform to this inequality.

V^n is by definition a symmetric matrix, which can be divided into 2×2 subblocks

$$V^n = \begin{pmatrix} A_k & C_{k-lm} \\ C_{k-lm}^T & B_{lm} \end{pmatrix}, \tag{7}$$

where A_k and B_{lm} are local high-order covariance matrices related, respectively, to the subsystems k and lm, and C_{k-lm} represents their correlation. Using Williamson's theorem and a suitable singular value decomposition [46], we can always transform Eq. (7) into the following standard form [42]

$$V_1^n = \begin{pmatrix} n_1 & 0 & s_1 & 0 \\ 0 & n_2 & 0 & s_2 \\ s_1 & 0 & m_1 & 0 \\ 0 & s_2 & 0 & m_2 \end{pmatrix}, \tag{8}$$

where the matrix elements satisfy the relations

$$\mathfrak{n}_2 \mathfrak{m}_1 = \mathfrak{n}_1 \mathfrak{m}_2, \quad 2(|s_1| - |s_2|) = \sqrt{\mathfrak{n}_1 \mathfrak{m}_1} - \sqrt{\mathfrak{n}_2 \mathfrak{m}_2}, \tag{9}$$

with $\mathfrak{n}_i = 2n_i - f_k^n$ and $\mathfrak{m}_i = 2m_i - f_{lm}^n$ $(i = 1, 2)$. $|s_i| = 0$ for Gaussian states as they do not present C_{k-lm} correlations under the condition of $\langle \hat{R}_i^n \rangle = 0$.

Note that these high-order covariance matrices from V_1^1 to V_1^n only describe the correlation information between subsystem k and lm. To fully characterize three-mode non-Gaussian state with zero local moments, such as TPS, we need three sets of higher-order covariance matrices. In addition, local symplectic transformations do not affect the separability of V^n, which implies that states with the same three sets of standard forms (8) have the same entanglement structure. With these preliminaries, we now present the main theorem about the separability of V_1^n.

Theorem 2.—The necessary and sufficient condition for the separability of high-order covariance matrix V_1^n is that the operators

$$\hat{u}_{k,lm}^n = g_{k,n}\hat{q}_k^n - \frac{s_1 \hat{q}_{lm}^n}{g_{k,n}|s_1|}, \quad \hat{v}_{k,lm}^n = g_{k,n}\hat{p}_k^n + \frac{s_2 \hat{p}_{lm}^n}{g_{k,n}|s_2|}, \tag{10}$$

satisfy the inequality (3), where $g_{k,n}^2 = \sqrt{\mathfrak{m}_1/\mathfrak{n}_1}$.

Proof.—Inequality (3) is already a necessary condition for the separability of $k - lm$, so we only need to prove its sufficiency. Substituting $\hat{u}_{k,lm}^n$ and $\hat{v}_{k,lm}^n$ of Eq. (10) into inequality (3) and using the standard form V_1^n, we obtain the inequality $g_{k,n}^2(\mathfrak{n}_1 + \mathfrak{n}_2) + g_{k,n}^{-2}(\mathfrak{m}_1 + \mathfrak{m}_2) - 4|s_1| - 4|s_2| \geq 0$. Combined with Eq. (9), one finds

$$2|s_i| \leq \sqrt{\mathfrak{n}_i \mathfrak{m}_i}. \tag{11}$$

Since the uncertainty principle is invariant under local standard transformations, the standard form V_1^n always satisfies Eq. (6), which can be further reduced to $\det(V_1^n + i\langle \Omega^n \rangle/2) \equiv (f_k^n)^2 (f_{lm}^n)^2 \det(V_G^n + i\langle \Omega \rangle/2) \geq 0$, where $\Omega = \mathrm{diag}(J_k^1, J_k^1)$ and V_G^n is given in [42]. This is equivalent to inserting a normalization coefficient into the commutation relations and it does not affect the separability of V_1^n. The standard form V_G^n and the uncertainty principle $V_G^n + i\langle \Omega \rangle/2 \geq 0$ suggest that V_G^n can be regarded as a two-mode Gaussian state in the canonical quadratures. Inequality (11) ensures that every matrix $V_G^n - 1/2$ is semipositive definite, implying that all Gaussian states represented by V_G^n are separable [47,48], which demonstrates that the original states V_1^n are separable. This completes the proof of Theorem 2.

Thus, we have the following result: A quantum state with zero local moments represented by $3n$ high-order covariance matrices V_1^n is fully inseparable if and only if three pairs of operators violate inequalities (4a)–(4c), respectively, with any n. Remarkably, under the condition $\langle \hat{R}_i^n \rangle = 0$, inequality (11) reveals the non-Gaussian character of our entanglement criterion, as for Gaussian states $|s_i| = 0$ and Equation (11) is always fulfilled. Hence, only states with high-order correlations -non-Gaussian states-can violate this inequality and thus inequality (3).

Theorem 2 is a natural extension of the Duan criterion [48] to the tripartite high-order correlation system. Notably, this framework is inherently scalable and can be easily extended to N-partite systems.

Full inseparability is, however, not the more general form of multipartite entanglement [49,50]. It can only exclude any biseparable case rather than the general one in which the state can be described as a mixture

$$\rho = P_1 \sum_i \eta_i^{(1)} \rho_{1,23}^i + P_2 \sum_t \eta_t^{(2)} \rho_{2,13}^t + P_3 \sum_j \eta_j^{(3)} \rho_{3,12}^j,$$

where $\Sigma_i P_i = 1$. If a tripartite state can not be described by this equation, it is said to be genuinely entangled. Genuine entanglement and full inseparability are equivalent for pure states, but for mixed states, the former is more strict than the latter [43,50]. Substituting the above equation into inequality (4), we derive a series of genuine tripartite entanglement criteria in the Supplemental Material [42], which can be stated as follows:

Theorem 3.—A tripartite state is genuinely entangled if the inequality

$$\begin{aligned} W_n \equiv F_1^n + F_2^n + F_3^n + 4\langle \hat{q}_1^n \hat{q}_{23}^n \rangle_\rho - 4\langle \hat{p}_1^n \hat{p}_{23}^n \rangle_\rho \\ + 2(\langle \hat{a}_1^{\dagger n} \hat{a}_1^n \rangle_\rho + \langle \hat{a}_2^{\dagger n} \hat{a}_2^n \rangle_\rho \langle \hat{a}_3^{\dagger n} \hat{a}_3^n \rangle_\rho) \geq 0 \end{aligned} \tag{12}$$

is violated for any n.

Let us now discuss the main features of the proposed criteria. For any tripartite continuous variable state, violations of criteria (4) and (12) are sufficient to confirm fully inseparable and genuine tripartite entanglement, respectively. On the other hand, with $\langle \hat{R}_i^n \rangle = 0$, the matrix elements in block C_{k-lm} can be decomposed into the superposition of 3rd-order standard moments—coskewness—when $n = 1$, i.e., $\langle \hat{p}_k^1 \hat{p}_{lm}^1 \rangle = \langle \hat{p}_k^1 \hat{p}_l^1 \hat{q}_m^1 \rangle + \langle \hat{p}_k^1 \hat{q}_l^1 \hat{p}_m^1 \rangle$ and $\langle \hat{q}_k^j \hat{q}_{lm}^j \rangle = \langle \hat{q}_k^1 \hat{q}_l^1 \hat{q}_m^1 \rangle - \langle \hat{q}_k^1 \hat{p}_l^1 \hat{p}_m^1 \rangle$, recently measured experimentally for TPS [27]. Inequality (11) indicates that even if there are non-Gaussian correlations among the three modes, they may be separable, which implies that nonzero high-order standard moments are a necessary but not sufficient condition for diagnosing non-Gaussian entanglement. As commented above, this necessary condition is not fulfilled by Gaussian states. However, based on three sets of standard form V_1^n, we found that violations of criteria (4) and (12) imply fully inseparable and *genuine tripartite non-Gaussian entanglement*, respectively. Besides, the elements in block C_{k-lm} have state-independent properties [42], which simplify significantly the complexity of experimental measurements compared to other cases where determining non-Gaussian entanglement requires two different measurement protocols, non-Gaussianity and inseparability [24–26].

Other criteria based on three-mode correlations function composed of high-order creation or annihilation operators can also effectively determine the inseparability of TPS [51]. However, the eigenspectra of the observables in these criteria are discrete, so what they reveal is the high-order moments entanglement of Fock states. The picture is especially evident when considering the two-mode squeezed vacuum, where the entanglement conditions $|\langle \hat{a}^n \hat{b}^n \rangle| > [\langle \hat{a}^{\dagger n} \hat{a}^n \rangle \langle \hat{b}^{\dagger n} \hat{b}^n \rangle]^{1/2}$ are always satisfied for any n [42]. This indicates that the high-order moments entanglement of Fock states revealed by these conditions is only 2nd-order moment entanglement—Gaussian entanglement—from the perspective of continuous variables. Therefore, the high-order moments in these criteria do not distinguish between Gaussian and non-Gaussian entanglement, which is clearly different from those in our criteria.

Moreover, there are other types of genuine tripartite entanglement criteria derived from the uncertainty principle [43,45,50] and the Cauchy-Schwartz inequality [35]. No assumptions were made there about the statistical properties of the states in deriving these criteria, so, in principle, they apply to any tripartite continuous-variable state, Gaussian or non-Gaussian. In particular, these criteria are sufficient conditions for verifying entanglement rather than sufficient and necessary conditions. When we claim that some kind of entanglement does not exist, we need to use necessary conditions of entanglement to support this conclusion, instead of sufficient conditions. In other words, comparing two sufficient conditions for entanglement with two different states does not negate the existence of either Gaussian or non-Gaussian entanglement. Therefore, what was proposed in [35] is a state-independent generalized entanglement condition, which cannot unambiguously reveal the novel notion of genuine tripartite non-Gaussian entanglement. In contrast, our criteria can capture tripartite non-Gaussian entanglement in moments of order $3n$.

To conclude this Letter, we present the numerical verification of the proposed criteria. Using the Hamiltonian (1), the master equation $\dot{\rho}(t) = -i[\hat{H}_I, \rho(t)]/\hbar$ is solved numerically to deduce the final state of system at time t considering that the initial state is vacuum for the triplets and a coherent mode α_p for the pump [36]. Figure 1(a) shows the evolution of F_1^n versus the interaction strength $\xi = \kappa \alpha_p t$, where $g_{1,n}^2 = \sqrt{\mathfrak{m}_1/\mathfrak{n}_1}$. $F_1^n = F_2^n = F_3^n$ as expected from the

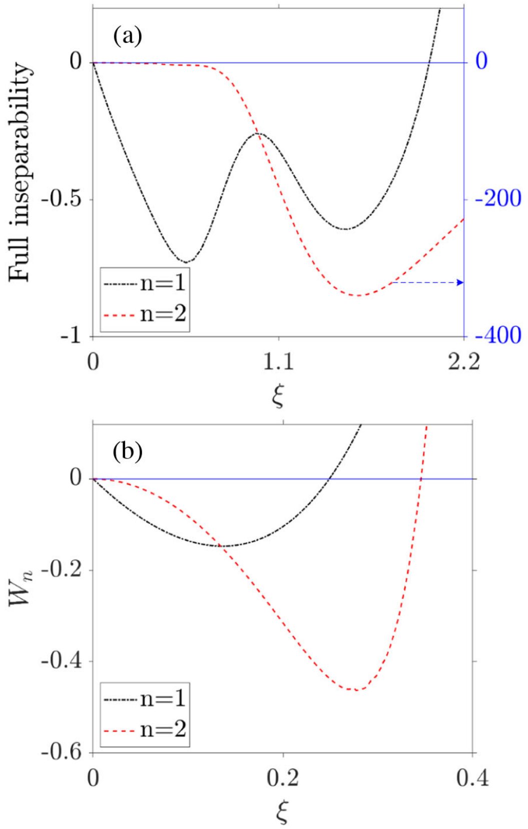

FIG. 1. Evolution of F_1^n (a) and W_n (b) as a function of the interaction strength ξ. $F_1^n < 0$ and $W_n < 0$ indicate full inseparability and genuine entanglement, respectively. $\xi = \kappa\alpha_p t$ and $\alpha_p = \sqrt{25}$. Note that $F_1^n = F_2^n = F_3^n$.

symmetry of the TPS. $F_1^{n=1,2} < 0$ in the parameter region demonstrating full inseparability related to the 3rd- and 6th-order covariance matrices of the TPS. Interestingly, when $\xi > 2$, the entanglement carried by the 3rd-order covariance matrices ($n = 1$) disappears, while the entanglement carried by the 6th-order covariance ($n = 2$) matrices remains. There are also higher order entanglements that we have not considered ($n \geq 3$). Figure 1(b) shows the evolution of W_n versus the interaction strength. $W_n < 0$ in the parameter region demonstrating genuine entanglement. Starting from the vacuum, the genuine tripartite non-Gaussian entanglement is firstly loaded on the 3rd-order covariance matrices and then gradually transitioned to the 6th-order covariance matrices with the increase of ξ. Similarly, genuine tripartite non-Gaussian entanglement still exists in higher order covariance matrices ($n \geq 3$).

In summary, we proposed a set of sufficient and necessary conditions for the separability of higher-order moments for spontaneously-generated tripartite non-Gaussian states, which provides a systematic framework for characterizing non-Gaussian entanglement. Besides, a series of genuine tripartite non-Gaussian entanglement criteria was proposed. Compared with others, our proposal has the following advantages: First, our criteria are more general since they can directly answer whether quantum states possess tripartite non-Gaussian entanglement, including full inseparability and genuine tripartite non-Gaussian entanglement. Second, our strategy is platform-agnostic, as it works for any physical system as long as the information is encoded in continuous variables. Third, our framework is naturally scalable and it constitutes a stepping stone to more sophisticated states and more than three parties. Fourth, the physical quantities involved in our criteria are all experimentally accessible without quantum tomography. In future work, we will study some interesting problems such as quantifying the entanglement of triple-photon states and exploring their potential advantages in quantum information tasks such as quantum teleportation. Moreover, we will study the multipartite non-Gaussian entanglement generated by triple-photon states and beam-splitter operations.

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