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A Solution to the Two-Dimensional Inverse Heat Conduction Problem

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A simple method is developed in this paper to solve two-dimensional nonlinear steady inverse heat conduction problems. The unknown boundary conditions can be numerically obtained by using the iteration and modification method. The effect of measurement errors of the wall temperature on the algorithm is numerically tested. The results prove that this method has the advantages of fast convergence, high precision, and good stability. The method is successfully applied to estimate the convective heat transfer coefficient in the case of a fluid flowing in an electrically heated helically coiled tube. © 2000 Scripta Technica, Heat Trans Asian Res, 29(2): 113–119, 2000

Key words: Inverse heat conduction problem, nonlinear, two-dimensional

1. Introduction

The direct heat conduction problem is to obtain the interior temperature distribution of a body on the basis of the given boundary conditions. However, the boundary conditions on the entire surface of the body are not available in some cases so that the measured temperature on some local positions is used. The unknown boundary conditions have to be determined and, in this case, the unknown boundary condition will be calculated with the measured temperature. This is one type of the inverse heat conduction problem. The problem is ill posed and the solution is sensitive to the measurement errors of temperature.

The inverse heat conduction problem of determining the unknown boundary conditions exists widely in scientific research and engineering. Although there has been much research into the problem, most of it has been focused on the one- or two-dimensional linear problems and little has been done on the multidimensional nonlinear problems. Beck's group conducted some outstanding work on the one-dimensional inverse problem [1]. The Monte Carlo method [2] and integral method [3] were developed but they are not available for nonlinear problems. Huang and Ozisik [4] proposed an improvement by combining the regular gradient method with the modified conjugate gradient method to determine the unknown heat flux for laminar flow through a parallel plate duct. The convergence speed of the iteration with this method was relatively fast but the method was restricted by the initial value of the iteration. The two-dimensional inverse problem was studied mathematically by Li [5] by using the normed linear space theory and a solution was suggested. Obviously, this method could only be used for the linear problem.

In this paper, an iteration method is developed by means of designing iterated direction and steplength of the unknown boundary condition. It is successfully applied to determine the unknown heat transfer coefficient for fluid flow in electrically heated coiled tubes.

2. Mathematical Model and Discrete Equation

For a fluid flowing inside an electrically heated helical coil tube, the heat transfer coefficients between the fluid and the coil are to be determined. The geometrical structure of the coil is illustrated in Fig. 1, where t and R stand for pitch and diameter of the coil, respectively. Because of the great difficulty measuring the boundary condition on Γ_1, thermocouples are installed to measure the wall temperature distribution on the outside of the wall surface. The mathematical equations describing the problem are given by

$$\frac{1}{\gamma_1\gamma_2\gamma_3}\left[\frac{\partial}{\partial X_1}\left(k\frac{\lambda_2\lambda_3}{\lambda_1}\frac{\partial T}{\partial X_1}\right)+\frac{\partial}{\partial X_2}\left(k\frac{\lambda_1\lambda_3}{\lambda_2}\frac{\partial T}{\partial X_2}\right)\right]+\dot{S}=0 \tag{1}$$

$$k\frac{\partial T}{\gamma_1\partial X_1}=f(X_2) \qquad \text{for } X_1=r_o \tag{2}$$

$$k\frac{\partial T}{\gamma_1\partial X_1}=h(X_2)(T-T_f) \qquad \text{for } X_1=r_i \tag{3}$$

where k is the thermal conductivity of the coil, which is a function of temperature. $\dot{S}$ is the interior heat source, and T_f is the bulk temperature of the fluid. λ_1, λ_2, and λ_3 are Lamé coefficients, which are defined by the following equations for the case of the coils:

$$\lambda_1 = 1 \tag{4}$$

$$\lambda_2 = r \tag{5}$$

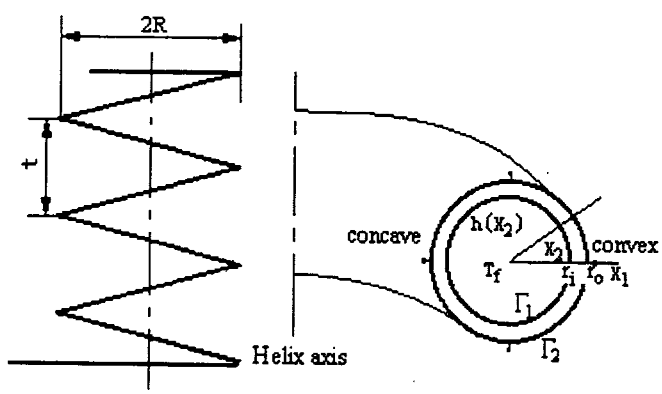

Fig. 1. Geometrical model (▼thermocouple).

$$\lambda_3 = \{[(t/2\pi)^2 + (R + X_1\cos X_2)^2/\cos^2\alpha]/[R^2 + (t/2\pi)^2]\}^{0.5} \tag{6}$$

$$\alpha = tg^{-1}[(X_1\sin\beta\sin X_2)/(R + X_1\cos X_2)] \tag{7}$$

$$\beta = tg^{-1}(t/2\pi R) \tag{8}$$

In Eq. (3), $h(X_2)$ is the unknown local convective heat transfer coefficient and is to be estimated. The thermocouples give some information about the measured temperature

$$T(X_1, X_{2i}) = Y_i,\ i = 1, 2, \ldots, N_m \text{ for } X_1 = r_o \tag{9}$$

where N_m is the number of thermocouples.

The boundary conditions on Γ_1 and Γ_2 can be of other types according to the practical situations.

The calculated domain is divided using the interior node method [6]. Integrating Eq. (1) over the control volume P, which is illustrated in Fig. 2, the discrete equation is obtained as follows:

$$a_P T_P = a_W T_W + a_E T_E + a_S T_S + a_N T_N + b \tag{10}$$

where the concrete equations of a_P, a_W, a_E, a_S, a_N, and b are available in many reference sources including Ref. 6. Any kinds of boundary conditions are processed by the attachment source method [6], which are included in b.

3. Solution Method and Computational Procedures

With a supposed boundary condition, $h(X_2)$, a temperature field of the coil can be calculated by using the direct heat conduction solution method. Obviously, the temperature fields vary according to the different boundary conditions. However, the boundary condition can be taken as the final estimated value only when the calculated temperatures are equal to the measured ones at all locations.

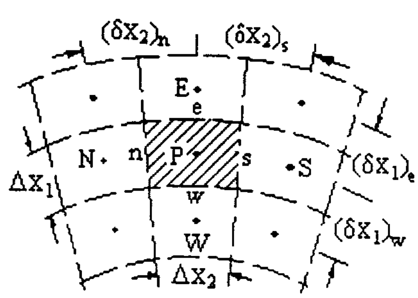

Fig. 2. Control volume P.

Therefore, an iteration solution method for the inverse problem can be obtained. The first boundary condition, $h(X_2)$, is selected arbitrarily as the initial iterated value, and then the temperature field can be calculated. If the measured temperature is higher than the calculated one at one location, it means that the heat transfer between the fluid and the wall is lower and the heat transfer coefficient will be modified greater at the next iteration. If the measured temperature is lower than the calculated one, it means that the heat transfer is greater and the heat transfer coefficient will be modified smaller at the next iteration. The modified equation of the estimated boundary condition can be given by

$$h(X_{2i}) = h(X_{2i})^* + a(T_i - Y_i) \text{ for } i = 1, 2, \ldots, N_m \tag{11}$$

where $h(X_{2i})^*$ is the last iteration, a is the iterated steplength and greater than zero, $(T_i - Y_i)$ is the iterated direction, and T_i is the calculated temperature at the location of Y_i. If the following equation is true, the iteration terminates and the results are output, otherwise, iteration repeats with the modified boundary condition.

$$|T_i - Y_i|_{\max} \leq \varepsilon \tag{12}$$

4. Measurement Errors and Numerical Test

It is well known that the error of the measured temperature has a great effect on the accuracy of the estimated boundary condition. However, the measurement error of the wall temperature cannot be avoided. Therefore, its effect must be taken into condition before the algorithm is applied.

Some examples for two-dimensional problems are solved with this method. The structure of the coil tube is $r_i = 5.5$ mm, $r_o = 7.5$ mm, $R = 128$ mm, and $t = 60$ mm. Its thermal conductivity, k, is 14.282(1 + 0.001 T) W/(m °C). The electrical power per meter of the coil is 8.7 kW/m. The bulk temperature of the fluid is 113.4 °C. The insulation efficiency of the heated coil is 0.95. Eight thermocouples are installed evenly on the periphery of the outer wall as illustrated in Fig. 1. The distribution of the heat transfer coefficient is supposed as follows:

1. Triangular distribution

$$h(X_2) = \begin{cases} 7000 + 14000/\pi X_2, & 0 \leq X < \pi \\ 35000 - 14000/\pi X_2, & \pi \leq X_2 < 2\pi \end{cases} \tag{13}$$

2. Trapezoidal distribution

$$h(X_2) = \begin{cases} 14000, & 0 \leq X < \pi \\ 35000, & \pi \leq X_2 < 2\pi \end{cases} \tag{14}$$

3. Sine wave distribution

$$h(X_2) = 7000(1.5 + 0.5 \sin X_2) \tag{14}$$

Table 1. Effect of the Measurement Error on the Solution Method

Measurement error	–0.5	–0.2	0.0	0.2	0.5
Triangle	3.4	0.7	0.1	2.0	4.9
Trapezoid	2.1	0.7	0.5	1.8	3.4
Sine wave	2.3	0.6	0.4	1.8	3.4

By using the supposed boundary condition, the temperature distribution on the periphery of the tube can be obtained by the direct solution method. To estimate the effects of the measurement error on the estimated boundary condition, the calculated temperature on the outer wall is taken as the real value and the measured temperature is replaced by the sum of the real value and the measured errors. The measured errors of the wall temperature are supposed as 0, ±0.2, ± 0.5 °C. The unknown boundary conditions are estimated by the above-mentioned method. The maximum relative errors, E_h, between the estimated heat transfer coefficient and the real value are employed to describe the advantage of the method,

$$E_h = \frac{|h_{estimated} - h_{true}|_{\max}}{h_{true,max}} \times 100\% \tag{16}$$

The results are shown in Table 1. If the measured temperature has no errors, the estimated heat transfer coefficients are very close to the real values. The difference between the real value and the estimated value results from numerical discrete errors and the rounding errors. When the error occurs in the measured temperature, the estimated heat transfer coefficients stray from the real value. The greater the measured error is, the greater the error of the estimated boundary condition. The estimated error depends linearly on the measured error of the wall temperature. When the measured error of the wall temperature is less than 0.5 °C, the maximum estimated error of the unknown boundary condition is less than 5%. Therefore, the solution method has the advantage of good stability.

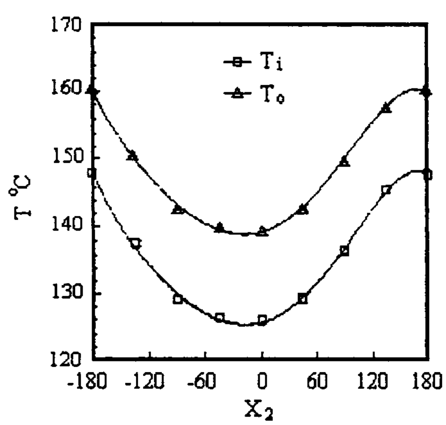

Fig. 3. Tube wall temperature distributions.

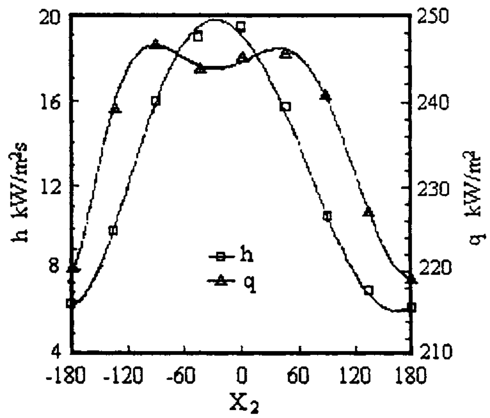

Fig. 4. The distributions of the heat transfer coefficients and heat flux.

In this paper, a numerical test case is given according to experimental data. The test run is $P = 0.5$ MPa, $\Delta T_{sub} = 40$ °C, $G = 2250$ kg/(m^{2}s), $q = 240$ kW/m^2. The results obtained with the algorithm are shown in Figs. 3 and 4. The distribution of the temperature on the inner surface of the tube wall is similar to that on the outer surface of the tube wall. The maximum difference between the temperature on the outer and the inner surface is 14 °C. When X_2 is equal to 0°, the tube wall temperature is the greatest and the local heat transfer coefficient and the local heat flux have the smallest values. When X_2 is equal to 180°, the tube wall temperature is the lowest, the local heat transfer coefficient is the greatest, but the local heat flux does not have the greatest value. The calculated results are closely related to the phenomena of the fluid flowing in the coil. It can be seen from Fig. 1 that when X_2 is equal to 0°, it is at the convex of the coil and for π is at the concave of the coil. As a result of the centrifugal force of the fluid flowing in the coil, the fluid velocity on the convex is greater than that at other locations. Therefore, the heat transfer on the convex has the maximum value. But the heat flux on the convex is not the maximum because the heat conductivity of the wall is a function of the temperature and does not distribute evenly. The numerical test shows that the heat conduction of the coil should be taken as the two-dimensional model.

5. Conclusions

A two-dimensional nonlinear inverse heat conduction problem can be solved by the iteration and modification method. The method is successfully applied to estimate the unknown heat transfer coefficient between the fluid and the electrically heated helical coil tube. The effect of the measurement error of the wall temperature on the algorithm is numerically tested. The results of the numerical tests prove that this method has the advantages of fast convergence, high precision, and good stability. The unknown boundary condition can be obtained with accuracy although measurement errors occur in the measured temperatures.

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